is at hand.[14] For such a theory it can be shown that *any* mode of statement composition, other than the truth functions, is referentially opaque. For, let ϕ and ψ be any statements alike in truth value, and let $\Phi(\phi)$ be any true statement containing ϕ as a part. What is to be shown is that $\Phi(\psi)$ will also be true, unless the context represented by 'Φ' is referentially opaque. Now the class named by $\hat{\alpha}\phi$ is either V or Λ, according as ϕ is true or false; for remember that ϕ is a statement, devoid of free α. (If the notation $\hat{\alpha}\phi$ without recurrence of α seems puzzling, read it as $\hat{\alpha}(\alpha = \alpha \,.\, \phi)$.) Moreover ϕ is logically equivalent to $\hat{\alpha}\phi = V$. Hence, by (a), since $\Phi(\phi)$ is true, so is $\Phi(\hat{\alpha}\phi = V)$. But $\hat{\alpha}\phi$ and $\hat{\alpha}\psi$ name one and the same class, since ϕ and ψ are alike in truth value. Then, since $\Phi(\hat{\alpha}\phi = V)$ is true, so is $\Phi(\hat{\alpha}\psi = V)$ unless the context represented by 'Φ' is referentially opaque. But if $\Phi(\hat{\alpha}\psi = V)$ is true, then so in turn is $\Phi(\psi)$, by (a).

[14] See *From a Logical Point of View*, pp. 27, 87.

II

MODALITY AND DESCRIPTION

ARTHUR F. SMULLYAN

THERE are logicians who maintain that modal logic violates Leibniz's principle that if x and y are identical, then y has every property of x. The alleged difficulty is illustrated in the following example due to Quine.[1]

(a) It is logically necessary that 9 is less than 10.
(b) 9 = the number of the planets.
(c) Therefore, it is logically necessary that the number of the planets is less than 10.

The premisses of this argument are true, the conclusion is false, and yet the conclusion appears to be derived by means of the logical precept that if x is y then any property of x is a property of y. Such is the paradox of modal logic. But the difficulty is obviated if we draw a distinction. We must distinguish between statements of the following forms:

(d) The so-and-so satisfies the condition that it is necessary that Fx.
and
(e) It is necessary that the so-and-so satisfies the condition that Fx.

The reader at this stage is bound to feel as though he were being asked to distinguish between Tweedledum and Tweedledee. Possibly, it will be of assistance to him to remark that statements of type (d) are sometimes synthetic, whereas those of type (e) are never synthetic. I will ask the reader to believe that James is now thinking of the number 3. If, now, some one were to remark, 'there is one and only one integer which James is now thinking of and that integer is necessarily odd', then he would be stating a contingent truth. For that there is just one integer which James now thinks of, is only an empirical fact. This statement could just as well be expressed in the form, (d), 'The integer, which James is now

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[1] W. Quine, 'Notes on Existence and Necessity', *Journal of Philosophy*, XL (1943), 113–27.
We assume, at this stage of the discussion, that '9' and '10', as they occur in the illustration, are *names* of familiar logical *properties*. This is in order to simplify the introductory discussion. In due course we shall consider the problem in a more general way.

thinking of, satisfies the condition that it is necessarily odd.' In contrast, the statement, 'It is necessary that James's integer is odd', which is of the form, (e) is an impossible statement and not a contingent one. If not necessary, then necessarily not necessary: at least, so we assume.

The conclusion, (c) is of the form, (e), and it does not follow logically from (a) and (b). Leibniz's law does not require that (a) and (b) entail (c). What Leibniz's law does permit us to infer from the premisses (a) and (b) is the statement,

(f) As a matter of brute fact, the number of planets satisfies the condition that it is necessary that x is less than 10.

It is to be noted that this sentence (f) is true, synthetic and not paradoxical. On the other hand, the statement (c) is not only incorrectly inferred from the premisses, but is, moreover, logically impossible. For it is false, and, as we have already said, a false sentence which attributes necessity is *logically* false. We have just noted that (c) is of the form (e), whereas the valid conclusion is of the form (d).

In order to show that the difficulty is not essentially connected with Leibniz's principle, we shall consider another specious but equally instructive argument. We shall assume the truth of a synthetic sentence

(1) $E!(\imath x)(Fx)$

and we assume the principle

(2) $(x)[N(x = x)]$, where 'N' means what we express by the idiom, 'it is necessary that'.

The critics of modal logic might very well allege that we are by that logic committed to infer from (1) and (2) the statement,

(3) $N[(\imath x)(Fx) = (\imath x)(Fx)]$.

The absurdity of this derivation will soon be made apparent to the reader. What we are committed by modal logic to infer is the quite different statement,

(4) $(\exists x)\,(Fu \equiv_u u = x \,.\, N(x = x))$.

This statement is not at all equivalent to

(5) $N[(\exists x)\,(Fu \equiv_u u = x \,.\, x = x)]$, which is logically equivalent to (3).

The reader will note that if in place of (1) we had written '$N[E!(\imath x)(Fx)]$' we could then have deduced (5) which says that it is necessary that $(\imath x)(Fx)$ is self-identical or, in other words, that it is necessary that there exists

just one instance of the property F. But from the premisses (1) and (2) we may validly infer only (4) which asserts that *in fact* one and only one instance of F exists satisfying the condition that it is necessarily self-identical. The reader will already have observed that (4) is a statement of the sort (d), whereas (3) and (5) are statements of type (e).

The student of logic will at this stage remaind us of theorem *14.18 of *Principia Mathematica* which is the sentence,

$$\vdash :.\ E!(\imath x)(\varphi x) \,.\supset :\ (x) \,.\, \psi x \,.\supset\, \psi(\imath x)(\varphi x).$$

Does not this formula enable us to deduce (5), which is equivalent to (3), from (1) and (2)? Surely, all we need to do is to perform these substitutions on *14.18, viz., substitute 'f' for 'φ', '$N(x = x)$' for 'ψx', and use *modus ponens* twice? But the student who so objects is committing the subtle fallacy of misreading the scope of the description, '$(\imath x)(\varphi x)$'. In *Principia Mathematica* it is assumed that the scope of a description is the smallest formula containing that description, unless it is to the contrary indicated. In the case of *14.18, the scope of the second occurrence of the descriptive phrase is '$\psi(\imath x)(\varphi x)$'. It is only by neglecting this consideration that one is led to deduce (3) from (1) and (2) in place of the correct deduction of (4).

But the tireless objector will try once more. 'Surely', we can hear him saying, 'in case "$E!(\imath x)(\varphi x)$" is true, the scope of the descriptive phrase can be ignored and "$(\imath x)(\varphi x)$" may be treated as a name.' But this is an error which impedes the development of modal logic. It cannot be demonstrated [because it is not so] that if f is *any* function of propositions, then

$$E!(\imath x)(\varphi x) :\supset: f\{[(\imath x)(\varphi x)] \,.\, \chi(\imath x)(\varphi x) \,.\,\} \,.\equiv.\, [(\imath x)(\varphi x)] \,.\, f\{\chi(\imath x)(\varphi x)\}.$$

What *can* be regarded as established is *14.3 in *Principia Mathematica* which asserts that when '$E!(\imath x)(\varphi x)$' is true and when '$(\imath x)(\varphi x)$' occurs in a *truth-functional* context, then the scope of '$(\imath x)(\varphi x)$' does not affect the truth value of the sentence in which it occurs. *14.3 reads as follows:

$$\vdash :.\ p \equiv q \supset_{p,q} :.\ f(p) \equiv f(q) : E!(\imath x)(\varphi x) :\supset: f\{[(\imath x)(\varphi x)] \,.\, \chi(\imath x)(\varphi x) \,.\,\} \,.\equiv.\, [(\imath x)(\varphi x)] \,.\, f\{\chi(\imath x)(\varphi x)\}.$$

However, in non-truth-functional contexts the scope of the description, even when $E!(\imath x)(\varphi x)$, does matter to the truth value of the context. From (4) we cannot deduce (5). From (f) we cannot deduce (c).

One of the possible sources of the confusion which we are trying to eliminate is to be found in *Principia Mathematica* itself where the authors inadvertently assert, on p. 186, vol. I,

> 'It should be observed that the proposition in which $(\iota x)(\varphi x)$ has the larger scope always implies the corresponding one in which it has the smaller scope, . . .'

It is evident that this pronouncement holds good only when truth functional contexts are in question. In non-truth-functional contexts, the contention fails to hold. We cannot, for example, derive from the sentence $[(\iota x)(\varphi x)] \,.\, N(\psi(\iota x)(\varphi x))$., the sentence, $N(\psi(\iota x)(\varphi x))$, although the description in the latter proposition has the smaller scope. This is an important difference between intensional and extensional contexts.

Let us now return to the specious argument with which we began, in which (c) was derived from (a) and (b). This argument may be taken as illustrating the abstract form

$$\begin{array}{l} N(Fy) \\ \underline{y = (\iota x)(\varphi x)} \\ \therefore N[F(\iota x)(\varphi x)] \end{array}$$

The fallacy implicit in this mode of argument consists in taking the scope of the description in the conclusion to be '$F(\iota x)(\varphi x)$'. That is to say, the valid argument-form is rather

$$\begin{array}{l} N(Fy) \\ \underline{y = (\iota x)(\varphi x)} \\ \therefore [(\iota x)(\varphi x)] \,.\, N(F(\iota x)(\varphi x)). \end{array}$$

For the second premiss of this argument is, by definition, equivalent to

$$(\exists x)(\varphi z \equiv_z z = x : y = x),$$

which, in conjunction with the first premise, yields

$$(\exists x)(\varphi z \equiv_z z = x : y = x \,.\, N(Fy)).$$

This, by Leibniz's law gives

$$(\exists x)(\varphi z \equiv_z z = x \,.\, N(Fx))$$

which is the same proposition as

$$[(\iota x)(\varphi x)] \,.\, N(F(\iota x)(\varphi x)).$$

The reader should note that to obtain the conclusion, $N(F(\iota x)(\varphi x))$, it would be possible to strengthen the second of the premisses. I.e., the following argument-form is valid:

$$\begin{array}{l} N(Fy) \\ \underline{N(y = (\iota x)(\varphi x))} \\ \therefore N(F(\iota x)(\varphi x)) \end{array}$$

Similarly,

$$\begin{array}{l} (x)N(Fx) \\ \underline{N(E!(\iota x)(\varphi x))} \\ \therefore N(F(\iota x)(\varphi x)) \end{array}$$

and

$$\begin{array}{l} (x)N(Fx) \\ \underline{E!(\iota x)(\varphi x)} \\ \therefore [(\iota x)(\varphi x)] \,.\, N(F(\iota x)(\varphi x)) \end{array}$$

are valid argument-forms.

The intention of this paper is to show that the unrestricted use of modal operators in connection with statements *and* matrices embedded in the framework of a logical system such as *Principia Mathematica* does not involve a violation of Leibniz's principle. In order to show this, we have, of course, utilized Russell's method of contextual definition of descriptive phrases and we shall, in the sequel, and for the same reason, adopt contextual definitions of class abstracts.

In the light of our discussion so far, it may suggest itself to the reader that the modal paradoxes arise not out of any intrinsic absurdity in the use of the modal operators but rather out of the assumption that descriptive phrases are names. It may indeed be the case that the critics of modal logic object primarily not to the use of modal operators but to the method of contextual definition as employed, e.g., in Russell's theory of definite descriptions. In this case, however, it would be in the interest of clarity to indicate the prior grounds on which their objections to the theory of descriptions are based.

It is natural in this connection to refer to the reviews of Alonzo Church which antedate important parts of this discussion by about five years.[2] In his review of Quine's essay, to which we principally refer, Church argues that the modal paradoxes indicate that if modal operators are used in a system in which descriptions and class abstracts are construed as names, then these operators must be prefixed not to sentences in the

[2] A. Church, *Journal of Symbolic Logic*, 7 (1942), 100.

system but to the names of their *senses*. One may agree with this hypothetical proposition of Church and use it to defend the contention that *since* we *do* ordinarily prefix modal operators to sentences we are by this fact committed to a logical system in which descriptions *are* contextually defined. This is not to deny, of course, the legitimacy of other constructions. In particular, Church's preference is to prefix modal operators to the names of the senses of sentences rather than to the sentences themselves. But the theory of descriptions appears to have a peculiar relevance to our ordinary use of modal notions. Other sections of Church's review may be adduced as corroboration of our contention that the logical modalities need not involve paradox when they are referred to a system in which descriptions and class abstracts are contextually defined.

We have been discussing logical systems in which descriptions are treated in accordance with the theory of descriptions. But there are logicians who will say that the theory of descriptions is really a 'proposal to do without descriptions'. This is Church's view in the review cited. And it *is* a possible interpretation to make; but to many it will seem preferable to regard a logical system as containing not only formulae in primitive notation but also the abbreviations of those formulae which are effected by the use of descriptive phrases and other defined *idioms*. In semantical discussions this latter interpretation is particularly useful because although we may wish to deny that descriptions are names we may yet wish to insist that some of them *describe* objects. Being a meaningless noise is not the only alternative to being a name.

Our discussion of the argument (abc) may seem to have depended, at certain crucial points, upon our decision to interpret the expression, 'the number of the planets' as a definite description. What would have been the situation had the phrase in question been interpreted as a class abstract, as synonymous with '$\hat{x}$(x is equinumerous with the set of planets)' for example?

The analysis which we have been expounding applies to this interpretation also provided that class abstracts are not treated as names but as incomplete symbols. We shall briefly indicate how this may be shown in the case of systems which use class variables in addition to property variables. Essentially the same analysis holds more generally for systems such as *Principia Mathematica* which have only property variables and which seek to dispense with the assumption of classes. But there are subtleties[3] connected with such reductive techniques which are not germane

[3] See R. Carnap, *Meaning and Necessity* (Chicago, Ill.: Univ. of Chicago Press, 1947), pp. 147–50, where certain inelegancies of the *Principia* approach to class theory are indicated.

to our thesis, and, accordingly, we shall restrict our discussion to logical systems in which class variables are used.

One procedure for treating class abstracts is that followed by Quine,[4] namely to use expressions of the form '$\hat{x}(A)$' as abbreviations of the correlative descriptions '$(\iota\alpha)(A \equiv_x x \epsilon \alpha)$'. If this procedure is followed our entire preceding analysis becomes directly relevant.

A second but essentially equivalent procedure is to provide a contextual definition, in a manner reminiscent of *Principia Mathematica*, defining all contexts of the form '$f\hat{x}(Gx)$'. In order to eliminate confusions of the sort with which this paper is concerned it is essential to employ scope prefixes in connection with class abstracts. This provision is automatically secured by the first method which explicitly interprets class abstracts as singular descriptions. But if the idiom of abstraction is to be introduced in the second way it should be done with explicit reference made to the scope of the abstraction. Thus:

$$[\hat{x}(Gx)] \cdot F\hat{x}(Gx) =_{\mathrm{df}} (\exists\alpha)(Gx \equiv_x x \epsilon \alpha \,.\, F\alpha)$$

Of course it is not customary to use scope prefixes in connection with class abstracts. In extensional systems they would be of little use particularly if it were assumed that every correctly formed matrix determines a class. However, in modal logic, their importance is readily seen from a study of the modal paradoxes.

Since we are here concerned to combine modalities with a logic which assumes the existence of classes, it appears natural to stipulate

(s1) $N(\exists\alpha)(\varphi x \equiv_x x \epsilon \alpha)$

(s2) $N[x \epsilon \alpha \equiv_x x \epsilon \beta \supset \alpha = \beta]$

We shall find it convenient for the present purpose informally to assume the theory of types. Also, the symbol for identity is to be so construed as to satisfy Leibniz's law.

It should be noted that if in place of (s1) we had stipulated

(s3) $(\exists\alpha)N[\varphi x \equiv_x x \epsilon \alpha]$

we would then be able to derive paradoxical results. This may be shown as follows:

Assume that '$Fx \equiv_x Gx$' is a contingent truth. By (s3) we have both

$(\exists\alpha)(N[Fx \equiv_x x \epsilon \alpha])$

[4] W. Quine, 'New Foundations for Mathematical Logic', *The American Mathematical Monthly*, 44 (1937), 70–80.

and

$$(\exists\beta)(\mathrm{N}[Gx \equiv_x x\epsilon\beta])$$

which jointly imply

$$(\exists\alpha)(\exists\beta)[\mathrm{N}(Fx \equiv_x x\epsilon\alpha) \,.\, \mathrm{N}(Gx \equiv_x x\epsilon\beta)].$$

This statement implies

$$(\exists\alpha)(\exists\beta)[\mathrm{N}(Fx \equiv_x x\epsilon\alpha) \,.\, \mathrm{N}(Gx \equiv_x x\epsilon\beta) : Fx \equiv_x x\epsilon\alpha : Gx \equiv_x x\epsilon\beta].$$

We may now insert the empirical truth, '$Fx \equiv_x Gx$', inside the scope of the quantifiers to obtain

$$(\exists\alpha)(\exists\beta)[\mathrm{N}(Fx \equiv_x x\epsilon\alpha) \,.\, \mathrm{N}(Gx \equiv_x x\epsilon\beta) : Fx \equiv_x x\epsilon\alpha : Gx \equiv_x x\epsilon\beta : Fx \equiv_x Gx].$$

Using (s2) we derive

$$(\exists\alpha)(\exists\beta)[\mathrm{N}(Fx \equiv_x x\epsilon\alpha) \,.\, \mathrm{N}(Gx \equiv_x x\epsilon\beta) \,.\, \alpha = \beta],$$

which is equivalent to

$$(\exists\alpha)(\exists\beta)[\mathrm{N}(Fx \equiv_x x\epsilon\alpha : Gx \equiv_x x\epsilon\beta) \,.\, \alpha = \beta].$$

Using Leibniz's law, we have

$$(\exists\alpha)(\exists\beta)[\mathrm{N}(Fx \equiv_x x\epsilon\alpha : Gx \equiv_x x\epsilon\alpha)],$$

which in turn implies $\mathrm{N}(Fx \equiv_x Gx)$, which contradicts the hypothesis that '$Fx \equiv_x Gx$' is a contingent truth.

Thus it is that (s3) does give rise to the paradox that no true formal equivalences are contingent. It is for this reason that (s1) was preferred. It is presumed that (s1) gives rise to no such untoward consequences.

The modal paradox (abc) may now equally well be viewed as illustrating the following argument-form:

$$\frac{\begin{array}{l}\hat{x}(Ax) = \hat{x}(Bx)\\ \mathrm{N}[f\hat{x}(Ax)]\end{array}}{\therefore \mathrm{N}[f\hat{x}(Bx)]}.$$

The scope of the abstract is to be understood to be the shortest formula containing the abstract unless it is otherwise indicated.

It will be duly noted that the conclusion cannot be deduced from the premisses. However, if the operator, 'N', is prefixed to the first premiss then the conclusion can correctly be obtained. Again, if in the second premise and conclusion the class abstract is given maximum scope, the conclusion

would be valid. This is simply to say again what was said before in reference to descriptions. It is not, essentially, the unrestricted use of modal operators which violates Leibniz's law. It is rather that the modal paradoxes arise out of neglect of the circumstance that in modal contexts the scopes of incomplete symbols, such as abstracts or descriptions, affect the truth value of those contexts.

It is of course possible to design languages in such wise that modal operators would be employed in a different way than that considered here. It is not the intention of this paper to discuss every use or meaning of the idiom 'it is necessary that—' but only to show that one usage, so far as we yet know, does not incur paradoxes nor, in particular, violation of Leibniz's law.