Exercises — 5

5.1 Reduce the following wff to MCNF. Where a wff passes the test give a sketch proof using the method described on p. 107. Where it does not pass the test use the method described on pp. 105-107 to construct a falsifying S5-model.

(a) $L(p \vee (q \wedge (r \vee Ls)))$
(b) $M(p \wedge q) \supset L(L(Lp \supset Lq) \supset Mq)$
(c) $L(p \supset (q \supset L(p \supset q))) \supset (\sim L(p \supset q) \supset L(p \supset \sim q))$
(d) $L(\sim p \wedge \sim q) \supset (L(L(p \vee q) \supset r) \wedge (r \supset L(p \supset p)))$
(e) $L(p \supset q) \supset L(M(p \wedge \sim Lp) \supset M(q \wedge L(p \supset Lp)))$
(f) $L(p \supset L(q \supset r)) \supset (q \supset L(p \supset r))$
(g) $L(L(p \equiv q) \supset Mq) \supset L(L(p \equiv q) \supset q)$
(h) $L(L(p \supset Lp) \supset Lp) \supset (MLp \supset Lp)$
(i) $(L(L(p \supset q) \supset q) \supset p) \supset M(Lq \supset p)$
(j) $L(L(Lp \supset Lq) \supset L(p \supset q))$

5.2 Prove that $M(p \wedge M{\sim}p)$ is not equivalent in S4 to any first-degree wff.

Notes

[1] The name 'modal conjunctive normal form' is ours, but the idea derives from Carnap 1946. Carnap calls the formula in MCNF to which a wff α can be reduced the *MP-reductum* of α. In Wajsberg 1933 a slightly more complicated normal form is described in which each disjunct consists of L or $\sim L$ followed by a disjunction of variables (negated or unnegated). Schumm 1975 points out that Wajsberg's method has to be adapted to deal with unmodalized disjuncts. One can apply the method of MCNF to some systems in which reduction to first degree is not possible by forming a CNF whose atoms are PC wff or wff of the form $L\alpha$ or $M\alpha$, and then reducing α to a CNF with similar atoms and so on. See Ōhama 1982.

[2] Makinson 1966a uses a generalization of this wff to show that a system containing S4, and therefore S4 itself, has infinitely many non-equivalent modal functions of a single variable, with no upper limit therefore on their modal degree.

[3] This definition is given in Parry 1939, p. 144.

6

COMPLETENESS

In this chapter we shall prove the completeness of K, D, T, S4, B and S5. But the technique we use will generalize to all modal systems of a certain class, and we shall begin by making a few remarks about systems and validity in general.

The first point to note is that we can define a modal system in two ways, in terms of its axiomatic basis, or in terms of its theorems. For instance, in our discussion of the system D we showed that in place of the wff **D**, $(Lp \supset Mp)$, we could have chosen $M(p \supset p)$, and have obtained exactly the same theorems. Although it would be possible to call the two different ways of axiomatizing D two different systems, for most purposes nothing is to be gained by this, and we shall say that S and S′ are the same system iff they have the same theorems. In fact it is convenient to *define* a system S as simply a class of wff, and then $\vdash_S \alpha$ and $\alpha \in S$, are just alternative ways of saying the same thing.

Of course not just *any* collection of wff of modal logic will count as a system. We shall, in most of this book, be interested in extensions of K. This class of systems is the class of what are called *normal* systems. A normal system of modal propositional logic is a class S of wff of modal propositional logic which contains all PC-valid wff and **K**, and has the property that if α and β are in S then so is anything obtainable from them by the use of US, MP and N.

This means that every modal system may be expressed as $K + \Lambda$, using the notation introduced on p. 39, since Λ could be simply S itself. But typically we can choose Λ to be much smaller, often a single wff (or, what comes to the same thing, a finite set of wff − since we may always form the single wff which is their conjunction).

In defining validity for a system S we have done so in terms of a class $\mathscr{E}$ of frames. Let us use the notation $\mathscr{E}$-valid, to mean, of a wff α, that for every $\langle W,R\rangle \in \mathscr{E}$, and every model $\langle W,R,V\rangle$ based on $\langle W,R\rangle$, $V(\alpha,w) = 1$ for every $w \in W$.

Where $\langle W,R,V\rangle$ is a particular model, then it is convenient to say that α is *valid* in $\langle W,R,V\rangle$ iff $V(\alpha,w) = 1$ for all $w \in W$. We must be careful about this use of 'valid' since, e.g., there will be models in which the single variable p is valid, and if we wish validity to mean truth for every value of the variables then validity in a model will not capture this in all models. Despite this, we shall speak of validity in a model, and in fact many of the models we shall be using will have the property that if α is valid in that model so is every substitution-instance of α.

The key result of the present chapter is that for every (consistent) normal modal system S there is a special kind of model, called the *canonical model* of S, which has the remarkable property that a wff α is valid in the canonical model of S iff $\vdash_S \alpha$.

The connection between this fact and completeness is this. Suppose that we have a class $\mathscr{E}$ of frames, and we wish to show that a wff α of a system S is $\mathscr{E}$-valid iff $\vdash_S \alpha$. We need to show first that S is sound with respect to $\mathscr{E}$, i.e. that every theorem of S is $\mathscr{E}$-valid. This we do by showing that the axioms are $\mathscr{E}$-valid, for theorem 2.1 on p. 39 then assures us that all the theorems will be. Now suppose that we can establish that the frame of the canonical model of S is in $\mathscr{E}$. If α is $\mathscr{E}$-valid then α will be valid on the frame of the canonical model for S, and so *a fortiori* valid in the canonical model itself. But that means that $\vdash_S \alpha$. So if α is $\mathscr{E}$-valid then $\vdash_S \alpha$, which is what the completeness of S with respect to $\mathscr{E}$ means.

In all of this procedure the part that is specific to each system is to establish that the frame of the canonical model is indeed in $\mathscr{E}$. For K this is immediate for $\mathscr{E}$ in the case of K is the class of all frames. For D, $\mathscr{E}$ is the class of serial frames and so we must show that the frame of the canonical model for D is serial; for T we must show that it is reflexive; for S4, B, S5 that it is reflexive and, respectively, transitive, symmetrical, and both transitive and symmetrical.

Although establishing that the frame of the canonical model is in $\mathscr{E}$ is sufficient to give completeness it is not necessary in that $\mathscr{E}$ need not contain the frame of the canonical model. Indeed we shall in Part II look at some systems where although, as guaranteed by the results of the present chapter, every theorem is valid on the canonical model itself, not every theorem is valid on the frame of the canonical model.

Be all that as it may, our task is now to construct, for any system S, the canonical model of S. As we observed on p. 37 the worlds in a model can be anything we please. One very tempting candidate is to make the worlds *sets of wff*. For then we could think of a wff as true in a world iff that wff is in the set of wff which constitutes that world. However, if we do this only certain sets will be able to count as worlds. For instance, since any wff α is either true or false at a world, and since $\sim\alpha$ is true iff α is false, then the set which is that world will have to contain either α or $\sim\alpha$, but not both. And it will have to contain $\alpha \vee \beta$ iff it contains at least one of α and β. Sets like this are described in the next section.

Maximal consistent sets of wff

Where Λ is a set of wff of modal logic we say that Λ is *S-inconsistent* iff there are $\alpha_1, \ldots, \alpha_n \in \Lambda$ such that

$$\vdash_S \sim(\alpha_1 \wedge \ldots \wedge \alpha_n)$$

The idea is that in S you can prove that a contradiction arises from the members of Λ. Λ is then *consistent* if there is *no* finite collection $\{\alpha_1, \ldots \alpha_n\} \subseteq \Lambda$, i.e. no $\alpha_1, \ldots, \alpha_n \in \Lambda$, such that

$$\vdash_S \sim(\alpha_1 \wedge \ldots \wedge \alpha_n)$$

In the case of a finite set, say $\{\beta_1, \ldots, \beta_k\}$ this definition simply means that

$$\dashv_S \sim(\beta_1 \wedge \ldots \wedge \beta_k)$$

(where $\dashv$ means 'not $\vdash$'). In the case of a single wff γ, $\{\gamma\}$ is consistent iff $\dashv_S \sim\gamma$. Thus $\{\sim\gamma\}$ is consistent iff $\dashv_S \sim\sim\gamma$, i.e. iff $\dashv_S \gamma$. (In the above definitions $\subseteq$ is the symbol for *class inclusion*. Where A and B are any classes then $A \subseteq B$ iff every a in A is also in B. I.e., if $a \in A$ then $a \in B$. $\subseteq$ and $\in$ should not be confused. One important difference is that $A \subseteq A$ for every A, while $A \in A$ is false in most set theories.)

A set Γ of wff is said to be *maximal* iff for every wff α either $\alpha \in \Gamma$ or $\sim\alpha \in \Gamma$. Γ is said to be *maximal consistent* with respect to a system S (or *maximal S-consistent*) iff it is both maximal and S-consistent. We now establish a lemma which shows that in respect of the PC-operators, a maximal consistent set of wff does indeed look like a world, at which the true wff are the wff in the set.

LEMMA 6.1 Suppose that Γ is any maximal consistent set of wff with respect to S. Then

6.1a for any wff α, exactly one member of $\{\alpha, \sim\alpha\}$ is in Γ;
6.1b $\alpha \vee \beta \in \Gamma$ iff either $\alpha \in \Gamma$ or $\beta \in \Gamma$;
6.1c $\alpha \wedge \beta \in \Gamma$ iff $\alpha \in \Gamma$ and $\beta \in \Gamma$;
6.1d if $\alpha \in \Gamma$ and $\alpha \supset \beta \in \Gamma$ then $\beta \in \Gamma$.

Proof: One half of 6.1a, viz. that at least one member of $\{\alpha, \sim\alpha\}$ is in Γ, is directly given by Γ's maximality. The other half, that they are not both in Γ, follows directly from its consistency; for if both were in Γ, then $\{\alpha, \sim\alpha\}$ would be a subset of Γ; but $\{\alpha, \sim\alpha\}$ is inconsistent since $\vdash_S \sim(\alpha \wedge \sim\alpha)$, and therefore Γ itself would be inconsistent. To prove 6.1b, suppose first that $\alpha \vee \beta$ is in Γ but that neither α nor β is. Then by 6.1a, $\sim\alpha$ and $\sim\beta$ would both be in Γ, and hence $\{\alpha \vee \beta, \sim\alpha, \sim\beta\}$ would be a subset of Γ. But this would again make Γ inconsistent, since by PC, $\vdash_S \sim((\alpha \vee \beta) \wedge \sim\alpha \wedge \sim\beta)$. Suppose next that one of α and β, say α, is in Γ but that $\alpha \vee \beta$ is not. Then $\{\alpha, \sim(\alpha \vee \beta)\}$ would be a subset of Γ. But this would make Γ inconsistent since $\vdash_S \sim(\alpha \wedge \sim(\alpha \vee \beta)$. The proof of 6.1c is analogous using the definition of $\alpha \wedge \beta$ as $\sim(\sim\alpha \vee \sim\beta)$. 6.1d holds because if we had $\alpha \in \Gamma$, $\alpha \supset \beta \in \Gamma$ but not $\beta \in \Gamma$ then $\{\alpha, \alpha \supset \beta, \sim\beta\}$ would be a subset of Γ. But this would make Γ inconsistent since $\vdash_S \sim(\alpha \wedge (\alpha \supset \beta) \wedge \sim\beta)$. This proves lemma 6.1.

The next lemma illustrates an important connection between maximal consistent sets and theorems of S.

LEMMA 6.2 Suppose that Γ is any maximal consistent set of wff with respect to S. Then

6.2a if $\vdash_S \alpha$ then $\alpha \in \Gamma$;
6.2b if $\alpha \in \Gamma$ and $\vdash_S \alpha \supset \beta$ then $\beta \in \Gamma$.

Proof: For 6.2a, if $\vdash_S \alpha$ then $\{\sim\alpha\}$ is S-inconsistent. So $\sim\alpha$ cannot be in Γ and so α must be. 6.2b follows immediately from 6.2a and 6.1d. This proves lemma 6.2.

Maximal consistent extensions

The idea behind the kind of model we are about to construct is this. The worlds of the model are maximal consistent sets of wff with respect to

some particular system S. Lemma 6.2a guarantees that if $\vdash_S \alpha$ then α is in every maximal consistent set of wff. But we said that the canonical model validates *all and only* theorems of S. This means that if α is *not* a theorem of S then there ought to be a maximal S-consistent set Γ such that $\alpha \notin \Gamma$. Now if α is not a theorem of S then $\{\sim\alpha\}$ is S-consistent, since otherwise $\vdash_S \sim\sim\alpha$ and so $\vdash_S \alpha$. The result we are about to prove guarantees that *every* S-consistent set Λ, whether finite like $\{\sim\alpha\}$ or infinite, can be extended to a maximal S-consistent set Γ. So if $\{\sim\alpha\}$ is consistent then there will be a maximal consistent Γ such that $\sim\alpha \in \Gamma$, and so, by lemma 6.1a, $\alpha \notin \Gamma$.

THEOREM 6.3 Suppose that Λ is an S-consistent set of wff. Then there is a maximal S-consistent set of wff Γ such that $\Lambda \subseteq \Gamma$.

Proof: Let us assume that the wff of modal propositional logic are arranged in some determinate order and labelled α_1, α_2, ... and so on. The idea behind the proof is that we make the set maximal by adding in turn every wff or its negation. We define a sequence Γ_0, Γ_1, ... of sets of wff in the following way.

(1) Γ_0 is Λ itself.
(2) Given Γ_n we let Γ_{n+1} be $\Gamma_n \cup \{\alpha_{n+1}\}$ if this is S-consistent and let Γ_{n+1} be $\Gamma_n \cup \{\sim\alpha_{n+1}\}$ otherwise.

(The symbol $\cup$ means that where A and B are classes $A \cup B$ is their *union*, the class of things in either A or B. I.e. $a \in A \cup B$ iff $a \in A$ or $a \in B$. So in the present case $\Gamma \cup \{\alpha_{n+1}\}$ means Γ together with α_{n+1}, and $\Gamma \cup \{\sim\alpha_{n+1}\}$ means Γ together with $\sim\alpha_{n+1}$.)

We next show that, for any n, if Γ_n is S-consistent then so is Γ_{n+1}. The proof is that if Γ_{n+1} is not S-consistent this means that neither $\Gamma_n \cup \{\alpha_{n+1}\}$ nor $\Gamma_n \cup \{\sim\alpha_{n+1}\}$ is S-consistent. This in turn means that there are some wff $\beta_1, \ldots, \beta_m$ in Γ_n such that

$$\vdash_S \sim(\beta_1 \wedge \ldots \wedge \beta_m \wedge \alpha_{n+1}) \quad \text{(i)}$$

and also some wff $\gamma_1, \ldots, \gamma_k$ in Γ_n such that

$$\vdash_S \sim(\gamma_1 \wedge \ldots \wedge \gamma_k \wedge \sim\alpha_{n+1}) \quad \text{(ii)}$$

Now from (i) and (ii) it follows by PC that

$$\vdash_S \sim(\beta_1 \wedge \ldots \wedge \beta_m \wedge \gamma_1 \wedge \ldots \wedge \gamma_k)$$

i.e. that $\{\beta_1, \ldots, \beta_m, \gamma_1, \ldots, \gamma_k\}$ is S-inconsistent. But this is a subset of Γ_n, and therefore Γ_n is itself inconsistent.

Now let Γ be the union of all the Γ_ns. Then (a) Γ is consistent. For if it were not then some finite subset of Γ would be inconsistent. But clearly every finite subset of Γ is a subset of some Γ_n, and we have shown that no Γ_n is inconsistent. (b) Γ is maximal. For consider any wff α_i. By the construction of Γ_i, either $\alpha_i \in \Gamma_i$ or $\sim\alpha_i \in \Gamma_i$; and so, since $\Gamma_i \subseteq \Gamma$, either $\alpha_i \in \Gamma$ or $\sim\alpha_i \in \Gamma$. This completes the proof of theorem 6.3.

Consistent sets of wff in modal systems

All the results we have proved so far depend only on the fact that S contains PC. They therefore hold for any system, whether modal or not, which contains PC. We now go on to consider features of maximal consistent sets which have to do with their modal properties. In particular, in constructing a model in which the worlds are maximal consistent sets of wff we will have to specify when one world is accessible from another. Now if a set Γ is to see a set Δ then one thing that is required is that if a wff β is *necessary* in Γ, i.e., if $L\beta \in \Gamma$, then β must be *true* in Δ, ie. $\beta \in \Delta$. In fact we shall use this as a *definition* of R in the canonical model. We shall say that ΓRΔ iff for every wff β, if $L\beta \in \Gamma$, then $\beta \in \Delta$. In order to express this more succinctly we shall introduce some new notation. Suppose that Λ is any set of wff of modal logic. Then we write $L^-(\Lambda)$ to denote that set consisting precisely of every wff β for which $L\beta$ is in Λ. More formally expressed:

$$L^-(\Lambda) = \{\beta : L\beta \in \Lambda\}$$

where $\{\alpha : L\alpha \in \Lambda\}$ denotes the class whose members are precisely the αs such that $L\alpha \in \Lambda$. Using this notation we can say that ΓRΔ iff $L^-(\Gamma) \subseteq \Delta$. Our next lemma will depend on the modal properties of S. Its purpose is the following. If $\sim L\alpha$ is in a set Λ of wff, and that set is supposed to represent a world in a model, there had better be a set which represents an accessible world, and which contains $\sim\alpha$. We need a guarantee that it will always be consistent to suppose this, and that means that we need to know that $L^-(\Lambda)$ is consistent with $\sim\alpha$. The lemma can be stated as follows:

LEMMA 6.4 Let S be any normal system of propositional modal logic, and let Λ be an S-consistent set of wff containing $\sim L\alpha$. Then $L^-(\Lambda) \cup \{\sim\alpha\}$ is S-consistent.

Proof: We prove the lemma by showing that if $L^-(\Lambda) \cup \{\sim\alpha\}$ is *not* consistent then neither is Λ. So suppose that $L^-(\Lambda) \cup \{\sim\alpha\}$ is not S-consistent. This means that there is some finite subset $\{\beta_1, \ldots, \beta_n\}$ of $L^-(\Lambda)$ such that

$$\vdash_S \sim(\beta_1 \wedge \ldots \wedge \beta_n \wedge \sim\alpha)$$

hence by PC

$$\vdash_S (\beta_1 \wedge \ldots \wedge \beta_n) \supset \alpha$$

So by DR1 (p. 30)

$$\vdash_S L(\beta_1 \wedge \ldots \wedge \beta_n) \supset L\alpha$$

So by L-distribution (K3, p. 28) and Eq (p. 32),

$$\vdash_S (L\beta_1 \wedge \ldots \wedge L\beta_n) \supset L\alpha$$

and finally by PC,

$$\vdash_S \sim(L\beta_1 \wedge \ldots \wedge L\beta_n \wedge \sim L\alpha)$$

But this means that $\{L\beta_1, \ldots, L\beta_n, \sim L\alpha\}$ is not S-consistent; so, since it is a subset of Λ, Λ is not S-consistent, which is what we had to prove. (If Λ should happen to contain no wff of the form $L\beta$ then $L^-(\Lambda)$ would be empty and so if $L^-(\Lambda) \cup \{\sim\alpha\}$ is not consistent then $\vdash_S \alpha$. But then by N $\vdash_S L\alpha$, and so Λ is inconsistent in this case also.) This ends the proof.

In conjunction with theorem 6.3 lemma 6.4 guarantees that there will be a maximal consistent set Γ such that $L^-(\Lambda) \subseteq \Gamma$ and $\sim\alpha \in \Gamma$. This means that for any wff β, if $L\beta \in \Lambda$ then $\beta \in \Gamma$ so if Λ is itself maximal consistent then ΛRΓ.

Canonical models

The canonical model for S is, like any other model for a normal propositional modal system, a triple ⟨W,R,V⟩. W is the set of all sets of maximal S-consistent sets of wff. I.e. $w \in W$ iff w is maximal S-consistent.[1] If w and w' are both in W then wRw' iff for every wff β if $L\beta \in w$ then $\beta \in w'$ – using the L^- notation wRw' iff $L^-(w) \subseteq w'$. Finally we define V in the canonical model for S by stipulating that $V(p,w) = 1$ iff $p \in w$. I.e., a variable is true in a world in the canonical model iff it is a member of that world, i.e., a member of that set of formulae.

Given the assignment to the variables, [V∼], [V∨] and [V*L*] then give a value in every world to every wff. Our aim is now to show that every wff – not merely every variable – is true in a world in the canonical model iff it is a member of that world. This will have the consequence that $\vdash_S \alpha$ iff α is valid in the canonical model, since, as we observed on p. 115, α is a theorem of S iff it is a member of every maximal S-consistent set. So α will be a theorem of S iff it is a member of every world in the canonical model of S. Therefore if being a *member* of w is equivalent to being *true* in w then α will be a theorem of S iff it is true in every world in the canonical model of S, i.e. iff it is valid in the canonical model of S.

In the case of the variables the V in the canonical model of S was defined so that a variable is true in a world iff it is a member of that world. In the case of other wff this has to be proved, and our next theorem is sometimes called the fundamental theorem for canonical models.

THEOREM 6.5 Let $\langle W,R,V \rangle$ be the canonical model for a normal propositional model system S. Then for any wff α and any $w \in W$, $V(\alpha,w) = 1$ iff $\alpha \in w$.

Proof: The result is defined to hold for the propositional variables. To show that it holds for all wff it will be sufficient to show the following:

(a) If the theorem holds for α then it holds for $\sim\alpha$;
(b) If the theorem holds for α and β then it holds for $\alpha \vee \beta$;
(c) If the theorem holds for α then it holds for $L\alpha$.

Since every wff (in primitive notation) is made up from the variables in one of the ways mentioned in (a)–(c) this will show that the theorem holds for all wff. This style of proof is often called a proof by *induction on the construction of a wff* (or sometimes on the *length* of a wff).[2] The hypothesis that the theorem holds for α (and β) is called the *hypothesis of the induction* or the *inductive hypothesis*.

As we have observed, if α is a variable the theorem holds by definition. We now prove each of (a)–(c) in turn.

(a) Consider a wff $\sim\alpha$ and any $w \in W$. By [V∼] we have $V(\sim\alpha,w) = 1$ iff $V(\alpha,w) = 0$. Since the theorem is assumed to hold for α we have $V(\alpha,w) = 0$ iff $\alpha \notin w$. But by lemma 6.1a, $\alpha \notin w$ iff $\sim\alpha \in w$. Hence finally we have $V(\sim\alpha,w) = 1$ iff $\sim\alpha \in w$ as required.

(b) Consider next $\alpha \vee \beta$. By [V∨] we have $V(\alpha \vee \beta,w) = 1$ iff either $V(\alpha,w) = 1$ or $V(\beta,w) = 1$. Since the theorem is assumed to hold for α and β we therefore have $V(\alpha \vee \beta,w) = 1$ iff either $\alpha \in w$ or $\beta \in W$. Hence by lemma 6.1b we have $V(\alpha \vee \beta,w) = 1$ iff $\alpha \vee \beta \in w$, as required.

(c) Consider finally $L\alpha$. (A) Suppose that $L\alpha \in w$. Then by definition of R we have $\alpha \in w'$ for every w' such that wRw'. Since the theorem is assumed to hold for α we therefore have $V(\alpha,w') = 1$ for every w' such that wRw'. Hence by [V*L*], $V(L\alpha,w) = 1$. (B) Suppose now that $L\alpha \notin w$. Then by lemma 6.1a, $\sim L\alpha \in w$. Hence by lemma 6.4, $L^-(w) \cup \{\sim\alpha\}$ is S-consistent. So by theorem 6.3 and the definition of W, there is some $w' \in W$ such that $L^-(w) \cup \{\sim\alpha\} \subseteq w'$, and therefore such that (i) $L^-(w) \subseteq w'$ and (ii) $\sim\alpha \in w'$. Now (i) gives us wRw', by the definition of R, and by lemma 6.1a (ii) gives us $\alpha \notin w'$; and so, since the theorem is assumed to hold for α, $V(\alpha,w') = 0$. So by [V*L*] we have $V(L\alpha,w) = 0$.

This completes the proof of theorem 6.5.

COROLLARY 6.6. Any wff α is valid in the canonical model of S iff $\vdash_S \alpha$.

Proof: Let $\langle W,R,V \rangle$ be the canonical model of S. First suppose $\vdash_S \alpha$. Then by lemma 6.2a α is in every maximal S-consistent set of wff. Hence α is in every $w \in W$, and so, by theorem 6.5, $V(\alpha,w) = 1$ for every $w \in W$; i.e. α is valid in $\langle W,R,V \rangle$. Suppose now that $\dashv_S \alpha$. Then $\{\sim\alpha\}$ is S-consistent and so, by theorem 6.3 there is some maximal S-consistent set – i.e. some $w \in W$ – such that $\sim\alpha \in w$ and hence $\alpha \notin w$. So by theorem 6.5, $V(\alpha,w) = 0$. So in this case α is not valid in $\langle W,R,V \rangle$.

The completeness of K, T, D, B, S4 and S5

Let us take stock of the position we have now reached. We assume we have a normal system S and a class $\mathscr{C}$ of frames. To say that S is complete with respect to $\mathscr{C}$ is to say that every $\mathscr{C}$-valid wff α is a theorem of S; where a $\mathscr{C}$-valid wff is a wff that is valid on every frame in $\mathscr{C}$, which in turn means that where $\langle W,R,V\rangle$ is any model such that $\langle W,R\rangle \in \mathscr{C}$, and w is any member of W, $V(\alpha,w) = 1$. Now if the frame of the canonical model is in $\mathscr{C}$ then every $\mathscr{C}$-valid wff is valid on that frame, and therefore valid in the canonical model itself. But in that case, by corollary 6.6, that wff will be a theorem of S.

This should make it clear that in order to prove the completeness of S by the canonical model method it will be sufficient to prove that the canonical model of S is based on a frame in $\mathscr{C}$. This means that we have immediately a completeness result for K, since in the case of K, $\mathscr{C}$ is the class of all frames, and the frame of the canonical model in this case is, trivially, in $\mathscr{C}$.

THEOREM 6.7 T is complete with respect to the class of all reflexive frames.

All we have to prove is that in the canonical model for T, R is reflexive, i.e. for every $w \in W$, wRw. By the definition of R in the canonical model this means that we must prove that for any wff α, if $L\alpha \in w$ then $\alpha \in w$. But from **T** and US we have $\vdash_S L\alpha \supset \alpha$, and so the result follows by lemma 6.2b.

THEOREM 6.8 D is complete with respect to the class of all serial frames.

To prove that D is complete it is sufficient to prove that R in its canonical model is serial. By D1 $M(p \supset p)$ is a theorem of D and so, for any w in the canonical model of D, $M(p \supset p) \in w$. So, by theorem 6.5 $V(M(p \supset p),w) = 1$. So there must be some w' such that wRw', and so R is serial as required.

THEOREM 6.9 S4 is complete with respect to the class of all reflexive and transitive frames.

We prove that the canonical model of S4 is based on a frame which is reflexive and transitive. Since S4 contains **T** the proof of theorem 6.7 establishes that it is reflexive. For transitivity suppose that wRw' and $w'Rw''$. To show that wRw'' we must show that for any wff α, if $L\alpha \in w$ then $\alpha \in w''$. Now $\vdash_{S4} L\alpha \supset LL\alpha$, and so, by lemma 6.2b, if $L\alpha \in w$ then $LL\alpha \in w$, and then since wRw', by the definition of R, $L\alpha \in w'$ and so, since $w'Rw''$, again by the definition of R, $\alpha \in w''$ as required. (Note that this proof also gives us the result that the system K4, mentioned on p. 64, is complete with respect to the class of transitive frames, whether or not R in those frames is reflexive.)

THEOREM 6.10 B is complete with respect to the class of all reflexive and symmetrical frames.

Reflexiveness is as for T. For symmetry suppose that wRw'. To show that $w'Rw$ we must show that, for any wff α, if $L\alpha \in w'$ then $\alpha \in w$. So suppose $\alpha \notin w$. Then $\sim\alpha \in w$, and, since $\vdash_B \sim\alpha \supset L\sim L\alpha$, by lemma 6.2b, $L\sim L\alpha \in w$, and since wRw', by the definition of R, $\sim L\alpha \in w'$ and so $L\alpha \notin w'$. (The proof also establishes that KB is complete with respect to the class of all symmetrical frames, whether or not they are reflexive.)

THEOREM 6.11 S5 is complete with respect to the class of all equivalence frames.

The presence in S5 of **T**, **4** and **B** means that the completeness of S5 follows from the proofs of theorems 6.7, 6.9 and 6.10.

Triv and Ver again

At the end of Chapter 3 we mentioned the Trivial system and the Verum system. What we said there has the consequence that the Trivial system is sound with respect to reflexive one-world frames, and the Verum system is sound with respect to irreflexive one-world frames (or what comes to the same thing, one-world frames in which the world is a dead end). Now in fact the frame of the canonical model for neither of these systems is a one-world frame. However we can show, and quite easily too, that every world in the frame of the canonical model of Triv can see itself, and itself alone, and that every world in the canonical model of Ver is a dead end. Clearly if α is valid on every model based on a one-world reflexive frame then it will be valid on a frame all of whose worlds are like this, and so will be valid on the frame of the canonical model of Triv, and so $\vdash \alpha$ similarly if α is valid on every model based on a dead end it will be valid in the canonical model of Ver.

It is easy to show that the canonical model of Triv contains only reflexive end points (i.e. worlds which can see themselves and themselves alone). From $\vdash Lp \equiv p$ we have $\vdash Lp \supset p$ and so the frame is reflexive, so suppose that in the canonical model for Triv there is a world w such that $w\mathrm{R}w'$ but $w \neq w'$. Then there will be a wff α such that $\alpha \in w$ but $\alpha \notin w'$. Now $\alpha \in w$ and $\vdash \alpha \supset L\alpha$, so $L\alpha \in w$. But $w\mathrm{R}w'$ and so $\alpha \in w'$ which is a contradiction. For Ver we note that $\vdash_{\mathrm{Ver}} L(p \wedge {\sim}p)$. But that can only happen at a dead end, and so every world in the canonical model of Ver is a dead end. Each world in the canonical model of Triv can be thought of as based on the one-world reflexive frame, and each world in the canonical model of Ver as based on the one-world dead end frame and so these frames respectively characterize Triv and Ver.

When we look at Triv and Ver in this way we see why it is that they collapse into PC. For in one-world frames there is no way of making a distinction between wff which are true in one world but false in another. Further, since there are only two one-world frames, one in which the single world can see itself, and one in which it cannot, we can see why it is that there are only two ways in which a normal modal system can collapse into PC.

Exercises — 6

6.1 Call Γ *maximal consistent** iff Γ is consistent and for every wff α, if $\Gamma \cup \{\alpha\}$ is consistent then $\alpha \in \Gamma$. Prove that Γ is maximal consistent* iff Γ is maximal consistent as defined in this chapter.

6.2 Prove that if Γ is maximal consistent then $\alpha \supset \beta \in \Gamma$ iff $\alpha \notin \Gamma$ or $\beta \in \Gamma$.

6.3 Let Γ and Λ both be maximal consistent. Show that if $\Lambda \subseteq \Gamma$ then $\Lambda = \Gamma$.

6.4 Show that if Γ and Λ are both maximal consistent then $\{\alpha : L\alpha \in \Gamma\} \subseteq \Lambda$ iff $\{M\alpha : \alpha \in \Lambda\} \subseteq \Gamma$.

6.5 Show that if Λ is consistent and $M\alpha \in \Lambda$ then $\mathrm{L}^{-}(\Lambda) \cup \{\alpha\}$ is consistent.

6.6 Let $w\mathrm{R}^{n}w'$ mean that w can see w' in n R-steps. Where S is any normal modal system show that in the canonical model of S $w\mathrm{R}^{n}w'$ iff $\{\alpha : L^{n}\alpha \in w\} \subseteq w'$.

6.7 Where S contains S4 show that if $\{L\gamma_1, \ldots, L\gamma_n, {\sim}L\beta\}$ is S-consistent, so is $\{L\gamma_1, \ldots, L\gamma_n, {\sim}\beta\}$.

6.8 Show that $\mathrm{K} + Mp \supset Lp$ is complete with respect to the class of frames in which each world can see at most one world, itself or another.

6.9 Let W2 be T with the additional axiom

W2 $(p \wedge q \wedge M(p \wedge {\sim}q)) \supset Lp$

Show that any wff α is a theorem of W2 iff it is valid in all models in which every world can see at most one other world besides itself.

6.10 Show that $\mathrm{K} + L(Lp \supset q) \vee L(Lq \supset p)$ is complete with respect to the class of frames in which if $w_1\mathrm{R}w_2$ and $w_1\mathrm{R}w_3$ then either $w_2\mathrm{R}w_3$ or $w_3\mathrm{R}w_2$.

6.11 Show that $\mathrm{K} + p \supset Lp$ is complete with respect to the class of frames in which every world is either a dead end or can see only itself.

6.12 Show that K + **E** is complete with respect to frames which satisfy the condition stated in exercise 3.11.

6.13 Consider the class of frames in which R is replaced by a subset N of W and $V(L\alpha, w) = 1$ iff $V(\alpha, w') = 1$ for every $w' \in \mathrm{N}$. Prove that K + **E** is characterized by frames of this kind.

Notes

[1] The use of maximal consistent sets in proving the completeness of systems of modal logic goes back at least as far as Bayart 1959. Other early works are Kaplan 1966, Makinson 1966b and Lemmon and Scott 1977. Completeness proofs of a different kind are found in Kripke 1959 and 1963a. The method was originally used for non-modal predicate logic in Henkin 1949.

[2] Although we have not used the word 'induction' before we have used this method of proof in earlier chapters, for instance in our proof of Eq on p. 32 and in the proof of lemma 3.1 on p. 66. A proof by induction, more precisely *mathematical induction*, applies when we have a class of objects made up from simple parts by a finite number of steps. So, for instance, the natural numbers are all obtained from 0 by the successor operation, the operation of adding 1, or as here any wff is obtained from the primitive symbols by successive operations of the formation rules. If we wish to show that every member of such a class has a certain property it is sufficient to show that the simple members of the class have

it, and that anything made up from members which have the property also has the property. Other examples of inductive proofs are in soundness proofs such as that for K on pp. 39-41.

Part II

NORMAL MODAL SYSTEMS