CMU HoTT Seminar – 9.11.2020 – S. Awodey

Equivariant Cubical QMS

Outline

1. Recall problem: model structure on cSet not equiv. to "spaces" = sSet w/ KQMS.

2. Idea: take the symmetries of the cubes into account in the filling conditions.

3. Do this systematically by first building a model str. on $cSet^{\Sigma}$ for Σ = "finite sym. groupoid", using the same "TT approach" as for cSet.

4. Transfer the model str. along the functor
$$\Delta : cSet \longrightarrow cSet^{\Sigma}.$$

5. Get a new "equivariant model str." on cSet.

NB: Sattler: $(\text{cSet w/ equiv.}) \simeq (\text{sSet w/ Kan})$.

1. cSets

Recall that we have a model str. on Cartesian cubical sets:

$$c\mathsf{Set} = \mathsf{Set}^{\mathbb{C}^{op}} .$$

In cSet, the representables are "cubes"

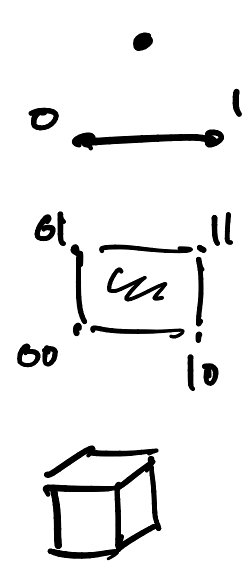
$$I^n = I \times \cdots \times I$$

I^0

I^1

I^2

I^3

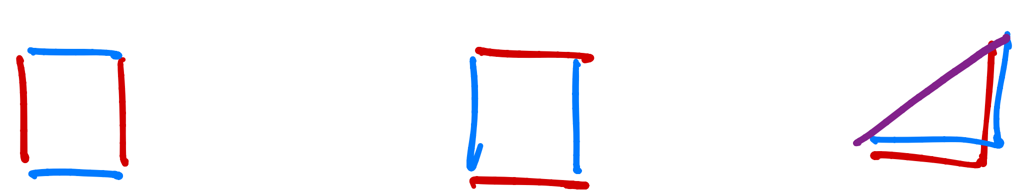


etc.

The "swap" map $I^2 \xrightarrow{\sigma} I^2$ exchanges the coordinates, so the coequalizer

$$I^2 \underset{1}{\overset{\sigma}{\rightrightarrows}} I^2 \longrightarrow Q$$

looks like:



Geometric realization $c\mathsf{Set} \longrightarrow \mathsf{Top}$ takes Q to a contractible space, but $Q \longrightarrow 1$ is not a weak equiv. in cSet.

"Type theoretic" model structure on cSet given by:

$$\mathcal{C} = \text{monos classified by } 1 \xrightarrow{\top} \Phi \subseteq \Omega.$$

$$\mathcal{F} = \text{uniform (unbiased) Kan fibrations},$$

$$\mathcal{W} = \underbrace{(\mathrm{TFib})}_{= \mathcal{C}^{\pitchfork}} \circ \underbrace{(\mathrm{TCof})}_{= {}^{\pitchfork}\mathcal{F}}.$$

"Type theoretic" because:

- Both TFib = $\mathcal{C}^{\pitchfork}$ & $\mathcal{F}$ have a "type of uniform filling structure".
- This is used to construct a universe $\mathcal{U}$.
- One then shows univalence & fibrancy of $\mathcal{U}$.
- From that, we finally get 3 for 2 for $\mathcal{W}$.

Recall furthermore:

- TFib can be determined algebraically as the algebras for the partial map classifier monad on the slice categories:

$$\mathcal{E}/X \longrightarrow \mathcal{E}/X$$

$$A \longmapsto \sum_{\varphi : \Phi} A^{[\varphi]} = A^{+}.$$

- $f : Y \longrightarrow X \in \mathcal{F}$ iff $(\delta \Rightarrow f) \in \mathrm{TFib}$, where $\delta : 1 \longrightarrow \mathrm{I}$ is generic in $\mathcal{E}/\mathrm{I}$.

In more detail, $f : Y \to X$ is a <u>fibration</u> if in $\mathcal{E}/I$,

$$
\begin{tikzcd}
Y^I \arrow[dr, dashed, "\delta \Rightarrow f"] \arrow[drr, bend left, "\varepsilon_Y"] \arrow[ddr, "f^I"'] & & \\
& \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
& X^I \arrow[r, "\varepsilon_X"'] & X
\end{tikzcd}
$$

$(\delta \Rightarrow f)$

$\in \mathrm{TFib}$

$\Longleftrightarrow$ f.all I^n, $c : C \rightarrowtail I^n$, $(z : I^n \to I)$,

$$
\begin{tikzcd}
B \arrow[r, "y"] \arrow[d, tail, "c \otimes \delta"'] & Y \arrow[d, "f"] \\
I^n \times I \arrow[r, "x"'] \arrow[ur, dashed, "j"] & X
\end{tikzcd}
$$

$\forall x, y$

$\exists j = j(c, x, y)$

and <u>uniform</u> in $I^m \xrightarrow{\alpha} I^n$:

$$
\begin{tikzcd}
B' \arrow[r] \arrow[d, tail, "c' \otimes \delta"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "y"] \arrow[d, tail, "c \otimes \delta"] & Y \arrow[d, "f"] \\
I^m \times I \arrow[r, "\alpha \times 1"'] \arrow[urr, dashed, "j'"] & I^n \times I \arrow[r, "x"'] \arrow[ur, dashed, "j"] & X
\end{tikzcd}
$$

$$j' = j \cdot (\alpha \times 1) .$$

Here $B \rightarrowtail I^n \times I$ is the <u>push-out product</u>

$$
\begin{tikzcd}
C \times 1 \arrow[r] \arrow[d, tail] & C \times I \arrow[d] & \\
I^n \times 1 \arrow[r] & B \arrow[r, tail, "c \otimes \delta"] \arrow[ul, phantom, "\ulcorner", very near start] & I^n \times I
\end{tikzcd}
$$

.

2. New idea: Equivariance

(i) in the gen. triv. cof.s:

$$\begin{tikzcd} B \arrow[d, tail, "C\otimes\delta"'] \\ I^n \times I^k \end{tikzcd} \quad \text{replace} \quad \begin{tikzcd} 1 \arrow[d, "\delta"'] \\ I \end{tikzcd} \quad \text{by} \quad \begin{tikzcd} 1 \arrow[d, "\delta^k"] \\ I^k \end{tikzcd} \quad \text{f. } k \geq 1.$$

(ii) require the fillers j to also be <u>equivariant</u>, i.e. uniform w/resp. to all symmetries

$$\sigma : I^k \to I^k \qquad \sigma \in \Sigma_k .$$

$$\begin{tikzcd}
B \arrow[r, "\sim"] \arrow[d, tail, "C\otimes\delta^k"'] \arrow[r, phantom, "\perp"'] & B \arrow[r, "y"] \arrow[d, tail, "C\otimes\delta^k"'] & Y \arrow[d, "f"] \\
I^n \times I^k \arrow[r, "1\times\sigma"'] \arrow[urr, dashed, "j"] & I^n \times I^k \arrow[r, "x"'] \arrow[ur, dashed, "j'"'] & X
\end{tikzcd}$$

$$j' \cdot (1 \times \sigma) = j .$$

<u>Note</u> this solves our original problem by making $1 \to I^2/\sigma = Q$ a TCof, so $Q \to 1 \in \mathcal{W}$.

In more detail:

since all $f : Y \to X \in \mathcal{I}$ are equivariant,

$$\left(\begin{array}{c} 0 \\ \downarrow \\ 1 \end{array}\right) \otimes \delta_2 = \begin{tikzcd} 1 \arrow[r, "\sim"] \arrow[d, "\delta_2"'] & 1 \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ I^2 \arrow[r, "\sim"', "\sigma"] & I^2 \arrow[r] & X \end{tikzcd} ,$$

we have:

$$\begin{tikzcd} 1 \arrow[r, "\sim"] \arrow[d, "\delta_2"'] & 1 \arrow[r, "\sim"] \arrow[d] & 1 \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ I^2 \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "1"'] & I^2 \arrow[r] & I^2/\sigma \arrow[r] & X \end{tikzcd}$$

- similarly for all I^k/Σ_k, $k \geq 2$.
- We'll have a more systematic view of what's going on soon.
- First note also that the equivariant filling condition can also be reformulated algebraically.

Consider first the following in $\mathcal{E}/I^k$, as before:

$$\begin{tikzcd}
Y^{I^k} \arrow[dr, dashed, "\delta_k \Rightarrow f"] \arrow[ddr, "f^{I^k}"'] \arrow[rrd, bend left] & & \\
& P \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "f"] \\
& X^{I^k} \arrow[r] & X
\end{tikzcd}$$

We asked for $(\delta_k \Rightarrow f) \in \mathcal{C}^{+}$ uniformly, i.e. a +Algebra, but now what about equivariance?

Take:

$$\begin{tikzcd}
Y^{I^k} \arrow[rrr, "Y^{\sigma^{-1}}"] \arrow[dr, "\delta_k \Rightarrow f"] \arrow[dd, "f^{I^k}"'] & & & Y^{I^k} \arrow[dl, "\delta_k \Rightarrow f"'] \arrow[dd, "f^{I^k}"] \\
& P \arrow[r, dashed, "\sigma_P"'] \arrow[dl] & P \arrow[dr] & \\
X^{I^k} \arrow[rrr, "X^{\sigma^{-1}}"'] & & & X^{I^k}
\end{tikzcd}$$

where as before

$$P = X^{I^k} \times_X Y .$$

Then $\sigma_P = \left(X^{\sigma^{-1}} \times_X Y \right)$ makes this commute.

Now take the upper square and consider $(\delta_k \Rightarrow f)^+$:

$$
\begin{tikzcd}
YI^k \arrow[rrr, "Y^{\sigma^{-1}}"] \arrow[dr, "\eta"'] \arrow[dd, "\delta_k \Rightarrow f"'] & & & YI^k \arrow[dl, "\eta"] \arrow[dd, "\delta_k \Rightarrow f"] \\
& \cdot \arrow[r, dashed, "\sigma_+"'] \arrow[dl, "+"] \arrow[ul, dashed, bend right, red, "\alpha"'] & \cdot \arrow[dr, "+"'] \arrow[ur, dashed, bend left, red, "\alpha"] & \\
P \arrow[rrr, "\sigma_P"'] & & & P
\end{tikzcd}
$$

We get σ_+ by pulling back, since $+$ is stable.

<u>Prop.</u> The uniform filling structure associated to an algebra structure $\alpha : (\delta_k \Rightarrow f)^+ \longrightarrow (\delta_k \Rightarrow f)$ is Σ_k-equivariant iff α makes $(*)$ commute,

$$\alpha \circ \sigma^+ = Y^{\sigma^{-1}} \circ \alpha .$$

<u>Remark</u> All of the other maps in $(*)$ are equivariant. Thus f (in $\mathcal{E}$) is a Σ_k-equivariant fibration in the sense of unif. equiv. lifting iff $(\delta_k \Rightarrow f)$ is a $+$Algebra in $c\mathrm{Set}^{\Sigma_k}$, the topos of <u>Σ_k-equivariant cSets</u>.

3. Equivariant Sets

Let Σ_k be the k-element symmetric group,

and $\Sigma := \coprod_k \Sigma_k$ the "symmetric groupoid".

The presheaf topos of Σ-sets,

$$\mathrm{Set}^{\Sigma} \simeq \prod_k \mathrm{Set}^{\Sigma_k},$$

is <u>atomic</u>, meaning the functor

$$\Delta : \mathrm{Set} \longrightarrow \mathrm{Set}^{\Sigma}$$

preserves:

- the SOC Ω
- exponentials B^A
- LCC structure $\Pi_A B$
- limits & colimits, since it has $L \dashv \Delta \dashv \Pi$.

<u>Facts:</u>

- the functors

$$\Delta_k : \mathrm{Set} \longrightarrow \mathrm{Set}^{\Sigma} \longrightarrow \mathrm{Set}^{\Sigma_k}$$

are in addition full & faithful.

- The left adjoints $L : \mathrm{Set}^{\Sigma_k} \longrightarrow \mathrm{Set}$ preserve monos, and the right adjoints

$$R : \mathrm{Set}^{\Sigma_k} \longrightarrow \mathrm{Set}$$

preserve the SOC Ω.

Now let us make a base change along the geom. morph. $\mathrm{Set}^{\mathbb{C}^{op}} \longrightarrow \mathrm{Set}$.

We have a pullback of toposes & g.m.s:

$$\begin{tikzcd}
\mathrm{Set}^{(\mathbb{C}^{op}\times\Sigma)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Set}^{\Sigma} \arrow[d] \\
\mathrm{Set}^{\mathbb{C}^{op}} \arrow[r] & \mathrm{Set}
\end{tikzcd}$$

It follows that $c\mathrm{Set}^{\Sigma} \longrightarrow c\mathrm{Set}$ also atomic.

We'll call $c\mathrm{Set}^{\Sigma}$ the equivariant cubical sets.

- the objects are cSets X equipped w/ an action (in cSet) by Σ: for each $\sigma \in \Sigma$

$$X \xrightarrow[\sim]{\sigma} X,$$

s.th.

$$\begin{aligned} (\sigma\cdot\tau)\cdot x &= \sigma(\tau\cdot x), \\ i\cdot x &= x, \qquad \text{etc.} \end{aligned}$$

- the maps are those in cSet that respect the action.

- There are <u>constant</u> objects ΔX w/ trivial action,

$$\Delta : cSet \longrightarrow cSet^{\Sigma},$$

as well as adjoints $L \dashv \Delta \dashv \Pi$.

As before, Δ preserves Ω, B^A, $\Pi_B A$, $\varprojlim$, ...

- There are also the useful adjunctions:

$$\begin{tikzcd}
cSet \arrow[r, shift left=3, "u_!"] \arrow[r, phantom, shift left=1.5, "\perp"] \arrow[r, phantom, shift right=1.5, "\perp"] \arrow[r, shift right=3] & cSet^{\Sigma} \arrow[l] \arrow[r, "p_k"] \arrow[r, phantom, shift right=1.5, "\perp"] & cSet^{\Sigma_k} \arrow[l, shift right=3] \arrow[l, phantom, shift left=1.5, "\perp"] \arrow[l, shift left=3]
\end{tikzcd}$$

where

$$\begin{aligned}
u_! X &= \text{free } \Sigma\text{-action} \\
p_k E &= p_k(E_1, E_1, \dots) = E_k \\
p_{k!} X &= (0 \cdots \underset{k}{X} \cdots) \\
p_{k*} X &= (1 \cdots X \cdots) .
\end{aligned}$$

- For the representables in $cSet^{\Sigma}$ we have

$$y([n], *_k) = (0 \cdots \overset{k}{\underset{k}{u_! I^n}} \cdots 0) .$$

- In each $cSet^{\Sigma_k}$ there's the <u>equivariant k-cube</u>

$$\mathbb{I}_k = (I^k, \Sigma_k)$$

- In $cSet^{\Sigma}$ there's the special object

$$\mathbb{I} = (\mathbb{I}_0, \mathbb{I}_1, \mathbb{I}_2, \dots) .$$

4. Model structure on $c\mathrm{Set}^{\Sigma}$

We're going to use the "axiomatic" construction that we gave for $c\mathrm{Set}$ in the new setting of $c\mathrm{Set}^{\Sigma}$. Thereby we first need:

<u>Lemma</u>. $\mathbb{I}$ is tiny in $c\mathrm{Set}^{\Sigma}$.

Pf. Show it for each $\mathbb{I}_K$ in Set^{Σ_K} by calculation.

Now define the <u>"TTQMS" on $c\mathrm{Set}^{\Sigma}$</u>:

Let:

- $\mathcal{C}$ = monos
- $\mathrm{TFib} = \mathcal{C}^{\pitchfork} = {}^{+}\mathrm{Alg}$ $\Big\}$ AWFS.
- $\mathrm{Gen\,TCof} = \{ c \otimes \delta : B \rightarrowtail I^n \times \mathbb{I} \mid c \in \mathcal{C} \}$ where again $\delta : 1 \to \mathbb{I}$ in $c\mathrm{Set}^{\Sigma}/\mathbb{I}$.
- $\mathcal{F} = \{\mathrm{GTCof}\}^{\pitchfork}$ $= \{ f \mid (\delta \Rightarrow f) \in {}^{+}\mathrm{Alg} \text{ in } c\mathrm{Set}^{\Sigma}/\mathbb{I} \}$.
- $\mathcal{W} = \mathrm{TFib} \circ \mathrm{TCof}$.

<u>Thm</u> $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ is a QMS on $c\mathrm{Set}^{\Sigma}$.

Pf. Just as for $c\mathrm{Set}$.

5. Transferring the model structure

Now we are going to "transfer" the model structure along the adjunction

$$\begin{tikzcd} cSet \arrow[r, shift left, "\Delta"] & cSet^{\Sigma} \arrow[l, shift left, "L"] \end{tikzcd}$$

It would be fine to do this with a known "transfer theorem" – if only I knew one that gives what we want! Instead we do this:

Lemma. $\Delta : cSet \longrightarrow cSet^{\Sigma}$ preserves $(-)^{+}$.

It follows that Δ preserves & reflects +Algebras.

Def. The equivariant model structure on cSet is determined by:

- $\mathcal{C} = \text{Monos}$, $\qquad \text{TFib} = \mathcal{C}^{\pitchfork} = +\text{Alg}$,
- $\mathcal{I} = \Delta^{-1}(\mathcal{I}^{\Sigma})$
  $= \{ f : Y \to X \mid \Delta f \in \mathcal{I} \text{ in } cSet^{\Sigma} \}$,
- $W = \text{TFib} \circ \text{TCof}$, $\quad \text{TCof} = {}^{\pitchfork}\mathcal{I}$.

Main Thm. This is a QMS on cSet.

Remarks

- Since $\Delta : cSet \to cSet^{\Sigma}$ preserves $\mathcal{I}$, $L \dashv \Delta$ preserves TCof. Use this to get a gen. set

$$\{ L(B \rightarrowtail I^n \times \mathbb{I}) / \cdots \}$$

of TCof in cSet, for <u>small object argument</u>.

- $\Delta : cSet \to cSet^{\Sigma}$ preserves and reflects $\mathcal{C}$, $\mathcal{I}$, & TFib, but unfortunately it does <u>not</u> preserve TCof, and so also not $\mathcal{W}$. So there is work to do to prove 3 for 2.

- One approach is to repeat/modify the previous type-theoretic proof:
  - define the type of fibration structures
  - use tinyness of the cubes I^k to show that these types are stable
  - build a universe of fibrations
  - show it's univalent & fibrant
  - deduce 3 for 2 from the FEP.

- That is the approach used for the "TT proof", which was essentially done last year – and was <u>formalized</u> by Evan C. !

[NB: These equivariant fibrations agree w/ those defined before in terms of uniformity & equivariance of fillers j !]

- The goal here is to give an "algebraic" proof in the setting $cSet \longrightarrow cSet^{\Sigma}$, using the (easier) QMS on $cSet^{\Sigma}$.

For example:

<u>Lemma</u> The $(TCof, \mathcal{I})$ WFS on $cSet$ has the Frobenius property.

<u>Pf.</u> The statement holds for $(TCof^{\Sigma}, \mathcal{I}^{\Sigma})$ in $cSet^{\Sigma}$, by the original proof, verbatim. So $\mathcal{I}^{\Sigma}$ is closed under Π-functors. Since Δ preserves Π and preserves & reflects $\mathcal{I}$, it follows that $\mathcal{I} \subseteq cSet$ is also closed under Π. $\square$

Note that we <u>don't</u> need Δ to preserve $TCof$!

6. Equivariant Universe

$$\begin{array}{c} \mathcal{E}^\Sigma \\ \uparrow \delta \\ \mathcal{E} \end{array}$$

Have a universe $\dot{\mathcal{U}}^\Sigma \longrightarrow \mathcal{U}^\Sigma$ in $\mathcal{E}^\Sigma$, want to build one $\dot{\mathcal{U}} \longrightarrow \mathcal{U}$ in $\mathcal{E}$.

Recall a family $E \longrightarrow X$ in $\mathcal{E}^\Sigma$ has a type $\mathrm{Fib}^\Sigma(E) \longrightarrow X$ of fibration structures on E over X.

Moreover, $\mathrm{Fib}^\Sigma(-)$ is stable under pullback:

$$\mathrm{Fib}^\Sigma(\alpha^* E) \cong \alpha^* \mathrm{Fib}^\Sigma(E) \qquad \begin{tikzcd} \alpha^* E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ Y \arrow[r, "\alpha"] & X \end{tikzcd} \qquad \mathrm{Fib}^\Sigma(E)$$

In $\mathcal{E}$, we then have fibration structure on any family $B \longrightarrow A$ using the "dependent right adjoint":

$$\mathrm{Fib}(B) := \eta^* T(\mathrm{Fib}^\Sigma(\delta B)) \qquad \begin{tikzcd} B \arrow[r, "\eta"] \arrow[d] & T\delta(B) \arrow[d] \\ A \arrow[r, "\eta"] & T\delta(A) \end{tikzcd} \qquad T(\mathrm{Fib}^\Sigma(\delta B)).$$

This Fib is then also stable under pullback:

$$\alpha^* \mathrm{Fib}(B) = \mathrm{Fib}(\alpha^* B)$$

f. all $\alpha : A' \longrightarrow A$ in $\mathcal{E}$.

- In general, we can use such a stable fibration structure to construct a <u>universe</u>: Let $\dot{\mathcal{V}} \to \mathcal{V}$ be a universe of "small" families. Then we set

$$\begin{tikzcd}
\dot{\mathcal{U}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}} \arrow[d] \\
\mathcal{U} := \mathrm{Fib}(\dot{\mathcal{V}}) \arrow[r] & \mathcal{V}
\end{tikzcd}$$

To get a <u>universe of fibrations</u>.

- This construction can be used on $\mathcal{E} \xrightarrow{\Delta} \mathcal{E}^{\Sigma}$:

$$\mathrm{Fib}(-) = \eta^* T^! \mathrm{Fib}^{\Sigma}(\Delta -),$$

to get a universe $\dot{\mathcal{U}} \to \mathcal{U}$ in $\mathcal{E}$.

<u>Prop.</u> (i) $\dot{\mathcal{U}} \to \mathcal{U}$ is a universe of fibrations.
(ii) Moreover, $\mathrm{Fib}(-)$ is a <u>weak proposition</u>.

<u>Pf.</u> (i) Just as in the general case.
(ii) "Weak prop." = TFib if inhabited.
$\mathrm{Fib}^{\Sigma}(-)$ is always a weak prop., and the dep. rt. adj. preserves weak prop.s, by ...

Lemma Let
$$\begin{tikzcd} \mathcal{I} \arrow[r, hook] \arrow[dr] & \mathcal{E}^{\to} \arrow[d, "\mathrm{cod}"] \\ & \mathcal{E} \end{tikzcd}$$
be a subfibration of weak propositions.

Let $\mathcal{E} \underset{\Delta}{\overset{\Pi}{\underset{\leftarrow}{\overset{\top}{\to}}}} \mathcal{C}$ be an adjunction, with "dependent right adjoint":

$$\mathcal{E}/_{\Delta C} \xrightarrow{\Pi_d} \mathcal{C}/_C$$

$$\Delta C \vdash t : X \quad \cong \quad C \vdash \tilde{t} : \Pi_d X .$$

Then for any $\begin{matrix} X \\ \downarrow \\ \Delta C \end{matrix}$ in $\mathcal{I}$, the image $\begin{matrix} \Pi_d X \\ \downarrow \\ C \end{matrix}$ is again a weak prop. $\odot$

Pf. Suppose $X \to \Delta C$ is in $\mathcal{I}$ and so is a weak prop. Consider $\Pi_d X \to C$, which is the pullback:

$$\begin{tikzcd} \Pi_d X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi X \arrow[d] \\ C \arrow[r, "\eta"'] & \Pi \Delta C \end{tikzcd}.$$

If $T_d X \to C$ has a section,
then so does $X \to \delta C$:

$$\frac{C \vdash s : T_d X}{\delta C \vdash \tilde{s} : X} .$$

Since X is a weak prop., $X \to \delta C$ is then a TFib. Thus so is $T X \to T \delta C$, (T preserves TFib since Δ preserves monos).
So the pullback $\eta^* T X = T_d X \to C$ is also a TFib. $\boxed{\cdot}$

Left to do:

- Show that this $\mathcal{U}$ is univalent.
- Show that this $\mathcal{U}$ is fibrant.