Equivariant Cubical QMS

Outline

1. Recall problem: model structure on cSet not equiv. to "spaces" = sSet w/KQMS.

2. Idea: take the symmetries of the cubes into account in the filling conditions.

3. Do this systematically by first building a model str. on cSet^Z for Z = "finite Segm groupoids", using the same "TT approach" as for cSet.

4. Transfer the model str. along the functor
   \[ \Delta : \text{cSet} \to \text{cSet}^Z. \]

5. Get a new "equivariant model str." on cSet.

NB: Setlem: (cSet w/equiv.) = (cSet w/kan).
1. cSets

Recall that we have a model str. on Cartesian cubical sets:

\[ \text{cSet} = \text{Set} \otimes \mathbb{C} \]

In \( \text{cSet} \), the representables are "cubes"

\[ I^n = I \times \ldots \times I \]

\[ I^0 \]

\[ I^1 \]

\[ I^2 \]

\[ I^3 \]

etc.

The "swap" map \( I^2 \xrightarrow{\sigma} I^2 \) exchanges the coordinates, so the coequalizer

\[ \xymatrix{ I^2 \ar[r]^-{\sigma} & I^2 \ar@<1ex>[r]^-{1} & Q } \]

looks like:

\[ \square \quad \square \quad \triangle \quad . \]

Geometric realization \( \text{cSet} \to \text{Top} \) takes \( Q \) to a contractible space, but \( Q \to I \) is not a weak equiv. in \( \text{cSet} \).
"Type theoretic" model structure on $cSet$ given by:

$E = \text{monos classified by } 1 \rightarrow \phi \in \Omega$,

$F = \text{uniform (unbiased) Kan fibrations}$,

$W = (TFib) \circ (Tcof)$.

"Type theoretic" because:

- Both $TFib = E^\perp \cap F$ have a "type of uniform filling structure".
- This is used to construct a universe $U$.
- One then shows univalence & fibrancy of $U$.
- From that, we finally get $3Sat$ for $W$.

Recall furthermore:

- $TFib$ can be determined algebraically as the algebras for the partial map classifier monad on the slice categories:

\[
\begin{align*}
\mathcal{E}/X & \rightarrow \mathcal{E}/X \\
A \rightarrow \Sigma A^{[\mathcal{E}]_{\Phi}} = A^\perp
\end{align*}
\]

- $f : Y \rightarrow X \in E$ iff $(S \Rightarrow f) \in TFib$,

where $S : 1 \rightarrow I$ is generic in $\mathcal{E}/I$. 
In more detail, \( f : Y \to X \) is a fibration if in \( \mathcal{B} \),
\[
\begin{array}{c}
\begin{tikzcd}
\delta \ar[r, shift right] & Y \ar[d, shift right, \varepsilon_Y] \\
\mathcal{B} \ar[r, x] \ar[d, shift left, \delta \circ f] & Y \\
\mathcal{C} \ar[r, x] \\
\end{tikzcd}
\end{array}
\]
and \( \forall \mathcal{I}, \mathcal{C} : \mathcal{C} \to \mathcal{I}^n, \mathcal{Z} : \mathcal{I}^n \to \mathcal{I} \),
\[
\begin{array}{c}
\begin{tikzcd}
\mathcal{B} \ar[r, x] \ar[d, shift left, \delta] & Y \ar[d, shift right, f] \\
\mathcal{I}^n \times \mathcal{I} \ar[r, x] \\
\end{tikzcd}
\end{array}
\]
and uniform in \( \mathcal{I}^n \xrightarrow{a} \mathcal{I}^n \):
\[
\begin{array}{c}
\begin{tikzcd}
\mathcal{B} \ar[r, x] \ar[d, shift left, \delta] & \mathcal{B} \ar[d, shift right, f] \\
\mathcal{I}^n \times \mathcal{I} \ar[r, x] \\
\end{tikzcd}
\end{array}
\]
\[
\tilde{j}' = j \cdot (\kappa \times 1)
\]
Here \( \mathcal{B} \to \mathcal{I}^n \times \mathcal{I} \) is the pushout product
\[
\begin{array}{c}
\begin{tikzcd}
\mathcal{C} \times 1 \ar[r] \ar[d, shift left, \gamma] & \mathcal{C} \times \mathcal{I} \\
\mathcal{I}^n \times \mathcal{I} \ar[r, c \circ \delta] \\
\end{tikzcd}
\end{array}
\]
2. New idea: Equivariance

(i) in the gen. triv. cof.s:

\[
\begin{align*}
\text{B} & \xrightarrow{\otimes \delta} \text{I} \times \text{I}^k \\
\text{C} & \xrightarrow{\otimes \delta^k} \text{I}^n \times \text{I}^k \\
& \text{replace} \quad \delta \quad \text{by} \quad \delta^k \quad f. \quad k > 1.
\end{align*}
\]

(ii) require the filtres \( j \) to also be \underline{equivariant}, i.e. uniform w/r esp. to all symmetries

\[
\sigma : \text{I}^k \rightarrow \text{I}^k \quad \sigma \in \Sigma_k.
\]

\[
\begin{align*}
B & \xrightarrow{\sim} B \xrightarrow{y} Y \\
\text{C} \otimes \delta^k & \xrightarrow{j} \text{I}^n \times \text{I}^k \xrightarrow{j'} X \\
\text{I}^n \times \text{I}^k & \xrightarrow{1 \times \delta} \text{I}^n \times \text{I}^k
\end{align*}
\]

\[
j' \cdot (1 \times \delta) = j.
\]

Note this solves our original problem by making \( 1 \rightarrow \text{I}^2/\sigma = Q \) a Tcof, so \( Q \rightarrow 1 \in W \).
In more detail:

since all \( f : Y \rightarrow X \) \( e \mathcal{E} \) are equivariant,

\[
\begin{array}{c}
\delta_2 \\
\downarrow \\
1
\end{array} \oplus \delta_2 = \delta_2 \downarrow \delta_2 \downarrow \delta_2 \downarrow \delta_2 \downarrow f
\]

\[
\begin{array}{cccccccc}
1 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & Y \\
\delta_2 & \downarrow & \delta_2 & \downarrow & \delta_2 & \downarrow & \delta_2 & \downarrow & f \\
I^2 & \rightarrow & I^2 & \rightarrow & I^2 & \rightarrow & X \\
\end{array}
\]

we have:

\[
\begin{array}{cccccccc}
1 & \rightarrow & 1 & \rightarrow & 1 & \rightarrow & Y \\
\delta_2 & \downarrow & \delta_2 & \downarrow & \delta_2 & \downarrow & \delta_2 & \downarrow & f \\
I^2 & \rightarrow & I^2 & \rightarrow & I^2 & \rightarrow & X \\
\end{array}
\]

Similarly for all \( I^k/\Sigma_k \), \( k \geq 2 \).

We'll have a more systematic view of what's going on soon.

First note also that the equivariant filling condition can also be reformulated algebraically.
Consider first the following in $\mathcal{E}/I^k$, as before:

$\begin{array}{ccc}
Y^I_k & \xrightarrow{\delta_k \Rightarrow f} & Y \\
\downarrow f^I_k & & \downarrow f \\
X^I_k & \xrightarrow{\alpha} & Y^I_k \\
\downarrow f^I_k & & \\
X^I_k & \xrightarrow{\delta} & X
\end{array}
$

We asked for $(\delta_k \Rightarrow f) \in \mathcal{C}^+$ uniformly, i.e. a $+$-Algebra, but now what about equivariance?

Take:

$\begin{array}{ccc}
Y^I_k & \xrightarrow{\gamma \delta^{-1}} & Y^I_k \\
\downarrow f^I_k & & \downarrow f^I_k \\
X^I_k & \xrightarrow{\delta_k \Rightarrow f} & Y^I_k \\
\downarrow f^I_k & & \\
X^I_k & \xrightarrow{\gamma^{-1} \delta} & Y^I_k
\end{array}
$

where as before

$P = X^I_k \times_X Y^I_k.
$

Then $\delta_P = (X^{\delta^{-1}} \times_Y X)$ makes this commute.
Now take the upper square and consider \((\delta_k \Rightarrow f)^+\):

\[
\begin{array}{ccc}
Y^I_k & \xrightarrow{\gamma} & Y^I_k \\
\downarrow h & & \downarrow h \\
\delta_k \Rightarrow f & \xrightarrow{\sigma} & \sigma_P \\
\end{array}
\]

We get \(\sigma_P\) by pulling back, since \(+\) is stable.

**Prop.** The uniform filling structure associated to an algebra structure \(\alpha : (\delta_k \Rightarrow f)^+ \rightarrow (\delta_k \Rightarrow f)\) is \(\Sigma_k\)-equivariant iff \(\alpha\) makes \((\ast)\) commute, where

\[\alpha \cdot \sigma^+ = \gamma \delta^{-1} \cdot \alpha\]

**Remark.** All of the other maps in \((\ast)\) are equivariant. Thus \(f\) (in \(\mathcal{E}\)) is a \(\Sigma_k\)-equivariant fibration in the sense of unif. equiv. lifting iff \((\delta_k \Rightarrow f)\) is a \(+\)-Algebra in \(\text{cSet}^{\Sigma_k}\), the topos of \(\Sigma_k\)-equivariant \(\text{cSets}\).
3. Equivariant Sets

Let $\Sigma_k$ be the $k$-element symmetric group, and $\Sigma := \biguplus_k \Sigma_k$ the "symmetric groupoid".

The presheaf topos of $\Sigma$-sets,

$$\mathbf{Set}^\Sigma = \prod_k \mathbf{Set}^{\Sigma_k},$$

is atomic, meaning the functor

$$\Delta : \mathbf{Set} \longrightarrow \mathbf{Set}^\Sigma$$

preserves:

- the soc $\mathcal{S}$
- exponential $\mathbb{B}^A$
- LLC structure $T_A B$
- limits & colimits, since it has $L$-$R\Delta$-$T$.

Facts:

- The functors
  $$\Delta_k : \mathbf{Set} \longrightarrow \mathbf{Set}^\Sigma \longrightarrow \mathbf{Set}^{\Sigma_k}$$
  are in addition full & faithful.

- The left adjoints $L : \mathbf{Set}^{\Sigma_k} \longrightarrow \mathbf{Set}$
  preserve monos, and the right adjoints $R : \mathbf{Set}^{\Sigma_k} \longrightarrow \mathbf{Set}$
  preserve the soc $\mathcal{S}$.
Now let us make a base change along the geometric morphism:

\[
\text{Set}^{\text{op}} \longrightarrow \text{Set}.
\]

We have a pullback of toposes & geometric morphisms:

\[
\begin{array}{ccc}
\text{Set}^{\text{op}} \times \Sigma & \longrightarrow & \text{Set}^\Sigma \\
\downarrow & & \downarrow \\
\text{Set}^{\text{op}} & \longrightarrow & \text{Set} \\
\end{array}
\]

It follows that \( \text{cSet}^\Sigma \longrightarrow \text{cSet} \) is atomic.

We'll call \( \text{cSet}^\Sigma \) the equivariant cubical sets.

- The objects are cSets \( X \) equipped with an action (in \( \text{cSet} \)) by \( \Sigma \): for each \( \sigma \in \Sigma \)

  \[
  X \xrightarrow{\sigma} X,
  \]

  such that \( (\sigma \cdot \tau) \cdot x = \sigma (\tau \cdot x) \),

  \( i \cdot x = x \), etc.

- The maps are those in \( \text{cSet} \) that respect the action.
There are constant objects $\Delta X$ with trivial action,

$$\Delta : \text{cSet} \to \text{cSet}^2,$$

as well as adjoints $L \dashv \Delta \dashv P_\Delta$.

As before, $\Delta$ preserves $\Omega$, $\mathbb{B}^A$, $\text{Proj} A$, $\text{lim}$, ... . There are also the useful adjunctions:

$$\text{cSet} \xrightarrow{\eta} \text{cSet} \xleftarrow{\mu} \text{cSet} \xrightarrow{\pi_k} \text{cSet}^\Sigma_k$$

where

$$\eta X = \text{free } \Sigma\text{-action}$$

$$\pi_k E = P_k (E_i, E_i, ...) = E_k$$

$$\pi_k ! X = (0 \ldots k \ldots 0)$$

$$\pi_k \mu X = (1 \ldots k \ldots 0).$$

For the representables in $\text{cSet}^\Sigma$ we have

$$y(\eta^! X, *_k) = (0 \ldots \eta^! X^k \ldots 0).$$

In each $\text{cSet}^\Sigma_k$ there's the equivariant $k$-cube

$$\Pi_k = (I^k, \Sigma_k).$$

In $\text{cSet}^\Sigma$ there's the special object

$$\Pi = (\Pi_0, \Pi_1, \Pi_2, \ldots).$$
4. Model structure on $\mathbf{cSet}^\Sigma$

We're going to use the "axiomatic" construction that we gave for $\mathbf{cSet}$ in the new setting of $\mathbf{cSet}^\Sigma$. Thereby we first need:

**Lemma.** $\Pi$ is tiny in $\mathbf{cSet}^\Sigma$.

pf. Show it for each $\Pi_k \in \mathbf{Set}^{\Sigma_k}$ by calculation.

Now define the "TTQMS" on $\mathbf{cSet}^\Sigma$:

Let:

- $\mathcal{E} = \text{monos}$
- $\mathcal{T}\text{Fib} = \mathcal{E}^{\dagger} = +\text{Alg}$ \(\in\) AWFS.
- $\text{Gen TCoF} = \{ c \otimes \delta : \mathbf{B} \to \Pi^x \Pi \mid c \in \mathcal{E} \}^\Sigma$
  where again $\delta : 1 \to \Pi \in \mathbf{cSet}^{\Sigma/\Pi}$.
- $\mathcal{I} = \{ \pi \in \text{Gen TCoF} \mid \}$
  $\pi = \{ f \mid (\delta \to f) \in +\text{Alg} \in \mathbf{cSet}^{\Sigma/\Pi} \}$.
- $W = \mathcal{T}\text{Fib} \circ \text{TCof}$.

Thus $(\mathcal{E}, W, \mathcal{I})$ is a QMS on $\mathbf{cSet}^\Sigma$.

pf. Just as for $\mathbf{cSet}$.
5. Transferring the model structure

Now we are going to "transfer" the model structure along the adjunction

\[
\text{cSet} \xrightarrow{\Delta} \text{cSet}^2 \xleftarrow{L} \text{cSet}
\]

It would be fine to do this with a known "transfer theorem" — if only I knew one that gives what we want! Instead we do this:

Lemma. \( \Delta : \text{cSet} \to \text{cSet}^2 \) preserves \((-)^+\). It follows that \( \Delta \) preserves & reflects \(+\text{Algebras}\).

Def. The equivalent model structure on \( \text{cSet} \) is determined by:

- \( \mathcal{E} = \text{Monos} \), \( \text{TFib} = \mathcal{E}^+ \)
- \( \mathcal{P} = \Delta^-(\mathcal{L}^2) \) = \( \{ f : y \to X \mid \delta f \in \mathcal{L} \in \text{cSet}^2 \} \)
- \( W = \text{TFib} \circ \text{TCoF} \), \( \text{TCoF} = +\mathcal{E} \).

Main Thm. This is a QMS on \( \text{cSet} \).
Remarks

• Since $\Delta : \mathbf{cSet} \to \mathbf{cSet}^\Sigma$ preserves $L$, $L \vdash \Delta$ preserves $\text{TCof}$. Use this to get a gen. set
  \[ \{ L(B) \to \mathbb{I}^k \times \mathbb{I} \} / \cdots \]
of $\text{TCof}$ in $\mathbf{cSet}$, for small object argument.

• $\Delta : \mathbf{cSet} \to \mathbf{cSet}^\Sigma$ preserves and reflects $C, L, \& \text{TFib}$, but unfortunately it does not preserve $\text{TCof}$, and so also not $W$.
So there is work to do to prove 3 for 2.

• One approach is to repeat/modify the previous type-theoretic proof:
  • define the type of fibration structures
  • use tininess of the cubes $I^k$ to show that these types are stable
  • build a universe of fibrations
  • show it's univalent & fibrant
  • deduce 3 for 2 from the FEP.
That is the approach used for the "TT proof", which was essentially done last year — and was formalized by Evan C.!

**NB:** These equivariant fibrations agree with those defined before in terms of uniformity & equivariance of fillers!

- The goal here is to give an "algebraic" proof in the setting $\text{cSet} \to \text{cSet}^\Sigma$, using the (easier) QMS on $\text{cSet}^\Sigma$.

For example:

**Lemma** The $(\text{TcOf}, E)$ WFS on $\text{cSet}$ has the Frobenius property.

**Pr.** The statement holds for $(\text{TcOf}^\Sigma, E^\Sigma)$ in $\text{cSet}^\Sigma$, by the original proof, verbatim. So $E^\Sigma$ is closed under $\text{T}$-functors. Since $\Delta$ preserves $\Pi$ and preserves & reflects $E$, it follows that $E \subseteq \text{cSet}$ is also closed under $\Pi$.

Note that we don't need $\Delta$ to preserve TcOf!
Have a universe \( \mathcal{U}^2 \to \mathcal{U}^2 \) in \( \mathcal{E}^2 \), want to build one \( \mathcal{U} \to \mathcal{U} \) in \( \mathcal{E} \).

Recall a family \( E \to X \) in \( \mathcal{E}^2 \) has a type \( \text{Fib}^2(E) \to X \) of fibration structures on \( E \) over \( X \).

Moreover, \( \text{Fib}^2(-) \) is stable under pullback:

\[
\begin{array}{c}
\text{Fib}^2(x^*E) \\
\downarrow \\
\times^*\text{Fib}^2(E) \\
\downarrow \\
Y \\
\alpha \to X
\end{array}
\]

In \( \mathcal{E} \), we then have fibration structure on any family \( B \to A \) using the "dependent right adjoint":

\[
\begin{array}{c}
\text{Fib}(B) \\
\downarrow \\
\pi^*T(\text{Fib}^2(\Delta B)) \\
\downarrow \\
A \\
\eta \to T\Delta(A)
\end{array}
\]

This \( \text{Fib} \) is then also stable under pullback:

\[ \alpha^*\text{Fib}(B) = \text{Fib}(\alpha^*B) \]

for all \( \alpha : A' \to A \) in \( \mathcal{E} \).
In general, we can use such a stable fibration structure to construct a universe:

Let \( \hat{V} \to V \) be a universe of "small" families. Then we set

\[
\begin{array}{ccc}
\hat{U} & \xrightarrow{1} & \hat{V} \\
\downarrow & & \downarrow \\
U \coloneqq \text{Fib}(\hat{V}) & \longrightarrow & V
\end{array}
\]

To get a universe of fibrations.

This construction can be used on \( \mathfrak{E} \xrightarrow{\delta} \mathfrak{E}^\Sigma \):

\[
\text{Fib}(\cdot) = \eta^* \text{T Fib}^{\mathfrak{E}^\Sigma}(\delta \cdot)
\]

to get a universe \( \hat{U} \to U \) in \( \mathfrak{E} \).

Prop. (i) \( \hat{U} \to U \) is a universe of fibrations.

(ii) Moreover, Fib(\cdot) is a weak proposition.

Pf. (i) Just as in the general case.

(ii) "Weak prop." = TFib if inhabited.

Fib^{\mathfrak{E}}(\cdot) is always a weak prop., and
the dep. rt. adj. preserves weak props., by ...
Lemma. Let $\mathcal{E} \xrightarrow{\varepsilon} \mathcal{C}$ be a subcategory of weak propositions. Let $\mathcal{E} \xrightarrow{\tau} \mathcal{C}$ be an adjunction, with "dependent right adjoint":

$$\mathcal{E}/\mathcal{C} \xrightarrow{T_d} \mathcal{C}$$

Then for any $X$ in $\mathcal{E}$, the image

$$T_d X \quad \mathcal{E}/\mathcal{C}$$

is again a weak prop.

Proof. Suppose $X \rightarrow \mathcal{E}$ is in $\mathcal{E}$ and so is a weak prop. Consider $T_d X \rightarrow \mathcal{C}$, which is the pullback:

$$\xymatrix{ T_d X \ar[r] \ar[d] & T_! X \ar[d] \\
\mathcal{C} \ar[r]_n & T_! \mathcal{C} }$$
If \( T \downarrow X \to C \) has a section, then so does \( X \to \Delta C \):

\[
\Delta C \vdash \tilde{s} : X
\]

Since \( X \) is a weak prep., \( X \to \Delta C \) is then a TFib. Thus so is \( T \downarrow X \to T \downarrow \Delta C \) (\( T \) preserves TFib since \( \Delta \) preserves monos).

So the pullback \( \eta^* T \downarrow X = T \downarrow X \to C \) is also a TFib.

\]

Left to do:

- Show that this \( \mathcal{U} \) is univalent.
- Show that this \( \mathcal{U} \) is fibrant.