Homotopy Type Theory
and
Univalent Foundations of Mathematics

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CMU PAL

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3. The computational implementation of type theory allows computer verified proofs in homotopy theory.
4. The homotopy interpretation suggests new logical constructions and axioms.
5. Voevodsky's *Univalent Foundations* program combines these aspects into a new program of foundations for mathematics.

Type theory

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Formal calculus of terms and equations – like polynomials, only more complicated.

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This is also known as the *Curry-Howard correspondence*.

Identity types

According to the logical interpretation we have:

- **propositional logic**: $A \times B, A \to B$,
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On the **mathematical** side, the identity type admits a newly discovered geometric interpretation.

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The **elimination** rule is a form of Leibniz's law:

$$\frac{a : A \;\vdash\; d(a) : D\big(a, a, \mathtt{r}(a)\big)}{c : \mathtt{Id}_A(a, b) \;\vdash\; \mathtt{J}_d(a, b, c) : D(a, b, c)}$$

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Schematically:

$$D(a, a) \ \& \ \mathtt{Id}_A(a, b) \ \Rightarrow \ D(a, b)$$

Intensionality

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$$a = b$$

then they are also **identical**:

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It gives rise to a structure of great combinatorial complexity.

The homotopy interpretation

Suppose we have terms of ascending identity types:

$$\begin{aligned} a,\ b &: A \\ p,\ q &: \mathtt{Id}_A(a,b) \\ \alpha,\ \beta &: \mathtt{Id}_{\mathtt{Id}_A(a,b)}(p,q) \\ \ldots &: \mathtt{Id}_{\mathtt{Id}_{\mathtt{Id}_{\ldots}}}(\ldots) \end{aligned}$$

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Consider the following interpretation:

$$\begin{array}{rcl} \text{Types} & \rightsquigarrow & \text{Spaces} \\ \text{Terms} & \rightsquigarrow & \text{Maps} \\ a : A & \rightsquigarrow & \text{Points } a : 1 \longrightarrow A \\ p : \mathtt{Id}_A(a,b) & \rightsquigarrow & \text{Paths } p : a \Rightarrow b \\ \alpha : \mathtt{Id}_{\mathtt{Id}_A(a,b)}(p,q) & \rightsquigarrow & \text{Homotopies } \alpha : p \Rrightarrow q \\ \vdots & & \end{array}$$

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But topologically, it is a **lifting property**:

$$\begin{tikzcd}
B \arrow[d] & \overline{a} \arrow[r, dotted] & p_* \overline{a} \\
A & a \arrow[r, "p"'] & b
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This is the notion of a "fibration".

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Thus we continue the homotopy interpretation as follows:

$$\text{Dependent types} \quad x : A \vdash B(x) \quad \rightsquigarrow \quad \text{Fibrations} \quad \begin{tikzcd} B \arrow[d] \\ A \end{tikzcd}$$

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The type $B(a)$ is the fiber of $B \longrightarrow A$ over the point $a : A$

$$\begin{tikzcd} B(a) \arrow[r] \arrow[d] & B \arrow[d] \\ 1 \arrow[r, "a"'] & A. \end{tikzcd}$$

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Identity type $\quad x, y : A \vdash \mathtt{Id}_A(x, y) \quad \rightsquigarrow \quad$ Path space $\quad$

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$$\text{Identity type} \quad x, y : A \vdash \mathtt{Id}_A(x, y) \quad \rightsquigarrow \quad \text{Path space} \quad \begin{array}{c} A^I \\ \downarrow \\ A \times A \end{array}$$

The fiber $\mathtt{Id}_A(a, b)$ over a point $(a, b) \in A \times A$ is the space of paths from a to b in A.

$$\begin{tikzcd}
\mathtt{Id}_A(a,b) \arrow[r] \arrow[d] & A^I \arrow[d] \\
1 \arrow[r, "{(a,b)}"'] & A \times A.
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The homotopy interpretation

The path space A^I classifies homotopies $\vartheta : f \Rightarrow g$ between maps $f, g : X \to A$,

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So given any terms $x : X \vdash f, g : A$, an identity term

$$x : X \vdash \vartheta : \mathtt{Id}_A(f, g)$$

is interpreted as a homotopy between f and g.

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- Gives a wide range of different models.
- Includes classical homotopy of spaces and simplicial sets.
- Allows the use of standard methods from categorical logic.

Soundness of the homotopy interpretation

Theorem (Awodey & Warren 2008)

Martin-Löf type theory has a **sound** *interpretation into any Quillen model category.*

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- There is an issue of "coherence" of the interpretation, which requires a technical condition on the QMC.
- One doesn't need the full QMC structure, but only a *weak factorization system*.

Soundness and completeness

The logical notion of **soundness** means that a provable statement is always true under the specified interpretation:

$$\text{provable} \quad \xrightarrow{\textit{sound}} \quad \text{true in all models}$$

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The converse notion is **completeness**: a statement is provable if its interpretation is always true:

$$\text{provable} \quad \xLeftarrow{\textit{complete}} \quad \text{true in all models}$$

Completeness of the homotopy interpretation

Theorem (Gambino & Garner 2009)

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A benefit of the abstract semantics: the proof uses the standard method of *syntactic categories* to construct a canonical model.

Conclusion of Part I

Martin-Löf type theory provides a "logic of homotopy".

The Fundamental Groupoid of a Type

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What homotopically relevant facts, properties, and constructions are logically expressible?

One example: the topological **fundamental group** and its higher generalizations are logical constructions.

Fundamental groupoids

Let's return to the system of identity terms of various orders:

$$\begin{aligned} a,\ b &: A \\ p,\ q &: \mathtt{Id}_A(a,b) \\ \alpha,\ \beta &: \mathtt{Id}_{\mathtt{Id}_A(a,b)}(p,q) \\ \vartheta &: \mathtt{Id}_{\mathtt{Id}_{\mathtt{Id}_{\dots}}}(\alpha,\beta) \end{aligned}$$

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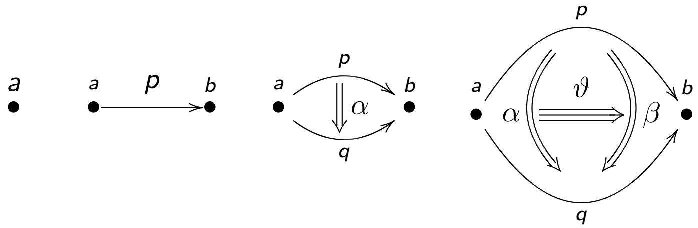
These can be represented suggestively as follows:



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As in topology, the terms of order 0 and 1, (“points” and “paths”),

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The laws of identity correspond to the **groupoid operations**:

$$\begin{array}{rll}
r : \mathtt{Id}(a, a) & \text{reflexivity} & a \longrightarrow a \\
s : \mathtt{Id}(a, b) \longrightarrow \mathtt{Id}(b, a) & \text{symmetry} & a \leftrightarrows b \\
t : \mathtt{Id}(a, b) \times \mathtt{Id}(b, c) \longrightarrow \mathtt{Id}(a, c) & \text{transitivity} & a \longrightarrow b \longrightarrow c
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This was first shown by Hofmann & Streicher (1998), who gave a model of intensional type theory using groupoids as types.

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But also just as in topology, the **groupoid equations** of associativity, inverse, and unit:

$$p \cdot (q \cdot r) = (p \cdot q) \cdot r$$
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This means they are witnessed by terms of the next higher order:

$$\vartheta : \mathtt{Id}_{\mathtt{Id}}\left(p^{-1} \cdot p, 1\right)$$

$$\begin{tikzcd}
 & b \arrow[dr, bend left, "p^{-1}"] \arrow[d, phantom, "\Downarrow \vartheta"] & \\
a \arrow[ur, bend left, "p"] \arrow[rr, bend right, "1"'] & \, & a
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Fundamental groupoids

The entire system of identity terms of all orders forms an infinite-dimensional graph, or "globular set":

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It has the structure of a **weak**, infinite-dimensional, groupoid (as defined by Batanin 1998 and occurring homotopy theory):

Theorem (Lumsdaine, Garner & van den Berg, 2009)

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Every type has a **fundamental groupoid**.

Conclusion of Part II

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- Logical methods suffice in principle to capture a great deal of homotopy theory.

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- the foregoing representation of homotopy theory in constructive type theory
- the well-developed implementations of type theory in computational proof assistants like Coq and Agda.

Allows computer verified proofs in homotopy theory, and related fields.

A computational example

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- Now the Eckmann-Hilton argument shows that the two structures on $\pi_2(X, b)$ agree, and are abelian.

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We can formalize this in type theory:

- the fundamental group $\pi_1(X, b)$ of a type X at basepoint $b : X$ consists of terms of type $\mathtt{Id}_X(b, b)$.
- the second homotopy group $\pi_2(X, b)$ consists of terms of type $\mathtt{Id}_{\mathtt{Id}_X(b,b)}(\mathtt{r}(b), \mathtt{r}(b))$.
- Each of these types has a group structure, and so the second one has *two* group structures that are compatible.
- Now the Eckmann-Hilton argument shows that the two structures on $\pi_2(X, b)$ agree, and are abelian.

This argument can be formalized in the automated proof assistant Coq and verified to be correct. In this way, we can use the homotopical interpretation to verify proofs in homotopy theory.

A computational example

```
(* An adaptation to Coq of Dan Licata's Agda proof that the higher homotopy groups are abelian,
   by Jeremy Avigad. This file depends on the library in the "UnivalentFoundations" directory of
   Andrej Bauer's Github repository. The code is written for Coq 8.3, which means that variables
   are introduced automatically.*)

Implicit Arguments homotopy_concat [A x y z p p' q q'].
Implicit Arguments idpath_left_unit [A x y].
Implicit Arguments idpath_right_unit [A x y].

Lemma map2 {A B C} {x x' : A} {y y' : B} (f : A -> B -> C)
  (p : x ~~> x') (p' : y ~~> y') : (f x y) ~~> (f x' y').
Proof. induction p; induction p'; trivial. Defined.

(* The next four lemmas are needed to prove the left and right identity laws,
    generalizing those laws to path spaces. *)

Lemma adjust_l {A} {x y : A} {p q : x ~~> y} (R : p ~~> q) :
  idpath x @ p ~~> idpath x @ q.
Proof. exact (idpath_left_unit p @ R @ !(idpath_left_unit q)). Defined.
(* induction R doesn't given a term that is explicit enough. *)

Lemma homotopy_concat_id_left {A} {x y : A} {p p' : x ~~> y}
  (R : p ~~> p') : homotopy_concat (idpath (idpath x)) R ~~> adjust_l R.
Proof. induction R; induction x0; trivial. Defined.

Lemma adjust_r {A} {x y: A} {p q : x ~~> y} (R : p ~~> q) :
  p @ idpath y ~~> q @ idpath y.
Proof. exact (idpath_right_unit p @ R @ !(idpath_right_unit q)). Defined.
```

```
Lemma homotopy_concat_id_right {A} {x y : A} {p p' : x ~~> y}
 (R : p ~~> p') : homotopy_concat R (idpath (idpath y)) ~~> adjust_r R.
Proof. induction R; induction x0; trivial. Defined.

Lemma concat_interchange {A} {x y z : A} {p q r : x ~~> y} {p' q' r' : y ~~> z}
  {R : p ~~> q} {S : q ~~> r} {T : p' ~~> q'} {U : q' ~~> r'} :
  homotopy_concat (R @ S) (T @ U) ~~>
    (homotopy_concat R T) @ (homotopy_concat S U).
Proof.
  induction R; induction S; induction T; induction U.
  induction x0; induction x1; trivial.
Defined.

(* Here is the standard proof. It is phrased in terms of Pi_2, but instantiating "A" and "base"
   accordingly yields the corresponding result for any n >= 2. *)

Section Pi2_Abelian.

Variables (A : Type) (base : A).

Definition Pi1 := (base ~~> base).
Definition Pi2 := (idpath base) ~~> (idpath base).

Notation "p @@ q" := (homotopy_concat p q) (at level 60).
Notation "[id]" := (idpath (idpath base)).

Lemma comp_left_unit {p : Pi2} : [id] @@ p ~~> p.
Proof.
  apply (concat (homotopy_concat_id_left p)).
  path_via (p @ [id]); apply idpath_left_unit.
Defined.
```

```
Lemma comp_right_unit {p : Pi2} : p @@ [id] ~~> p.
  apply (concat (homotopy_concat_id_right p)).
  path_via (p @ [id]); apply idpath_left_unit.
Defined.

Lemma comp_interchange  {a b c d : Pi2} :
  (a @ b) @@ (c @ d) ~~> (a @@ c) @ (b @@ d).
Proof. exact concat_interchange. Defined.

Lemma comp_same {a b : Pi2} : a @ b ~~> a @@ b.
Proof.
  path_via ((a @@ [id]) @ b). apply (!comp_right_unit).
  path_via ((a @@ [id]) @ ([id] @@ b)). apply (!comp_left_unit).
  path_via ((a @ [id]) @@ ([id] @ b)). apply (!comp_interchange).
  path_via (a @@ ([id] @ b)).
    apply map2; [apply idpath_right_unit | apply idpath].
  apply map2; [apply idpath | apply idpath_left_unit].
Defined.

(* Here path_via calls path_tricks, which decomposes "_ @ _ = _ @ _"  too aggressively. *)
Lemma Pi2_abelian {a b : Pi2} : a @ b ~~> b @ a.
Proof.
  apply @concat with (y := ([id] @@ a) @ b).
    path_tricks; apply (!comp_left_unit).
  apply @concat with (y := ([id] @@ a) @ (b @@ [id])).
    path_tricks; apply (!comp_right_unit).
  apply (concat (!comp_interchange)); apply (concat (!comp_same)); path_tricks.
Defined.

End Pi2_Abelian.
```

Conclusion of Part III

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- The program is being pursued by a group of researchers, formalizing parts of homotopy theory and other mathematics in this settting.
- Some new logical constructions and axioms are suggested by the homotopy interpretation.

Higher-dimensional inductive types

(Work in progress by Lumsdaine, Shulmann & others.)

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$$o : \mathbb{N}$$
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$$\frac{a : X \qquad f : X \to X}{\mathrm{rec}(a, f) : \mathbb{N} \to X}$$

such that:

$$\mathrm{rec}(a, f)(o) = a$$
$$\mathrm{rec}(a, f)(sn) = f(\mathrm{rec}(a, f)(n))$$

Higher-dimensional inductive types

This says just that $(\mathbb{N}, o, s)$ is the *free* structure of this type:

$$\begin{tikzcd} & 1 \arrow[dl, "o"'] \arrow[dr, "a"] & \\ \mathbb{N} \arrow["s"', loop left] \arrow[rr, dotted, "\mathsf{rec}"'] & & X \arrow["f", loop right] \end{tikzcd}$$

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The map $\mathsf{rec}(a, f) : \mathbb{N} \to X$ is unique with this property.

Higher-dimensional inductive types

The topological circle $\mathbb{S} = S^1$ can also be given as an inductive type, now involving a higher-dimensional generator:

$$b : \mathbb{S}$$
$$p : b \rightsquigarrow b$$

Here we have written $p : b \rightsquigarrow b$ for the "loop" $p : \mathtt{Id}_{\mathbb{S}}(b, b)$.

Higher-dimensional inductive types

There is an associated recursion property, captured again by an elimination rule:

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such that:

$$\mathrm{rec}(a, q)(b) = a$$
$$\mathrm{rec}(a, q)_1(p) = q$$

Here $\mathrm{rec}(a, q)_1$ is the effect of the map $\mathrm{rec}(a, q)$ on paths.

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This says that $(\mathbb{S}, b, p)$ is the *free* structure of this (higher) type:

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b \arrow[loop left, "p"] & & \mathbb{S} \arrow[rr, dotted, "\mathsf{rec}"'] & & X & & a \arrow[loop right, "q"]
\end{tikzcd}
$$

The map $\mathsf{rec}(a, q) : \mathbb{S} \to X$ is then unique up to homotopy.

Higher-dimensional inductive types

Here is a sanity check:

Theorem (Shulmann 2011)

The type-theoretic circle $\mathbb{S}$ has the correct homotopy groups: $\pi_1(\mathbb{S}) = \mathbb{Z}$, and $\pi_n(\mathbb{S}) = 0$ when $n \neq 1$.

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Theorem (Shulmann 2011)

The type-theoretic circle $\mathbb{S}$ *has the correct homotopy groups:* $\pi_1(\mathbb{S}) = \mathbb{Z}$, *and* $\pi_n(\mathbb{S}) = 0$ *when* $n \neq 1$.

The proof is implemented in Coq. It combines classical homotopy theory with methods from constructive type theory, and uses Voevodsky's new Univalence Axiom.

Higher-dimensional inductive types

The unit interval $\mathbb{I} = [0, 1]$ is also an inductive type, on the data:

$$\begin{aligned} 0, 1 &: \mathbb{I} \\ p &: 0 \rightsquigarrow 1 \end{aligned}$$

Again we have written $p : 0 \rightsquigarrow 1$ for the path $p : \mathtt{Id}_{\mathbb{I}}(0, 1)$.

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Remark. In topology, the interval is used to define the notion of a path. Here we have the notion of a path as a logical primitive, and can use it to define the interval.

Higher-dimensional inductive types

Many other basic spaces and constructions can be introduced in this way:

- the higher spheres S^n and disks D^n,
- the suspension ΣA of a space A,
- finite cell complexes, tori, cylinders, ...,
- homotopy algebras – i.e. algebraic structures with equations holding up to homotopy,
- the mapping cylinder of a map $f : A \to B$.

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- the mapping cylinder of a map $f : A \to B$.

Using higher-inductive types, one can show there is a rudimentary Quillen model structure in the type theory.

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New "invariant" foundations for mathematics, with geometric content and a computational character.

Conclusion

Under this new homotopy interpretation, constructive type theory captures a substantial amount of homotopy theory, permitting purely formal reasoning which can even be implemented on a computer.

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Under this new homotopy interpretation, constructive type theory captures a substantial amount of homotopy theory, permitting purely formal reasoning which can even be implemented on a computer.

The homotopy interpretation also suggests a new approach to foundations of math with intrinsic geometric content, capturing some forms of reasoning more naturally than traditional foundations in set theory.

References and Further Information

www.HomotopyTypeTheory.org