Lecture Notes: Introduction to Categorical Logic

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Chapter 1

Review of Category Theory

1.1 Categories

Definition 1.1.1 A *category* $\mathcal{C}$ consists of classes

$\mathcal{C}_0$ of objects A, B, C, ...
$\mathcal{C}_1$ of morphisms f, g, h, ...

such that:

- Each morphism f has uniquely determined *domain* $\mathsf{dom}\, f$ and *codomain* $\mathsf{cod}\, f$, which are objects. This is written as

$$f : \mathsf{dom}\, f \to \mathsf{cod}\, f$$

- For any morphisms $f : A \to B$ and $g : B \to C$ there exists a uniquely determined *composition* $g \circ f : A \to C$. Composition is associative:

$$h \circ (g \circ f) = (h \circ g) \circ f \ ,$$

  where domains are codomains are as follows:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

- For every object A there exists the *identity* morphism $1_A : A \to A$ which is a unit for composition:

$$1_A \circ f = f \ , \qquad\qquad g \circ 1_A = g \ ,$$

  where $f : B \to A$ and $g : A \to C$.

Morphisms are also called *arrows* or *maps*. Note that morphisms do not actually have to be functions, and objects need not be sets or spaces of any sort. We often write $\mathcal{C}$ instead of $\mathcal{C}_0$.

Definition 1.1.2 A category $\mathcal{C}$ is *small* when the objects $\mathcal{C}_0$ and the morphisms $\mathcal{C}_1$ are sets (as opposed to proper classes). A category is *locally small* when for all objects $A, B \in \mathcal{C}_0$ the class of morphisms with domain A and codomain B is a set.

We normally restrict attention to locally small categories, so unless we specify otherwise all categories are taken to be locally small. Next we consider several examples of categories.

1.1.1 The empty category $\mathsf{0}$

The empty category has no objects and no arrows.

1.1.2 The unit category $\mathsf{1}$

The unit category, also called the terminal category, has one object $\star$ and one arrow $1_\star$:

$$\begin{tikzcd} \star \arrow[loop right, "1_\star"] \end{tikzcd}$$

1.1.3 Other finite categories

There are other finite categories, for example the category with two objects and one (non-identity) arrow, and the category with two parallel arrows:

$$\begin{tikzcd} \star \arrow[r] & \bullet \end{tikzcd} \qquad\qquad \begin{tikzcd} \star \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

1.1.4 Groups as categories

Every group $(G, \cdot)$, is a category with a single object $\star$ and each element of G as a morphism:

$$\begin{tikzcd} \star \arrow[loop above, "b"] \arrow[loop right, "a"] \arrow[loop below, "c"] \end{tikzcd} \qquad\qquad a, b, c, \ldots \in G$$

The composition of arrows is given by the group operation:

$$a \circ b = a \cdot b$$

The identity arrow is the group unit e. This is indeed a category because the group operation is associative and the group unit is the unit for the composition. In order to get a category, we do not actually need to know that every element in G has an inverse. It suffices to take a *monoid*, also known as *semigroup*, which is an algebraic structure with an associative operation and a unit.

We can turn things around and *define* a monoid to be a category with a single object. A group is then a category with a single object in which every arrow is an isomorphism.

1.1.5 Posets as categories

Recall that a *partially ordered set*, or a *poset* $(P, \leq)$, is a set with a reflexive, transitive, and antisymmetric relation:

$$x \leq x \qquad \text{(reflexive)}$$
$$x \leq y \wedge y \leq z \Longrightarrow x \leq z \qquad \text{(transitive)}$$
$$x \leq y \wedge y \leq z \Longrightarrow x = y \qquad \text{(antisymmetric)}$$

Each poset is a category whose objects are the elements of P, and there is a single arrow $p \to q$ between $p, q \in P$ if, and only if, $p \leq q$. Composition of $p \to q$ and $q \to r$ is the unique arrow $p \to r$, which exists by transitivity of $\leq$. The identity arrow on p is the unique arrow $p \to p$, which exists by reflexivity of $\leq$.

Antisymmetry tells us that any two isomorphic objects in P are equal.[1] We do not need antisymmetry in order to obtain a category, i.e., a *preorder* would suffice.

Again, we may *define* a preorder to be a category in which there is at most one arrow between any two objects. A poset is a skeletal preorder. We allow for the possibility that a preorder or a poset is a proper class rather than a set.

A particularly important example of a poset category is the posets of open sets $\mathcal{O}X$ of a topological space X, ordered by inclusion.

1.1.6 Sets as categories

Any set S is a category whose objects are the elements of S and the only arrows are the identity arrows. A category in which the only arrows are the identity arrows is a *discrete category*.

1.1.7 Structures as categories

In general structures like groups, topological spaces, posets, etc., determine categories in which composition is composition of functions and identity morphisms are identity functions:

- Group is the category whose objects are groups and whose morphisms are group homomorphisms.
- Top is the category whose objects are topological spaces and whose morphisms are continuous maps.
- Set is the category whose objects are sets and whose morphisms are functions.[2]

[1] A category in which isomorphic object are equal is a *skeletal* category.

[2] A function between sets A and B is a relation $f \subseteq A \times B$ such that for every $x \in A$ there exists a unique $y \in B$ for which $\langle x, y \rangle \in f$. A morphism in Set is a triple $\langle A, f, B \rangle$ such that $f \subseteq A \times B$ is a function.

- Graph is the category of (directed) graphs an graph homomorphisms.
- Poset is the category of posets and monotone maps.

Such categories of structures are generally *large*.

1.1.8 Further Definitions

We recall some further basic notions in category theory.

Definition 1.1.3 A *subcategory* $\mathcal{C}'$ of a category $\mathcal{C}$ is given by a subclass of objects $\mathcal{C}'_0 \subseteq \mathcal{C}_0$ and a subclass of morphisms $\mathcal{C}'_1 \subseteq \mathcal{C}_1$ such that $f \in \mathcal{C}'_1$ implies $\mathsf{dom}\, f, \mathsf{cod}\, f \in \mathcal{C}'_0$, $1_A \in \mathcal{C}'_1$ for every $A \in \mathcal{C}'_0$, and $g \circ f \in \mathcal{C}'_1$ whenever $f, g \in \mathcal{C}'_1$ are composable.

A *full subcategory* $\mathcal{C}'$ of $\mathcal{C}$ is a subcategory of $\mathcal{C}$ such that, for all $A, B \in \mathcal{C}'_0$, if $f : A \to B$ is in $\mathcal{C}_1$ then it is also in $\mathcal{C}'_1$.

Definition 1.1.4 An *inverse* of a morphism $f : A \to B$ is a morphism $f^{-1} : B \to A$ such that

$$f \circ f^{-1} = 1_B \qquad \text{and} \qquad f^{-1} \circ f = 1_A \ .$$

A morphism that has an inverse is an *isomorphism*, or an *iso*. If there exists a pair of inverse morphisms $f : A \to B$ and $f^{-1} : B \to A$ we say that the objects A and B are *isomorphic*, written $A \cong B$.

The notation f^{-1} is justified because an inverse, if it exists, is unique. A *left inverse* is a morphism $g : B \to A$ such that $g \circ f = 1_A$, and a *right inverse* is a morphism $g : B \to A$ such that $f \circ g = 1_B$. A left inverse is also called a *retraction*, whereas a right inverse is called a *section*.

Definition 1.1.5 A *monomorphism*, or *mono*, is a morphism $f : A \to B$ that can be canceled on the left: for all $g : C \to A$, $h : C \to A$,

$$f \circ g = f \circ h \Longrightarrow g = h \ .$$

An *epimorphism*, or *epi*, is a morphism $f : A \to B$ that can be canceled on the right: for all $g : B \to C$, $h : B \to A$,

$$g \circ f = h \circ f \Longrightarrow g = h \ .$$

In Set monomorphisms are the injective functions and epimorphisms are the surjective functions. Isomorphisms in Set are the bijective functions. Thus, in Set a morphism is iso if, and only if, it is both mono and epi. However, this example is misleading! In general, a morphism can be mono and epi without being an iso. For example, the non-identity morphism in the category consisting of two objects and one morphism between them is both epi and mono, but it has no inverse. (See examples in the next section.)

A more realistic example of morphisms that are both epi and mono but are not iso occurs in the category Top of topological spaces and continuous map because not every continuous bijection is a homeomorphism.

A *diagram* of objects and morphisms is a directed graph whose vertices are objects of a category and edges are morphisms between them, for example:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[r, "h"] & C \arrow[dl, "j"] \\
D \arrow[ur, "m"] \arrow[r, "k"'] & E &
\end{tikzcd}$$

Such a diagram is said to *commute* when the composition of morphisms along any two paths with the same beginning and end gives equal morphisms. Commutativity of the above diagram is equivalent to the following two equations:

$$f = m \circ g \;, \qquad\qquad j \circ h \circ m = k \;.$$

From these we can derive $k \circ g = h \circ h \circ f$.

1.2 Functors

Definition 1.2.1 A *functor* $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ consists of functions

$$F_0 : \mathcal{C}_0 \to \mathcal{D}_0 \qquad \text{and} \qquad F_1 : \mathcal{C}_1 \to \mathcal{D}_1$$

such that, for all $f : A \to B$ and $g : B \to C$ in $\mathcal{C}$:

$$\begin{aligned} F_1 f : F_0 A \to F_0 B \;, \\ F_1(g \circ f) = (F_1 g) \circ (F_1 f) \;, \\ F_1(1_A) = 1_{F_0 A} \;. \end{aligned}$$

We usually write F for both F_0 and F_1.

A functor maps commutative diagrams to commutative diagrams because it preserves composition.

We may form the "category of categories" Cat whose objects are small categories and whose morphisms are functors. Composition of functors is composition of the corresponding functions, and the identity functor is one that is identity on objects and on morphisms. The category Cat is large and locally small.

Definition 1.2.2 A functor $F : \mathcal{C} \to \mathcal{D}$ is *faithful* when it is injective on morphisms: for all $f, g : A \to B$, if $Ff = Fg$ then $f = g$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is *full* when it is surjective on morphisms: for every $g : FA \to FB$ there exists $f : A \to B$ such that $g = Ff$.

We consider several examples of functors.

1.2.1 Functors between sets, monoids and posets

When sets, monoids, groups, and posets are regarded as categories, the functors turn out to be the *usual morphisms*, for example:

- A functor between sets S and T is a function from S to T.
- A functor between groups G and H is a group homomorphism from G to H.
- A functor between posets P and Q is a monotone function from P to Q.

Exercise 1.2.3 Verify that the above claims are correct.

1.2.2 Forgetful functors

For categories of structures Group, Top, Graphs, Poset, ..., there is a *forgetful* functor U which maps an object to the underlying set and a morphism to the underlying function. For example, the forgetful functor $U : \mathsf{Group} \to \mathsf{Set}$ maps a group $(G, \cdot)$ to the set G and a group homomorphism $f : (G, \cdot) \to (H, \star)$ to the function $f : G \to H$.

There are also forgetful functors that forget only part of the structure, for example the forgetful functor $U : \mathsf{Ring} \to \mathsf{Group}$ which maps a ring $(R, +, \times)$ to the additive group $(R, +)$ and a ring homomorphism $f : (R, +_R, \cdot_S) \to (S, +_S, \cdot_S)$ to the group homomorphism $f : (R, +_R) \to (S, +_S)$.

1.3 Constructions of Categories and Functors

1.3.1 Product of categories

Given categories $\mathcal{C}$ and $\mathcal{D}$, we form the *product category* $\mathcal{C} \times \mathcal{D}$ whose objects are pairs of objects $\langle C, D\rangle$ with $C \in \mathcal{C}$ and $D \in \mathcal{D}$, and whose morphisms are pairs of morphisms $\langle f, g\rangle : \langle C, D\rangle \to \langle C', D'\rangle$ with $f : C \to C'$ in $\mathcal{C}$ and $g : D \to D'$ in $\mathcal{D}$. Composition is given by $\langle f, g\rangle \circ \langle f', g'\rangle = \langle f \circ f', g \circ g'\rangle$.

There are evident *projection* functors

$$\begin{tikzcd} & \mathcal{C} \times \mathcal{D} \arrow[dl, "\pi_0"'] \arrow[dr, "\pi_1"] & \\ \mathcal{C} & & \mathcal{D} \end{tikzcd}$$

which act as indicated in the following diagrams:

$$\begin{tikzcd} & \langle C, D\rangle \arrow[dl, mapsto, "\pi_0"'] \arrow[dr, mapsto, "\pi_1"] & \\ C & & D \end{tikzcd} \qquad \begin{tikzcd} & \langle f, g\rangle \arrow[dl, mapsto, "\pi_0"'] \arrow[dr, mapsto, "\pi_1"] & \\ f & & g \end{tikzcd}$$

Exercise 1.3.1 Show that, for any categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$,

$$1 \times \mathcal{C} \cong \mathcal{C}$$
$$\mathcal{B} \times \mathcal{C} \cong \mathcal{C} \times \mathcal{B}$$
$$\mathcal{A} \times (\mathcal{B} \times \mathcal{C}) \cong (\mathcal{A} \times \mathcal{B}) \times \mathcal{C}$$

What does $\cong$ mean here?

1.3.2 Slice categories

Given a category $\mathcal{C}$ and an object $A \in \mathcal{C}$, the *slice* category $\mathcal{C}/A$ has as objects morphisms into A,

$$\begin{tikzcd} B \arrow[d, "f"] \\ A \end{tikzcd} \tag{1.1}$$

and as morphisms commutative diagrams over A,

$$\begin{tikzcd} B \arrow[rr, "g"] \arrow[dr, "f"'] & & B' \arrow[dl, "f'"] \\ & A & \end{tikzcd} \tag{1.2}$$

That is, a morphism from $f : B \to A$ to $f' : B' \to A$ is a morphism $g : B \to B'$ such that $f = f' \circ g$. Composition of morphisms in $\mathcal{C}/A$ is composition of morphisms in $\mathcal{C}$.

There is a forgetful functor $U_A : \mathcal{C}/A \to \mathcal{C}$ which maps an object (1.1) to its domain B, and a morphism (1.2) to the morphism $g : B \to B'$.

Furthermore, for each morphism $h : A \to A'$ in $\mathcal{C}$ there is a functor "composition by h",

$$\mathcal{C}/h : \mathcal{C}/A \to \mathcal{C}/A'$$

which maps an object (1.1) to the object $h \circ f : B \to A'$ and a morphisms (1.2) to the morphism

$$\begin{tikzcd} B \arrow[rr, "g"] \arrow[dr, "h \circ f"'] & & B' \arrow[dl, "h \circ f'"] \\ & A' & \end{tikzcd}$$

The construction of slice categories itself is a functor

$$\mathcal{C}/- : \mathcal{C} \to \mathsf{Cat}$$

provided that $\mathcal{C}$ is small. This functor maps each $A \in \mathcal{C}$ to the category $\mathcal{C}/A$ and each morphism $h : A \to A'$ to the functor $\mathcal{C}/h : \mathcal{C}/A \to \mathcal{C}/A'$.

Since Cat is a category, we may form the slice category $\mathsf{Cat}/\mathcal{C}$ for any small category $\mathcal{C}$. The slice functor $\mathcal{C}/-$ factors through the forgetful functor $U_{\mathcal{C}} : \mathsf{Cat}/\mathcal{C} \to \mathsf{Cat}$ via a functor $\overline{\mathcal{C}} : \mathcal{C} \to \mathsf{Cat}/\mathcal{C}$,

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\overline{\mathcal{C}}"] \arrow[dr, "\mathcal{C}/-"'] & \mathsf{Cat}/\mathcal{C} \arrow[d, "U_{\mathcal{C}}"] \\
& \mathsf{Cat}
\end{tikzcd}$$

where, for $A \in \mathcal{C}$, $\overline{C}A$ is the object

$$\begin{tikzcd}
\mathcal{C}/A \arrow[d, "U_A"] \\
\mathcal{C}
\end{tikzcd}$$

and, for $h : A \to A'$ in $\mathcal{C}$, $\overline{C}h$ is the morphism

$$\begin{tikzcd}
\mathcal{C}/A \arrow[rr, "\mathcal{C}/h"] \arrow[dr, "U_A"'] & & \mathcal{C}/A' \arrow[dl, "U_{A'}"] \\
& \mathcal{C} &
\end{tikzcd}$$

1.3.3 Opposite categories

For a category $\mathcal{C}$ the *opposite category* $\mathcal{C}^{\mathsf{op}}$ has the same objects as $\mathcal{C}$, but all the morphisms are turned around, that is, a morphism $f : A \to B$ in $\mathcal{C}^{\mathsf{op}}$ is a morphism $f : B \to A$ in $\mathcal{C}$. Composition and identity arrows in $\mathcal{C}^{\mathsf{op}}$ are the same as in $\mathcal{C}$. Clearly, the opposite of the opposite of a category is the original category.

A functor $F : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}$ is sometimes called a *contravariant functor* (from $\mathcal{C}$ to $\mathcal{D}$), and a functor $F : \mathcal{C} \to \mathcal{D}$ is a *covariant* functor.

For example, the opposite category of a preorder $(P, \leq)$ is the preorder P turned upside down, $(P, \geq)$.

Exercise 1.3.2 Given a functor $F : \mathcal{C} \to \mathcal{D}$, can you define a functor $F^{\mathsf{op}} : \mathcal{C}^{\mathsf{op}} \to \mathcal{D}^{\mathsf{op}}$ such that $-^{\mathsf{op}}$ itself becomes a functor? On what category is it a functor?

1.3.4 Representable functors

Let $\mathcal{C}$ be a locally small category. Then for each pair of objects $A, B \in \mathcal{C}$ the collection of all morphisms $A \to B$ forms a set, written $\mathsf{Hom}_{\mathcal{C}}(A, B)$, $\mathsf{Hom}(A, B)$

or $\mathcal{C}(A, B)$. For every $A \in \mathcal{C}$ there is a functor

$$\mathcal{C}(A, -) : \mathcal{C} \to \mathsf{Set}$$

defined by

$$\begin{aligned} &\mathcal{C}(A, B) = \{ f \in \mathcal{C}_1 \mid f : A \to B \} \\ &\mathcal{C}(A, g) : f \mapsto g \circ f \end{aligned}$$

where $B \in \mathcal{C}$ and $g : B \to C$. In words, $\mathcal{C}(A, g)$ is composition by g. This is indeed a functor because, for any morphisms

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \tag{1.3}$$

we have

$$\mathcal{C}(A, h \circ g)f = (h \circ g) \circ f = h \circ (g \circ f) = \mathcal{C}(A, h)(\mathcal{C}(A, g)f) ,$$

and $\mathcal{C}(A, 1_B)f = 1_A \circ f = f = 1_{\mathcal{C}(A,B)}f$. We may also ask whether $\mathcal{C}(-, B)$ is a functor. If we define its action on morphisms to be precomposition,

$$\mathcal{C}(f, B) : g \mapsto g \circ f ,$$

it becomes a *contravariant* functor,

$$\mathcal{C}(-, B) : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set} .$$

The contravariance is a consequence of precomposition; for morphisms (1.3) we have

$$\mathcal{C}(g \circ f, D)h = h \circ (g \circ f) = (h \circ g) \circ f = \mathcal{C}(f, D)(\mathcal{C}(g, D)h) .$$

A functor of the form $\mathcal{C}(A, -)$ is a *(covariant) representable functor*, and a functor of the form $\mathcal{C}(-, B)$ is a *(contravariant) representable functor*.

To summarize, hom-set is a functor

$$\mathcal{C}(-, -) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Set}$$

which maps a pair of objects $A, B \in \mathcal{C}$ to the set $\mathcal{C}(A, B)$ of morphisms from A to B, and it maps a pair of morphisms $f : A' \to A$, $g : B \to B'$ in $\mathcal{C}$ to the function

$$\mathcal{C}(f, g) : \mathcal{C}(A, B) \to \mathcal{C}(A', B')$$

defined by

$$\mathcal{C}(f, g) : h \mapsto g \circ h \circ f .$$

1.3.5 Group actions

A group $(G, \cdot)$ is a category with one object $\star$ and elements of G as the morphisms. Thus, a functor $F : G \to \mathsf{Set}$ is given by a set $F\star = S$ and for each $a \in G$ a function $Fa : S \to S$ such that, for all $x \in S$, $a, b \in G$,

$$(Fe)x = x\ , \qquad\qquad (F(a \cdot b))x = (Fa)((Fb)x)\ .$$

Here e is the unit element of G. If we write $a \cdot x$ instead of $(Fa)x$, the above two equations become the familiar requirements for a *left group action*:

$$e \cdot x = x\ , \qquad\qquad (a \cdot b) \cdot x = a \cdot (b \cdot x)\ .$$

Exercise 1.3.3 A *right group action* by a group $(G, \cdot)$ on a set S is an operation $\cdot : S \times G \to S$ that satisfies, for all $x \in S$, $a, b \in G$,

$$x \cdot e = x\ , \qquad\qquad x \cdot (a \cdot b) = (x \cdot a) \cdot b\ .$$

Exhibit right group actions as functors.

1.4 Natural Transformations and Functor Categories

Definition 1.4.1 Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta : F \Longrightarrow G$ from F to G is a map $\eta : \mathcal{C}_0 \to \mathcal{D}_1$ which assigns to every object $A \in \mathcal{C}$ a morphism $\eta_A : FA \to GA$, called the *component of* η *at* A, such that, for every $f : A \to B$, $\eta_B \circ Ff = Gf \circ \eta_A$, i.e., the following diagram commutes:

$$\begin{tikzcd} FA \arrow[r, "\eta_A"] \arrow[d, "Ff"'] & GA \arrow[d, "Gf"] \\ FB \arrow[r, "\eta_B"'] & GB \end{tikzcd}$$

As an example of a natural transformation, consider groups G and H as categories and two homomorphisms $f, g : G \to H$ as functors between them. A natural transformation $\eta : f \Longrightarrow g$ is given by a single element $\eta_\star = b \in H$ such that, for every $a \in G$, the following diagram commutes:

$$\begin{tikzcd} \star \arrow[r, "b"] \arrow[d, "fa"'] & \star \arrow[d, "ga"] \\ \star \arrow[r, "b"'] & \star \end{tikzcd}$$

This means that $b \cdot fa = (ga) \cdot b$, that is $ga = b \cdot (fa) \cdot b^{-1}$. In other words, a natural transformation $f \Longrightarrow g$ is a *conjugation* operation $b^{-1} \cdot - \cdot b$ which transforms f into g.

For every functor $F : \mathcal{C} \to \mathcal{D}$ there exists the *identity transformation* $1_F : F \Longrightarrow F$ defined by $(1_F)_A = 1_A$. If $\eta : F \Longrightarrow G$ and $\theta : G \Longrightarrow H$ are natural transformations, then their composition $\theta \circ \eta : F \Longrightarrow H$, defined by $(\theta \circ \eta)_A = \theta_A \circ \eta_A$ is also a natural transformation. Composition of natural transformations is associative because it is function composition. This leads to the definition of functor categories.

Definition 1.4.2 Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The *functor category* $\mathcal{D}^{\mathcal{C}}$ is the category whose objects are functors from $\mathcal{C}$ to $\mathcal{D}$ and whose morphisms are natural transformations between them.

A functor category may be quite large, too large in fact. In order to avoid problems with size we normally require $\mathcal{C}$ to be a locally small category. The "hom-class" of all natural transformations $F \Longrightarrow G$ is usually written as

$$\mathsf{Nat}(F, G)$$

instead of the more awkward $\mathsf{Hom}_{\mathcal{D}^{\mathcal{C}}}(F, G)$.

Suppose we have functors F, G, and H with a natural transformation $\theta : G \Longrightarrow H$, as in the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{E} \end{tikzcd}$$

Then we can form a natural transformation $\theta \circ F : G \circ F \Longrightarrow H \circ F$ whose component at $A \in \mathcal{C}$ is $(\theta \circ F)_A = \theta_{FA}$.

Similarly, if we have functors and a natural transformation

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{D} \arrow[r, "F"] & \mathcal{E} \end{tikzcd}$$

we can form a natural transformation $(F \circ \theta) : F \circ G \Longrightarrow F \circ H$ whose component at $A \in \mathcal{C}$ is $(F \circ \theta)_A = F\theta_A$.

A *natural isomorphism* is an isomorphism in a functor category. Thus, if $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ are two functors, a natural isomorphism between them is a natural transformation $\eta : F \Longrightarrow G$ whose components are isomorphisms. In this case, the inverse natural transformation $\eta^{-1} : G \Longrightarrow F$ is given by $(\eta^{-1})_A = (\eta_A)^{-1}$. We write $F \cong G$ when F and G are naturally isomorphic.

The definition of natural transformations is motivated in part by the fact that, for any small categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$,

$$\mathsf{Cat}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \cong \mathsf{Cat}(\mathcal{A}, \mathcal{C}^{\mathcal{B}}) \ . \tag{1.4}$$

The isomorphism takes a functor $F : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ to the functor $\widetilde{F} : \mathcal{A} \to \mathcal{C}^{\mathcal{B}}$ defined on objects $A \in \mathcal{A}$, $B \in \mathcal{B}$ by

$$(\widetilde{F}A)B = F\langle A, B\rangle$$

and on a morphism $f : A \to A'$ by

$$(\widetilde{F}f)_B = F\langle f, 1_B\rangle \ .$$

The functor $\widetilde{F}$ is called the *transpose* of F.

The inverse isomorphism takes a functor $G : \mathcal{A} \to \mathcal{C}^{\mathcal{B}}$ to the functor $\widetilde{G} : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$, defined on objects by

$$\widetilde{G}\langle A, B\rangle = (GA)B$$

and on a morphism $\langle f, g\rangle : A \times B \to A' \times B'$ by

$$\widetilde{G}\langle f, g\rangle = (Gf)_{B'} \circ (GA)g = (GA')g \circ (Gf)_B \ ,$$

where the last equation holds by naturality of Gf:

$$\begin{tikzcd}
(GA)B \arrow[r, "(Gf)_B"] \arrow[d, "(GA)g"'] & (GA')B \arrow[d, "(GA')g"] \\
(GA)B' \arrow[r, "(Gf)_{B'}"'] & (GA')B'
\end{tikzcd}$$

1.4.1 Directed graphs as a functor category

Recall that a *directed graph* G is given by a set of vertices G_V and a set of edges G_E. Each edge $e \in G_E$ has a uniquely determined *source* $\mathsf{src}_G\, e \in G_V$ and *target* $\mathsf{trg}_G\, e \in G_V$. We write $e : a \to b$ when a is the source and b is the target of e. A *graph homomorphism* $\phi : G \to H$ is a pair of functions $\phi_0 : G_V \to H_V$ and $\phi_1 : G_E \to H_E$, where we usually write ϕ for both ϕ_0 and ϕ_1, such that whenever $e : a \to b$ then $\phi_1 e : \phi_0 a \to \phi_0 b$. The category of directed graphs and graph homomorphisms is denoted by Graph.

Now let $\cdot \rightrightarrows \cdot$ be the category with two objects and two parallel morphisms, depicted by the following "sketch":

$$\begin{tikzcd}
E \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] & V
\end{tikzcd}$$

An object of the functor category $\mathsf{Set}^{\cdot \rightrightarrows \cdot}$ is a functor $G : (\cdot \rightrightarrows \cdot) \to \mathsf{Set}$, which consists of two sets GE and GV and two functions $Gs : GE \to GV$ and $Gt : GE \to GV$. But this is precisely a directed graph whose vertices are GV, the edges are GE, the source of $e \in GE$ is $(Gs)e$ and the target is $(Gt)e$. Conversely, any graph G is a functor $G : (\cdot \rightrightarrows \cdot) \to \mathsf{Set}$, defined by

$$GE = G_E \ , \qquad GV = G_V \ , \qquad Gs = \mathsf{src}_G \ , \qquad Gt = \mathsf{trg}_G \ .$$

If category theory is worth anything, it *should* be the case that the morphisms in $\mathsf{Set}^{\cdot \rightrightarrows \cdot}$ are precisely the graph homomorphisms. Indeed, a natural transformation $\phi : G \Longrightarrow H$ between graphs is a pair of functions,

$$\phi_E : G_E \to H_E \qquad \text{and} \qquad \phi_V : G_V \to H_V$$

whose naturality is expressed by the commutativity of the following two diagrams:

$$\begin{tikzcd} G_E \arrow[r, "\phi_E"] \arrow[d, "\mathsf{src}_G"'] & H_E \arrow[d, "\mathsf{src}_H"] \\ G_V \arrow[r, "\phi_V"'] & H_V \end{tikzcd} \qquad \begin{tikzcd} G_E \arrow[r, "\phi_E"] \arrow[d, "\mathsf{trg}_G"'] & H_E \arrow[d, "\mathsf{trg}_H"] \\ G_V \arrow[r, "\phi_V"'] & H_V \end{tikzcd}$$

This is precisely the requirement that $e : a \to b$ implies $\phi_E e : \phi_V a \to \phi_V b$.

1.4.2 The Yoneda Embedding

The example $\mathsf{Graph} = \mathsf{Set}^{\cdot \rightrightarrows \cdot}$ leads one to wonder which categories $\mathcal{C}$ can be represented as functor categories $\mathsf{Set}^{\mathcal{D}}$ for a suitably chosen $\mathcal{D}$ or, when that is not possible, at least as full subcategories of $\mathsf{Set}^{\mathcal{D}}$.

For a locally small category $\mathcal{C}$, there is the hom-set functor

$$\mathcal{C}(-,-) : \mathcal{C}^{\mathsf{op}} \times \mathcal{C} \to \mathsf{Set} \ .$$

By transposing it we obtain the functor

$$\mathsf{y} : \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$$

which maps an object $A \in \mathcal{C}$ to the functor

$$\mathsf{y}A = \mathcal{C}(-, A) : B \mapsto \mathcal{C}(B, A)$$

and a morphism $f : A \to A'$ in $\mathcal{C}$ to the natural transformation $\mathsf{y}f : \mathsf{y}A \Longrightarrow \mathsf{y}A'$ whose component at B is

$$(\mathsf{y}f)_B = \mathcal{C}(B, f) : g \mapsto f \circ g \ .$$

This functor is called the *Yoneda embedding.*

Theorem 1.4.3 (Yoneda embedding) *For any locally small category $\mathcal{C}$ the Yoneda embedding $\mathsf{y} : \mathcal{C} \to \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$ is full and faithful, and injective on objects. Therefore, $\mathcal{C}$ is a full subcategory of $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$.*

The proof of the theorem uses Yoneda Lemma.

Lemma 1.4.4 (Yoneda) *Every functor* $F : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ *is naturally isomorphic to the functor* $\mathsf{Nat}(\mathsf{y}-, F)$. *That is, for every* $A \in \mathcal{C}$,

$$\mathsf{Nat}(\mathsf{y}A, F) \cong FA ,$$

and this isomorphism is natural in A.

Proof. The desired natural isomorphism θ_A maps a natural transformation $\eta \in \mathsf{Nat}(\mathsf{y}A, F)$ to $\eta_A 1_A$. The inverse ${\theta_A}^{-1}$ maps an element $x \in FA$ to the natural transformation $({\theta_A}^{-1}x)$ whose component at B maps $f \in \mathcal{C}(B, A)$ to $(Ff)x$. To summarize, for $\eta : \mathcal{C}(-, A) \Longrightarrow F$, $x \in FA$ and $f \in \mathcal{C}(B, A)$, we have

$$\begin{aligned} &\theta_A : \mathsf{Nat}(\mathsf{y}A, F) \to FA , && {\theta_A}^{-1} : FA \to \mathsf{Nat}(\mathsf{y}A, F) , \\ &\theta_A \eta = \eta_A 1_A , && ({\theta_A}^{-1}x)_B f = (Ff)x . \end{aligned}$$

To see that θ_A and ${\theta_A}^{-1}$ really are inverses of each other, observe that

$$\theta_A({\theta_A}^{-1}x) = ({\theta_A}^{-1}x)_A 1_A = (F1_A)x = 1_{FA}x = x ,$$

and also

$$({\theta_A}^{-1}(\theta_A\eta))_B f = (Ff)(\theta_A\eta) = (Ff)(\eta_A 1_A) = \eta_B(1_A \circ f) = \eta_B f ,$$

where the third equality holds by the following naturality square for η:

$$\begin{tikzcd} \mathcal{C}(A, A) \arrow[r, "\eta_A"] \arrow[d, "\mathcal{C}(f, A)"'] & FA \arrow[d, "Ff"] \\ \mathcal{C}(B, A) \arrow[r, "\eta_B"'] & FB \end{tikzcd}$$

It remains to check that θ is natural, which amounts to establishing the commutativity of the following diagram, with $g : A \to A'$:

$$\begin{tikzcd} \mathsf{Nat}(\mathsf{y}A, F) \arrow[r, "\theta_A"] & FA \\ \mathsf{Nat}(\mathsf{y}A', F) \arrow[r, "\theta_{A'}"'] \arrow[u, "\mathsf{Nat}(\mathsf{y}g, F)"] & FA' \arrow[u, "Fg"'] \end{tikzcd}$$

The diagram is commutative because, for any $\eta : \mathsf{y}A' \Longrightarrow F$,

$$\begin{aligned} (Fg)(\theta_{A'}\eta) = (Fg)(\eta_{A'} 1_{A'}) = \eta_A(1_{A'} \circ g) = \\ \eta_A(g \circ 1_A) = (\mathsf{Nat}(\mathsf{y}g, F)\eta)_A 1_A = \theta_A(\mathsf{Nat}(\mathsf{y}g, F)\eta) , \end{aligned}$$

where the second equality is justified by naturality of η. ■

Proof. [Proof of Theorem 1.4.3] That the Yoneda embedding is full and faithful means that for all $A, B \in \mathcal{C}$ the map

$$\mathsf{y} : \mathcal{C}(A, B) \to \mathsf{Nat}(\mathsf{y}A, \mathsf{y}B)$$

which maps $f : A \to B$ to $\mathsf{y}f : \mathsf{y}A \Longrightarrow \mathsf{y}B$ is an isomorphism. But this is just Yoneda Lemma applied to the case $F = \mathsf{y}B$. Indeed, with notation as in the proof of Yoneda Lemma and $g : C \to A$, we see that the isomorphism

$$\theta_A^{-1} : \mathcal{C}(A, B) = (\mathsf{y}B)A \to \mathsf{Nat}(\mathsf{y}A, \mathsf{y}B)$$

is in fact y:

$$({\theta_A}^{-1} f)_C g = ((\mathsf{y}A)g)f = f \circ g = (\mathsf{y}f)_C g \ .$$

Furthermore, if $\mathsf{y}A = \mathsf{y}B$ then $1_A \in \mathcal{C}(A, A) = (\mathsf{y}A)A = (\mathsf{y}B)A = \mathcal{C}(B, A)$ which can only happen if $A = B$. Therefore, y is injective on objects. ■

The following corollary is often useful.

Corollary 1.4.5 *For $A, B \in \mathcal{C}$, $A \cong B$ if, and only if, $\mathsf{y}A \cong \mathsf{y}B$ in $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$.*

Proof. Every functor preserves isomorphisms, and a full and faithful one also reflects them. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to *reflect* isomorphisms when $Ff : FA \to FB$ being an isomorphisms implies that $f : A \to B$ is an isomorphism. ■

Exercise 1.4.6 Prove that a full and faithful functor reflects isomorphisms.

Functor categories $\mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$ are important enough to deserve a name. They are called *presheaf categories*, and a functor $F : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ is a *presheaf* on $\mathcal{C}$. We also use the notation $\widehat{\mathcal{C}} = \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$.

1.4.3 Equivalence of Categories

An isomorphism of categories $\mathcal{C}$ and $\mathcal{D}$ in Cat consists of functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

such that $G \circ F = 1_{\mathcal{C}}$ and $F \circ G = 1_{\mathcal{D}}$. This is often too restrictive a notion. A more general notion which replaces the above identities with natural isomorphisms is required.

Definition 1.4.7 An *equivalence of categories* is a pair of functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

such that

$$G \circ F \cong 1_{\mathcal{C}} \qquad \text{and} \qquad F \circ G \cong 1_{\mathcal{D}} .$$

We say that $\mathcal{C}$ and $\mathcal{D}$ are *equivalent categories* and write $\mathcal{C} \simeq \mathcal{D}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is called an *equivalence functor* if there exists $G : \mathcal{D} \to \mathcal{C}$ such that F and G form an equivalence.

The point of equivalence of categories is that it preserves almost all categorical properties, but ignores those concepts that are not at interest from a categorical point of view, such as identity of objects.

The following proposition requires the Axiom of Choice as stated in general form. However, in many specific cases a canonical choice can be made without appeal to the Axiom of Choice.

Proposition 1.4.8 *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence functor if, and only if, F is full and faithful, and* essentially surjective *on objects, which means that for every $B \in \mathcal{D}$ there exists $A \in \mathcal{C}$ such that $FA \cong B$.*

Proof. It is easily seen that the conditions are necessary, so we only show they are sufficient. Suppose $F : \mathcal{C} \to \mathcal{D}$ is full and faithful, and essentially surjective on objects. For each $B \in \mathcal{D}$, *choose* an object $GB \in \mathcal{C}$ and an isomorphism $\eta_B : F(GB) \to B$. If $f : B \to C$ is a morphism in $\mathcal{D}$, let $Gf : GB \to GC$ be the unique morphism in $\mathcal{C}$ for which

$$F(Gf) = {\eta_C}^{-1} \circ f \circ \eta_B . \tag{1.5}$$

Such a unique morphism exists because F is full and faithful. This defines a functor $G : \mathcal{D} \to \mathcal{C}$, as can be easily checked. In addition, (1.5) ensures that η is a natural isomorphism $F \circ G \Longrightarrow 1_{\mathcal{D}}$.

It remains to show that $G \circ F \cong 1_{\mathcal{C}}$. For $A \in \mathcal{C}$, let $\theta_A : G(FA) \to A$ be the unique morphism such that $F\theta_A = \eta_{FA}$. Naturality of θ_A follows from functoriality of F and naturality of η. Because F reflects isomorphisms, θ_A is an isomorphism for every A. ∎

Example 1.4.9 As an example of equivalence of categories we consider the category of sets and partial functions and the category of pointed sets.

A *partial function* $f : A \rightharpoonup B$ is a function defined on a subset $\mathsf{supp}\, f \subseteq A$, called the *support*[3] of f, and taking values in B. Composition of partial functions $f : A \rightharpoonup B$ and $g : B \rightharpoonup C$ is the partial function $g \circ f : A \rightharpoonup C$ defined by

$$\begin{aligned} \mathsf{supp}\,(g \circ f) &= \left\{ x \in A \mid x \in \mathsf{supp}\, f \wedge fx \in \mathsf{supp}\, g \right\} \\ (g \circ f)x &= g(fx) \quad \text{for } x \in \mathsf{supp}\,(g \circ f) \end{aligned}$$

[3] The support of a partial function $f : A \rightharpoonup B$ is usually called its *domain*, but this terminology conflicts with A being the domain of f as a morphism.

Composition of partial functions is associative. This way we obtain a category Par of sets and partial functions.

A *pointed set* (A, a) is a set A together with an element $a \in A$. A *pointed function* $f : (A, a) \to (B, b)$ between pointed sets is a function $f : A \to B$ such that $fa = b$. The category $\mathsf{Set}_\bullet$ consists of pointed sets and pointed functions.

The categories Par and $\mathsf{Set}_\bullet$ are equivalent. The equivalence functor $F : \mathsf{Set}_\bullet \to \mathsf{Par}$ maps a pointed set (A, a) to the set $F(A, a) = A \setminus \{a\}$, and a pointed function $f : (A, a) \to (B, b)$ to the partial function $Ff : F(A, a) \rightharpoonup F(B, b)$ defined by

$$\mathsf{supp}\,(Ff) = \left\{x \in A \;\middle|\; fx \neq b\right\} , \qquad\qquad (Ff)x = fx .$$

The inverse equivalence functor $G : \mathsf{Par} \to \mathsf{Set}_\bullet$ maps a set $A \in \mathsf{Par}$ to the pointed set $GA = (A + \{\bot_A\}, \bot_A)$, where $\bot_A$ is an element that does not belong to A. A partial function $f : A \rightharpoonup B$ is mapped to the pointed function $Gf : GA \to GB$ defined by

$$(Gf)x = \begin{cases} fx & \text{if } x \in \mathsf{supp}\, f \\ \bot_B & \text{otherwise} . \end{cases}$$

A good way to think about the "bottom" point $\bot_A$ as a special "undefined value". Let us look at the composition of F and G on objects:

$$\begin{aligned} G(F(A, a)) = G(A \setminus \{a\}) = ((A \setminus \{a\}) + \bot_A, \bot_A) \cong (A, a) . \\ F(GA) = F(A + \{\bot_A\}, \bot_A) = (A + \{\bot_A\}) \setminus \{\bot_A\} = A . \end{aligned}$$

The isomorphism $G(F(A, a)) \cong (A, a)$ is easily seen to be natural.

Example 1.4.10 Another example of an equivalence of categories arises when we take the poset reflection of a preorder. Let $(P, \leq)$ be a preorder, If we think of P as a category, then $a, b \in P$ are isomorphic, when $a \leq b$ and $b \leq a$. Isomorphism $\cong$ is an equivalence relation, therefore we may form the quotient set $P/{\cong}$. The set $P/{\cong}$ is a poset for the order relation $\sqsubseteq$ defined by

$$[a] \sqsubseteq [b] \iff a \leq b .$$

Here $[a]$ denotes the equivalence class of a. We call $(P/{\cong}, \sqsubseteq)$ the *poset reflection* of P. The quotient map $q : P \to P/{\cong}$ is a functor when P and $P/{\cong}$ are viewed as categories. By Proposition 1.4.8, q is an equivalence functor. Trivially, it is faithful and surjective on objects. It is also full because $a \leq b$ in P implies $qa \sqsubseteq qb$ in $P/{\cong}$.

1.5 Adjoint Functors

The notion of adjunction is arguably the most important concept unveiled by category theory. It is a general logical and mathematical concept that occurs everywhere and often marks an important and interesting connection between two objects of interest. In logic, adjoint functors are pervasive, although this is only recognizable from the category-theoretic approach to logic.

1.5.1 Adjoint maps between preorders

Let us begin with a simple situation. We have already seen that a preorder $(P, \leq)$ is a category in which there is at most one morphism between any two objects. A functor between preorders is a monotone map. Suppose we have preorders P and Q with two monotone maps between them,

$$\begin{tikzcd} P \arrow[r, bend left, "f"] & Q \arrow[l, bend left, "g"] \end{tikzcd}$$

We say that f and g are *adjoint*, and write $f \dashv g$, when for all $x \in P$, $y \in Q$,

$$fx \leq y \iff x \leq gy \; . \tag{1.6}$$

Note that adjointness is *not* a symmetric relation. The map f is the *left adjoint* and g is the *right adjoint*.[4]

Equivalence (1.6) is more conveniently displayed as

$$\frac{fx \leq y}{\overline{\overline{\quad x \leq gy \quad}}}$$

The double line indicates the fact that this is a two-way rule: the top line implies the bottom line, and vice versa.

Let us consider two examples.

Conjunction is adjoint to implication

Consider a propositional calculus whose only logical operations are conjunction $\wedge$ and implication $\Rightarrow$.[5] The formulas of this calculus are built from variables x_0, x_1, x_2, …, the truth values $\bot$ and $\top$, and the logical connectives $\wedge$ and $\Rightarrow$. The logical rules are given in natural deduction style:

$$\frac{}{\top} \qquad \frac{\bot}{A} \qquad \frac{A \qquad B}{A \wedge B} \qquad \frac{A \wedge B}{A} \qquad \frac{A \wedge B}{B}$$

$$\frac{A \Rightarrow B \qquad A}{B} \qquad \begin{array}{c} [u : A] \\ \vdots \\ \dfrac{B}{A \Rightarrow B}\, u \end{array}$$

For example, we read the last two inference rules as "from $A \Rightarrow B$ and A we infer B" and "if from assumption A we infer B, then (without any assumptions)

[4] Remember it like this: the left adjoint stands on the *left* side of $\leq$, the right adjoint stands on the *right* side of $\leq$.

[5] Nothing changes if we consider a calculus with more connectives.

we infer $A \Rightarrow B$", respectively. We indicate assumptions by enclosing them in brackets. The symbol u in $[u : A]$ is a label for the assumption. When an assumption is *discharged* its label is written to the right of the inference rule that discharges it, as above.

Logical entailment $\vdash$ between formulas of the propositional calculus is the relation $A \vdash B$ which holds if, and only if, from assuming A we can prove B (by using only the inference rules of the calculus). It is trivially the case that $A \vdash A$, and also

$$\text{if } A \vdash B \text{ and } B \vdash C \text{ then } A \vdash C\ .$$

In other words, $\vdash$ is a reflexive and transitive relation on the set P of all propositional formulas so that $(\mathsf{P}, \vdash)$ is a preorder.

Let A be a propositional formula. Define $f : \mathsf{P} \to \mathsf{P}$ and $g : \mathsf{P} \to \mathsf{P}$ to be the maps

$$fB = (A \wedge B)\ , \qquad\qquad gB = (A \Rightarrow B)\ .$$

The maps f and g are functors because they respect entailment. Indeed, if $B \vdash B'$ then $A \wedge B \vdash A \wedge B'$ and $A \Rightarrow B \vdash A \Rightarrow B'$ by the following two derivations:

$$\dfrac{\dfrac{[A \wedge B]}{A} \qquad \begin{array}{c}\dfrac{[A \wedge B]}{B} \\ \vdots \\ B'\end{array}}{A \wedge B'} \qquad\qquad \dfrac{\begin{array}{c}\dfrac{[A \Rightarrow B] \qquad [u : A]}{B} \\ \vdots \\ B'\end{array}}{A \Rightarrow B'}\, u$$

We claim that $f \dashv g$. For this we need to prove that $A \wedge B \vdash C$ if, and only if, $B \vdash A \Rightarrow C$. The following two derivations establish the equivalence:

$$\dfrac{\begin{array}{c}\dfrac{[u : A] \qquad [B]}{A \wedge B} \\ \vdots \\ C\end{array}}{A \Rightarrow C}\, u \qquad\qquad \dfrac{\begin{array}{c}\dfrac{[A \wedge B]}{B} \\ \vdots \\ A \Rightarrow C\end{array} \qquad \dfrac{[A \wedge B]}{A}}{C}$$

Therefore, *conjunction is left adjoint to implication.*

Topological interior as an adjoint

Recall that a *topological space* $(X, \mathcal{O}X)$ is a set X together with a family $\mathcal{O}X \subseteq \mathcal{P}X$ of subsets of X which contains $\emptyset$ and X, and is closed under finite intersections and arbitrary unions. The elements of $\mathcal{O}X$ are called the *open sets.*

The *topological interior* of a subset $S \subseteq X$ is the largest open set contained in S:

$$\mathsf{int}\, S = \bigcup \left\{ U \in \mathcal{O}X \;\middle|\; U \subseteq S \right\}\ .$$

Both $\mathcal{O}X$ and $\mathcal{P}X$ are posets ordered by subset inclusion. The inclusion $i : \mathcal{O}X \to \mathcal{P}X$ is a monotone map, and so is the interior $\mathsf{int} : \mathcal{P}X \to \mathcal{O}X$:

$$\begin{tikzcd} \mathcal{O}X \arrow[r, bend left, "i"] & \mathcal{P}X \arrow[l, bend left, "\mathsf{int}"] \end{tikzcd}$$

For $U \in \mathcal{O}X$ and $S \in \mathcal{P}X$ we have

$$\frac{iU \subseteq S}{U \subseteq \mathsf{int}\, S}$$

Therefore, *topological interior is a right adjoint* to the inclusion of $\mathcal{O}X$ into $\mathcal{P}X$.

1.5.2 Adjoint Functors

Let us now generalize the notion of adjoint monotone maps to the general situation

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

with arbitrary categories and functors. For monotone maps $f \dashv g$, the adjunction is a bijection

$$\frac{fx \to y}{x \to gy}$$

between morphisms of the form $fx \to y$ and morphisms of the form $x \to gy$. This is the notion that generalizes the special case; for any $A \in \mathcal{C}$, $B \in \mathcal{D}$ we require a bijection between $\mathcal{D}(FA, B)$ and $\mathcal{C}(A, GB)$:

$$\frac{FA \to B}{A \to GB}$$

Definition 1.5.1 An *adjunction* $F \dashv G$ between functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

is a natural isomorphism θ between functors

$$\mathcal{D}(F-, -) : \mathcal{C}^{\mathsf{op}} \times \mathcal{D} \to \mathsf{Set} \qquad \text{and} \qquad \mathcal{C}(-, G-) : \mathcal{C}^{\mathsf{op}} \times \mathcal{D} \to \mathsf{Set}\ .$$

This means that for every $A \in \mathcal{C}$ and $B \in \mathcal{D}$ there is a bijection

$$\theta_{A,B} : \mathcal{D}(FA, B) \to \mathcal{C}(A, GB)\ ,$$

and naturality of θ means that for $f : A' \to A$ in $\mathcal{C}$ and $g : B \to B'$ in $\mathcal{D}$ the following diagram commutes:

$$\begin{tikzcd}
\mathcal{D}(FA, B) \arrow[r, "\theta_{A,B}"] \arrow[d, "\mathcal{D}(Ff,g)"'] & \mathcal{D}(A, GB) \arrow[d, "\mathcal{C}(f,Gg)"] \\
\mathcal{D}(FA', B') \arrow[r, "\theta_{A',B'}"'] & \mathcal{C}(A', GB')
\end{tikzcd}$$

Equivalently, for every $h : FA \to B$ in $\mathcal{D}$,

$$Gg \circ (\theta_{A,B}h) \circ f = \theta_{A',B'}(g \circ h \circ Ff) \ .$$

We say that F is a *left adjoint* and G is a *right adjoint*.

We have already seen examples of adjoint functors. For any category $\mathcal{B}$ we have functors $- \times \mathcal{B}$ and $-^{\mathcal{B}}$ from Cat to Cat. Recall the isomorphism (1.4),

$$\mathsf{Cat}(\mathcal{A} \times \mathcal{B}, \mathcal{C}) \cong \mathsf{Cat}(\mathcal{A}, \mathcal{C}^{\mathcal{B}}) \ .$$

This isomorphism is in fact natural so that

$$- \times \mathcal{B} \dashv -^{\mathcal{B}} \ .$$

Similarly, for any set $B \in \mathsf{Set}$ there are functors

$$- \times B : \mathsf{Set} \to \mathsf{Set} \ , \qquad\qquad -^{B} : \mathsf{Set} \to \mathsf{Set} \ ,$$

where $A \times B$ is the cartesian product of A and B, and C^B is the set of all functions from B to C. For morphisms, $f \times B = f \times 1_B$ and $f^B = f \circ -$. Then we have, for all $A, C \in \mathsf{Set}$, a natural isomorphism

$$\mathsf{Set}(A \times B, C) \cong \mathsf{Set}(A, C^B) \ ,$$

which maps a function $f : A \times B \to C$ to the function $(\widetilde{f}x)y = f\langle x, y\rangle$. Therefore, $- \times B \dashv -^B$.

Exercise 1.5.2 Verify that the definition (1.6) of adjoint monotone maps between preorders is a special case of Definition 1.5.1.

For another example, consider the forgetful functor

$$U : \mathsf{Cat} \to \mathsf{Graph} \ ,$$

which maps a category to the underlying directed graph. It has a left adjoint $P \dashv U$. The functor P is the *free* construction of a category from a graph; it maps a graph G to the *category of paths* $P(G)$. The objects of $P(G)$ are the vertices of G. The morphisms of $P(G)$ are finite paths

$$v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_2} \cdots \xrightarrow{e_n} v_{n+1}$$

of edges in G, composition is concatenation of paths, and the identity morphism on a vertex v is the empty path starting and ending at v.

By using Yoneda Lemma we can easily prove that adjoints are unique up to natural isomorphism.

Proposition 1.5.3 *Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. If $F \dashv G$, $F \dashv G'$ and $F' \dashv G$ then $F \cong F'$ and $G \cong G'$.*

Proof. Suppose $F \dashv G$ and $F \dashv G'$. By Yoneda Embedding, $GB \cong G'B$ if, and only if, $\mathcal{C}(-, GB) \cong \mathcal{C}(-, G'B)$, which holds because, for any $A \in \mathcal{C}$,

$$\mathcal{C}(A, GB) \cong \mathcal{D}(FA, B) \cong \mathcal{C}(A, G'B) \ .$$

Therefore, $G \cong G'$. That $F \dashv G$ and $F' \dashv G$ implies $F \cong F'$ is proved similarly, except that the Yoneda Embedding must be replaced by its covariant version.
■

1.5.3 The Unit of an Adjunction

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be adjoint functors, $F \dashv G$, and let $\theta : \mathcal{D}(F-, -) \to \mathcal{C}(-, G-)$ be the natural isomorphism witnessing the adjunction. For any object $A \in \mathcal{C}$ there is a distinguished morphism $\eta_A = \theta_{A,FA} 1_{FA} : A \to G(FA)$,

$$\frac{1_{FA} : FA \to FA}{\eta_A : A \to G(FA)}$$

The transformation $\eta : 1_{\mathcal{C}} \Longrightarrow G \circ F$ is natural. It is called the *unit of the adjunction* $F \dashv G$. In fact, we can recover θ from η as follows, for $f : FA \to B$:

$$\theta_{A,B} f = \theta_{A,B}(f \circ 1_{FA}) = Gf \circ \theta_{A,FA}(1_{FA}) = Gf \circ \eta_A \ ,$$

where we used naturality of θ in the second step. Schematically, given any $f : FA \to B$, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\theta_{A,B} f"'] & G(FA) \arrow[d, "Gf"] \\ & GB \end{tikzcd}$$

Since $\theta_{A,B}$ is a bijection, it follows that *every* morphism $g : A \to GB$ has the form $g = Gf \circ \eta_A$ for a *unique* $f : FA \to B$. We say that $\eta_A : A \to G(FA)$

is a *universal* morphism to G, or that η has the following *universal mapping property*: for every $A \in \mathcal{C}$, $B \in \mathcal{D}$, and $g : A \to GB$, there exists a *unique* $f : FA \to B$ such that $g = Gf \circ \eta_A$:

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "g"'] & G(FA) \arrow[d, "Gf"] & FA \arrow[d, dotted, "f"] \\ & GB & B \end{tikzcd}$$

This means that an adjunction can be given in terms of its unit. The isomorphism $\theta : \mathcal{D}(F-,-) \to \mathcal{C}(-,G-)$ is then recovered by

$$\theta_{A,B} f = Gf \circ \eta_A \ .$$

Proposition 1.5.4 *A functor $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$ if, and only if, there exists a natural transformation*

$$\eta : 1_{\mathcal{C}} \Longrightarrow G \circ F \ ,$$

called the unit *of the adjunction, such that, for all $A \in \mathcal{C}$ and $B \in \mathcal{D}$ the map $\theta_{A,B} : \mathcal{D}(FA, B) \to \mathcal{C}(A, GB)$, defined by*

$$\theta_{A,B} f = Gf \circ \eta_A \ ,$$

is an isomorphism.

Let us demonstrate how the universal mapping property of the unit of an adjunction appears as a well known construction in algebra. Consider the forgetful functor from monoids to sets,

$$U : \mathsf{Mon} \to \mathsf{Set} \ .$$

Does it have a left adjoint $F : \mathsf{Set} \to \mathsf{Mon}$? In order to obtain one, we need a "most economical" way of making a monoid FX from a given set X. Such a construction readily suggests itself, namely the *free monoid* on X, consisting of finite sequences of elements of X,

$$FX = \{x_1 \dots x_n \mid n \geq 0 \wedge x_1, \dots, x_n \in X\} \ .$$

The monoid operation is concatenation of sequences

$$x_1 \dots x_m \cdot y_1 \dots y_n = x_1 \dots x_m y_1 \dots y_n \ ,$$

and the empty sequence is the unit of the monoid. In order for F to be a functor, it should also map morphisms to morphisms. If $f : X \to Y$ is a function, define $Ff : FX \to FY$ by

$$Ff : x_1 \dots x_n \mapsto (fx_1) \dots (fx_n) \ .$$

There is an inclusion $\eta_X : X \to U(FX)$ which maps every element $x \in X$ to the singleton sequence x. This gives a natural transformation $\eta : 1_{\mathsf{Set}} \Longrightarrow U \circ F$.

The free monoid FX is "free" in the sense that for every every monoid M and a function $f : X \to UM$ there exists a unique homomorphism $\overline{f} : FX \to M$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & U(FX) \arrow[d, "U\overline{f}"] \\
& UM
\end{tikzcd}$$

This is precisely the condition required by Proposition 1.5.4 for η to be the unit of the adjunction $F \dashv U$. In this case, the universal mapping property of η is just the usual characterization of free monoid FX generated by the set X: a homomorphism from FX is uniquely determined by its values on the generators.

1.5.4 The Counit of an Adjunction

Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be adjoint functors, and let $\theta : \mathcal{D}(F-,-) \to \mathcal{C}(-,G-)$ be the natural isomorphism witnessing the adjunction. For any object $B \in \mathcal{D}$ there is a distinguished morphism $\varepsilon_B = \theta^{-1}_{GB,B} 1_{GB} : F(GB) \to B$,

$$\begin{array}{c} 1_{GB} : GB \to GB \\ \hline\hline \varepsilon_B : F(GB) \to B \end{array}$$

The transformation $\varepsilon : F \circ G \Longrightarrow 1_{\mathcal{D}}$ is natural and is called the *counit* of the adjunction $F \dashv G$. It is the dual notion to the unit of an adjunction. We state briefly the basic properties of counit, which are easily obtained by "turning around" all morphisms in the previous section and exchanging the roles of the left and right adjoints.

The bijection $\theta^{-1}_{A,B}$ can be recovered from the counit. For $g : A \to GB$ in $\mathcal{C}$, we have

$$\theta^{-1}_{A,B} g = \theta^{-1}_{A,B}(1_{GB} \circ g) = \theta^{-1}_{A,B} 1_{GB} \circ Fg = \varepsilon_B \circ Fg \ .$$

The universal mapping property of the counit is this: for every $A \in \mathcal{C}$, $B \in \mathcal{D}$, and $f : FA \to B$, there exists a *unique* $g : A \to GB$ such that $f = \varepsilon_B \circ Fg$:

$$\begin{tikzcd}
B & F(GB) \arrow[l, "\varepsilon_B"'] & & GB \\
& FA \arrow[ul, "f"] \arrow[u, "Fg"'] & & A \arrow[u, dotted, "g"']
\end{tikzcd}$$

The following is the dual of Proposition 1.5.4.

Proposition 1.5.5 *A functor $F : \mathcal{C} \to \mathcal{D}$ is left adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$ if, and only if, there exists a natural transformation*

$$\varepsilon : F \circ G \Longrightarrow 1_{\mathcal{D}} \ ,$$

called the counit *of the adjunction, such that, for all $A \in \mathcal{C}$ and $B \in \mathcal{D}$ the map $\theta^{-1}_{A,B} : \mathcal{C}(A, GB) \to \mathcal{D}(FA, B)$, defined by*

$$\theta^{-1}_{A,B} g = \varepsilon_B \circ Fg \ ,$$

is an isomorphism.

Let us consider again the forgetful functor $U : \mathsf{Mon} \to \mathsf{Set}$ and its left adjoint $F : \mathsf{Set} \to \mathsf{Mon}$, the free monoid construction. For a monoid $(M, \star) \in \mathsf{Mon}$, the counit of the adjunction $F \dashv U$ is a monoid homomorphism $\varepsilon_M : F(UM) \to M$, defined by

$$\varepsilon_M(x_1 x_2 \dots x_n) = x_1 \star x_2 \star \cdots \star x_n \ .$$

It has the following universal mapping property: for $X \in \mathsf{Set}$, $(M, \star) \in \mathsf{Mon}$, and a homomorphism $f : FX \to M$ there exists a unique function $\overline{f} : X \to UM$ such that $f = \varepsilon_M \circ F\overline{f}$, namely

$$\overline{f}x = fx \ ,$$

where in the above definition $x \in X$ is viewed as an element of the set X on the left-hand side, and as an element of the free monoid FX on the right-hand side. To summarize, the universal mapping property of the counit ε is the familiar piece of wisdom that a homomorphism $f : FX \to M$ from a free monoid is already determined by its values on the generators.

1.6 Limits and Colimits

1.6.1 Binary products

In a category $\mathcal{C}$, the *(binary) product* of objects A and B is an object $A \times B$ together with *projections* $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$ such that, for every object $C \in \mathcal{C}$ and all morphisms $f : C \to A$, $g : C \to B$ there exists a *unique* morphism $h : C \to A \times B$ for which the following diagram commutes:

$$\begin{tikzcd}
& C \arrow[dl, "f"'] \arrow[d, dotted, "h"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & B
\end{tikzcd}$$

We normally refer to the product $(A \times B, \pi_0, \pi_1)$ just by its object $A \times B$, but you should keep in mind that a product is given by an object *and* two projections.

The arrow $h : C \to A \times B$ is denoted by $\langle f, g\rangle$. The property

$$\forall\, C{:}\mathcal{C}\,.\,\forall\, f{:}C \to A\,.\,\forall\, g{:}C \to B\,.\,\exists!\, h{:}C \to A \times B\,.\\ (\pi_0 \circ h = f \wedge \pi_1 \circ h = g)$$

is the *universal mapping property* of the product $A \times B$. It characterizes the product of A and B uniquely up to isomorphism in the sense that if $(P, p_0{:}P \to A, p_1{:}P \to B)$ is another product of A and B then there exists a unique isomorphism $r : P \xrightarrow{\sim} A \times B$ such that $p_0 = \pi_0 \circ r$ and $p_1 = \pi_1 \circ r$.

If in a category $\mathcal{C}$ every two objects have a product, we can turn binary products into an operation[6] by *choosing* a product $A \times B$ for each pair of objects $A, B \in \mathcal{C}$. In general this requires the Axiom of Choice, but in many specific cases a particular choice of products can be made without appeal to the axiom of choice. When we view binary products as an operation, we say that "$\mathcal{C}$ has chosen products". The same holds for other specific and general instances of limits and colimits.

For example, in Set the usual cartesian product of sets is a product. In categories of structures, products are the usual construction: the product of topological spaces in Top is their topological product, the product of directed graphs in Graph is their cartesian product, the product of categories in Cat is their product category, and so on.

1.6.2 Terminal object

A *terminal object* in a category $\mathcal{C}$ is an object $1 \in \mathcal{C}$ such that for every $A \in \mathcal{C}$ there exists a *unique* morphism $!_A : A \to 1$.

For example, in Set an object is terminal if, and only if, it is a singleton. The terminal object in Cat is the unit category $\mathsf{1}$ consisting of one object and one morphism.

Exercise 1.6.1 Prove that if 1 and $1'$ are terminal objects in a category then they are isomorphic.

Exercise 1.6.2 Let Field be the category whose objects are fields and morphisms are field homomorphisms.[7] Does Field have a terminal object?

1.6.3 Equalizers

Given objects and morphisms

$$E \xrightarrow{\;e\;} A \underset{g}{\overset{f}{\rightrightarrows}} B$$

[6] More precisely, binary product is a functor from $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$, cf. Section 1.6.11.

[7] A field $(F, +, \cdot, ^{-1}, 0, 1)$ is a ring with a unit in which all non-zero elements have inverses. We also require that $0 \neq 1$. A homomorphism of fields preserves addition and multiplication, and consequently also 0, 1 and inverses.

we say that e *equalizes* f and g when $f \circ e = g \circ e$.[8] An *equalizer* of f and g is a *universal* equalizing morphism; thus $e : E \to A$ is an equalizer of f and g when it equalizes them and, for all $k : K \to A$, if $f \circ k = g \circ k$ then there exists a unique morphism $m : K \to E$ such that $k = e \circ m$:

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
K \arrow[u, dotted, "m"] \arrow[ur, "k"'] & &
\end{tikzcd}$$

In Set the equalizer of parallel functions $f : A \to B$ and $g : A \to B$ is the set

$$E = \{x \in A \mid fx = gx\}$$

with $e : E \to A$ being the subset inclusion $E \subseteq A$, $ex = x$. In general, equalizers can be thought of as those subobjects (subsets, subgroups, subspaces, ...) that can be defined by a single equation.

Exercise 1.6.3 Show that an equalizer is a monomorphism, i.e., if $e : E \to A$ is an equalizer of f and g, then, for all $r, s : C \to E$, $e \circ r = e \circ s$ implies $r = s$.

Definition 1.6.4 A morphism is a *regular mono* if it is an equalizer.

The difference between monos and regular monos is best illustrated in the category Top: a continuous map $f : X \to Y$ is mono when it is injective, whereas it is a regular mono when it is a topological embedding.[9]

1.6.4 Pullbacks

A *pullback* of $f : A \to C$ and $g : B \to C$ is an object P with morphisms $p_0 : P \to A$ and $p_1 : P \to B$ such that $f \circ p_0 = g \circ p_1$, and whenever $r_0 : R \to A$, $r_1 : R \to B$ are such that $f \circ r_0 = g \circ r_1$, then there exists a unique $h : R \to P$

[8] Note that this does *not* mean the diagram involving f, g and e is commutative!

[9] A continuous map $f : X \to Y$ is a topological embedding when, for every $U \in \mathcal{O}X$, the image $f[U]$ is an open subset of the image $\mathsf{im}(f)$; this means that there exists $V \in \mathcal{O}Y$ such that $f[U] = V \cap \mathsf{im}(f)$.

such that $r_0 = p_0 \circ h$ and $r_1 = p_1 \circ h$:

$$\begin{tikzcd}
R \arrow[drr, bend right, "r_0"'] \arrow[dr, dotted, "h"] \arrow[rrd, phantom] \arrow[drr, phantom] & & \\
& P \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

$$\begin{tikzcd}
R \arrow[ddr, bend right, "r_0"'] \arrow[dr, dotted, "h"] \arrow[drr, bend left, "r_1"] & & \\
& P \arrow[r, "p_1"] \arrow[d, "p_0"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

We indicate that P is a pullback by drawing a square corner next to it, as in the above diagram. Sometimes we denote the pullback P by $A \times_C B$.

In Set, the pullback of $f : A \to C$ and $g : B \to C$ is the set

$$P = \{\langle x, y\rangle \in A \times B \mid fx = gy\}$$

and the functions $p_0 : P \to A$, $p_1 : P \to B$ are the projections, $p_0\langle x, y\rangle = x$, $p_1\langle x, y\rangle = y$.

When we form the pullback of $f : A \to C$ and $g : B \to C$ we also say that we *pull back g along f* and draw the diagram

$$\begin{tikzcd}
f^*B \arrow[r] \arrow[d, "f^*g"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

We think of $f^*g : f^*B \to A$ as the inverse image of B along f. This terminology is explained by looking at the pullback of a subset inclusion $u : U \hookrightarrow C$ along a function $f : A \to C$ in the category Set:

$$\begin{tikzcd}
f^*U \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & U \arrow[d, hook, "u"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}$$

In this case the pullback is $\{\langle x, y\rangle \in A \times U \mid fx = y\} \cong \{x \in A \mid fx \in U\} = f^*U$, the inverse image of U along f.

Exercise 1.6.5 Prove that in a category $\mathcal{C}$, a morphism $f : A \to B$ is mono if, and only if, the following diagram is a pullback:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "1_A"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

1.6.5 Limits

Let us now define a general notion of a limit.

A *diagram of shape* $\mathcal{I}$ in a category $\mathcal{C}$ is a functor $D : \mathcal{I} \to \mathcal{C}$, where the category $\mathcal{I}$ is called the *index category*. We use letters i, j, k, … for objects of an index category $\mathcal{I}$, call them *indices*, and write D_i, D_j, D_k, … instead of Di, Dj, Dk, …

For example, if $\mathcal{I}$ is the category with three objects and three morphisms

$$\begin{tikzcd} & 1 \arrow[dl, "12"'] \arrow[d, "13"] \\ 2 \arrow[r, "23"'] & 3 \end{tikzcd}$$

where $13 = 23 \circ 12$ then a diagram of shape $\mathcal{I}$ is a commutative diagram

$$\begin{tikzcd} & D_1 \arrow[dl, "d_{12}"'] \arrow[d, "d_{13}"] \\ D_2 \arrow[r, "d_{23}"'] & D_3 \end{tikzcd} \tag{1.7}$$

Given an object $A \in \mathcal{C}$, there is a *constant diagram* of shape $\mathcal{I}$, which is the constant functor $\Delta_A : \mathcal{I} \to \mathcal{C}$ that maps every object to A and every morphism to 1_A.

Let $D : \mathcal{I} \to \mathcal{C}$ be a diagram of shape $\mathcal{I}$. A *cone* on D from an object $A \in \mathcal{C}$ is a natural transformation $\alpha : \Delta_A \Longrightarrow D$. This means that for every index $i \in \mathcal{I}$ there is a morphism $\alpha_i : A \to D_i$ such that whenever $u : i \to j$ in $\mathcal{I}$ then $\alpha_j = Du \circ \alpha_i$.

For a given diagram $D : \mathcal{I} \to \mathcal{C}$, we can collect all cones on D into a category $\mathsf{Cone}(D)$ whose objects are cones on D. A morphism between cones $f : (A, \alpha) \to (B, \beta)$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that $\alpha_i = \beta_i \circ f$ for all $i \in \mathcal{I}$. Morphisms in $\mathsf{Cone}(D)$ are composed as morphisms in $\mathcal{C}$. A morphism

$f : (A, \alpha) \to (B, \beta)$ is also called a factorization of the cone (A, α) through the cone (B, β).

A *limit* of a diagram $D : \mathcal{I} \to \mathcal{C}$ is a terminal object in $\mathsf{Cone}(D)$. Explicitly, a limit of D is given by a cone (L, λ) such that for every other cone (A, α) there exists a *unique* morphism $f : A \to L$ such that $\alpha_i = \lambda_i \circ f$ for all $i \in \mathcal{I}$. We denote a limit of D by one of the following:

$$\lim D \qquad\qquad \lim_{i\in\mathcal{I}} D_i \qquad\qquad \varprojlim_{i\in\mathcal{I}} D_i \ .$$

Limits are also called *projective limits*. We say that a category *has limits of shape* $\mathcal{I}$ when every diagram of shape $\mathcal{I}$ in $\mathcal{C}$ has a limit.

Products, terminal objects, equalizers, and pullbacks are all special cases of limits:

- a product $A \times B$ is the limit of the functor $D : \mathsf{2} \to \mathcal{C}$ where $\mathsf{2}$ is the discrete category on two objects 0 and 1, and $D_0 = A$, $D_1 = B$.

- a terminal object $\mathsf{1}$ is the limit of the (unique) functor $D : \mathsf{0} \to \mathcal{C}$ from the empty category.

- an equalizer of $f, g : A \to B$ is the limit of the functor $D : (\cdot \rightrightarrows \cdot) \to \mathcal{C}$ which maps one morphism to f and the other one to g.

- the pullback of $f : A \to C$ and $g : B \to C$ is the limit of the functor $D : \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is the category

$$\begin{tikzcd} & \bullet \arrow[d, "2"] \\ \bullet \arrow[r, "1"'] & \bullet \end{tikzcd}$$

  with $D1 = f$ and $D2 = g$.

It is clear how to define the product of an arbitrary family of objects

$$\{A_i \in \mathcal{C} \mid i \in I\} \ .$$

Such a family is a diagram of shape I, where I is viewed as a discrete category. A *product* $\prod_{i\in I} A_i$ is then given by an object $P \in \mathcal{C}$ and morphisms $\pi_i : P \to A_i$ such that, whenever we have a family of morphisms $\{f_i : B \to A_i \mid i \in I\}$ there exists a *unique* morphism $\langle f_i \rangle_{i\in I} : B \to P$ such that $f_i = \pi_i \circ f$ for all $i \in I$.

A *finite product* is a product of a finite family. As a special case we see that a terminal object is the product of an empty family. It is not hard to show that a category has finite products precisely when it has a terminal object and binary products.

A diagram $D : \mathcal{I} \to \mathcal{C}$ is *small* when $\mathcal{I}$ is a small category. A *small limit* is a limit of a small diagram. A *finite limit* is a limit of a diagram whose index category is finite.

Exercise 1.6.6 Prove that a limit, when it exists, is unique up to isomorphism.

The following proposition and its proof tell us how to compute arbitrary limits from simpler ones. We omit detailed proofs as they can be found in any standard textbook on category theory.

Proposition 1.6.7 *The following are equivalent for a category $\mathcal{C}$:*

1. *$\mathcal{C}$ has all pullbacks and a terminal object.*
2. *$\mathcal{C}$ has finite products and equalizers.*
3. *$\mathcal{C}$ has has finite limits.*

Proof. We only show how to get binary products from pullbacks and a terminal object. For objects A and B, let P be the pullback of $!_A$ and $!_B$:

$$\begin{tikzcd} P \arrow[r, "\pi_1"] \arrow[d, "\pi_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "!_B"] \\ A \arrow[r, "!_A"'] & 1 \end{tikzcd}$$

Then (P, π_0, π_1) is a product of A and B because, for all $f : X \to A$ and $g : X \to B$, it is trivially the case that $!_A \circ f = !_B \circ g$. ∎

Proposition 1.6.8 *The following are equivalent for a category $\mathcal{C}$:*

1. *$\mathcal{C}$ has small products and equalizers.*
2. *$\mathcal{C}$ has small limits.*

Proof. We indicate how to construct an arbitrary limit from a product and an equalizer. Let $D : \mathcal{I} \to \mathcal{C}$ be a small diagram of an arbitrary shape $\mathcal{I}$. First form an $\mathcal{I}_0$-indexed product P and an $\mathcal{I}_1$-indexed product Q

$$P = \prod_{i \in \mathcal{I}_0} D_i \;, \qquad\qquad Q = \prod_{u \in \mathcal{I}_1} D_{\mathsf{cod}\,u} \;.$$

By the universal property of products, there are unique morphisms $f : P \to Q$ and $g : P \to Q$ such that, for all morphisms $u \in \mathcal{I}_1$,

$$\pi^Q_u \circ f = Du \circ \pi^P_{\mathsf{dom}\,u} \;, \qquad\qquad \pi^Q_u \circ g = \pi^P_{\mathsf{cod}\,u} \;.$$

Let E be the equalizer of f and g,

$$\begin{tikzcd} E \arrow[r, "e"] & P \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Q \end{tikzcd}$$

For every $i \in \mathcal{I}$ there is a morphism $\varepsilon_i : E \to D_i$, namely $\varepsilon_i = \pi_i^P \circ e$. We claim that (E, ε) is a limit of D. First, (E, ε) is a cone on D because, for all $u : i \to j$ in $\mathcal{I}$,

$$Du \circ \varepsilon_i = Du \circ \pi_i^P \circ e = \pi_u^Q \circ f \circ e = \pi_u^Q \circ g \circ e = \pi_j^P \circ e = \varepsilon_j \ .$$

If (A, α) is any cone on D there exists a unique $t : A \to P$ such that $\alpha_i = \pi_i^P \circ t$ for all $i \in \mathcal{I}$. For every $u : i \to j$ in $\mathcal{I}$ we have

$$\pi_u^Q \circ g \circ t = \pi_j^P \circ t = t_j = Du \circ t_i = Du \circ \pi_i^P \circ t = \pi_u^Q \circ f \circ t \ ,$$

therefore $g \circ t = f \circ t$. This implies that there is a unique factorization $k : A \to E$ such that $t = e \circ k$. Now for every $i \in \mathcal{I}$

$$\varepsilon_i \circ k = \pi_i^P \circ e \circ k = \pi_i^P \circ t = \alpha_i$$

so that $k : A \to E$ is the required factorization of the cone (A, α) through the cone (E, ε). To see that k is unique, suppose $m : A \to E$ is another factorization such that $\alpha_i = \varepsilon_i \circ m$ for all $i \in \mathcal{I}$. Since e is mono it suffices to show that $e \circ m = e \circ k$, which is equivalent to proving $\pi_i^P \circ e \circ m = \pi_i^P \circ e \circ k$ for all $i \in \mathcal{I}$. This last equality holds because

$$\pi_i^P \circ e \circ k = \pi_i^P \circ t = \alpha_i = \varepsilon_i \circ m = \pi_i^P \circ e \circ m \ .$$

■

A category is *(small) complete* when it has all small limits, and it is *finitely complete* or *lex* when it has finite limits.

Limits of presheaves

Let $\mathcal{C}$ be a locally small category. Then the presheaf category $\widehat{\mathcal{C}} = \mathsf{Set}^{\mathcal{C}^{\mathsf{op}}}$ has all small limits and they are computed pointwise, e.g., $(P \times Q)A = PA \times QA$ for $P, Q \in \widehat{\mathcal{C}}$, $A \in \mathcal{C}$. To see that this is really so, let $\mathcal{I}$ be a small index category and $D : \mathcal{I} \to \widehat{\mathcal{C}}$ a diagram of presheaves. Then for every $A \in \mathcal{C}$ the diagram D can be instantiated at A to give a diagram $DA : \mathcal{I} \to \mathsf{Set}$, $(DA)_i = D_i A$. Because Set is small complete, we can define a presheaf L by computing the limit of DA:

$$LA = \lim DA = \lim_{i \in \mathcal{I}} D_i A \ .$$

We should keep in mind that $\lim DA$ is actually given by an object $(\lim DA)$ and a natural transformation $\delta A : \Delta_{(\lim DA)} \Longrightarrow DA$. The value of LA is supposed to be just the object part of $\lim DA$. From a morphism $f : A \to B$ we obtain for each $i \in \mathcal{I}$ a function $D_i f \circ (\delta A)_i : LA \to D_i B$, and thus a cone $(LA, Df \circ \delta A)$ on DB. Presheaf L maps the morphism $f : A \to B$ to the unique factorization $Lf : LA \Longrightarrow LB$ of the cone $(LA, Df \circ \delta A)$ on DB through the limit cone LB on DB.

For every $i \in \mathcal{I}$, there is a function $\Lambda_i = (\delta A)_i : LA \to D_i A$. The family $\{\Lambda_i\}_{i \in \mathcal{I}}$ is a natural transformation from Δ_{LA} to DA. This gives us a cone

(L, Λ) on D, which is in fact a limit cone. Indeed, if (S, Σ) is another cone on D then for every $A \in \mathcal{C}$ there exists a unique function $\phi_A : SA \to LA$ because SA is a cone on DA and LA is a limit cone on DA. The family $\{\phi_A\}_{A\in\mathcal{C}}$ is the unique natural transformation $\phi : S \Longrightarrow L$ for which $\Sigma = \phi \circ \Lambda$.

1.6.6 Colimits

Colimits are the dual notion of limits. Thus, a *colimit* of a diagram $D : \mathcal{I} \to \mathcal{C}$ is a limit of the dual diagram $D^{\mathsf{op}} : \mathcal{I}^{\mathsf{op}} \to \mathcal{C}^{\mathsf{op}}$ in the dual category $\mathcal{C}^{\mathsf{op}}$:

$$\mathrm{colim}(D : \mathcal{I} \to \mathcal{C}) = \lim(D^{\mathsf{op}} : \mathcal{I}^{\mathsf{op}} \to \mathcal{C}^{\mathsf{op}}) \;.$$

Equivalently, the colimit of a diagram $D : \mathcal{I} \to \mathcal{C}$ is the initial object in the category of *cocones* $\mathsf{Cocone}(D)$ on D. A cocone (A, α) on D is a natural transformation $\alpha : D \Longrightarrow \Delta_A$. It is given by an object $A \in \mathcal{C}$ and, for each $i \in \mathcal{I}$, a morphism $\alpha_i : D_i \to A$, such that $\alpha_i = \alpha_j \circ Du$ whenever $u : i \to j$ in $\mathcal{I}$. A morphism between cocones $f : (A, \alpha) \to (B, \beta)$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that $\beta_i = f \circ \alpha_i$ for all $i \in \mathcal{I}$.

Explicitly, a colimit of $D : \mathcal{I} \to \mathcal{C}$ is given by a cocone (C, ζ) on D such that, for every other cocone (A, α) on D there exists a unique morphism $f : C \to A$ such that $\alpha_i = f \circ \zeta_i$ for all $i \in D$. We denote a colimit of D by one of the following:

$$\mathrm{colim}\, D \qquad\qquad \mathrm{colim}_{i\in\mathcal{I}}\, D_i \qquad\qquad \underrightarrow{\mathrm{colim}}_{i\in\mathcal{I}} D_i \;.$$

Colimits are also called *inductive limits*.

Exercise 1.6.9 Formulate the dual of Proposition 1.6.7 and Proposition 1.6.8 for colimits (coequalizers are defined in Subsection 1.6.9).

1.6.7 Binary Coproducts

In a category $\mathcal{C}$, the *(binary) coproduct* of objects A and B is an object $A + B$ together with *injections* $\iota_0 : A \to A + B$ and $\iota_1 : B \to A + B$ such that, for every object $C \in \mathcal{C}$ and all morphisms $f : A \to C$, $g : B \to C$ there exists a *unique* morphism $h : A + B \to C$ for which the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\iota_0"] \arrow[dr, "f"'] & A + B \arrow[d, dotted, "h"] & B \arrow[l, "\iota_1"'] \arrow[dl, "g"] \\
 & C &
\end{tikzcd}$$

The arrow $h : A + B \to C$ is denoted by $[f, g]$.

The coproduct $A + B$ is the colimit of the diagram $D : 2 \to \mathcal{C}$, where $\mathcal{I}$ is the discrete category on two objects 0 and 1, and $D_0 = A$, $D_1 = B$.

In Set the coproduct is the disjoint union, defined by

$$X + Y = \{\langle 0, x\rangle \mid x \in X\} \cup \{\langle 1, y\rangle \mid x \in Y\} \ ,$$

where 0 and 1 are distinct sets, for example $\emptyset$ and $\{\emptyset\}$. Given functions $f : X \to Z$ and $g : Y \to Z$, the unique function $[f, g] : X + Y \to Z$ is the usual *definition by cases*:

$$[f, g]u = \begin{cases} fx & \text{if } u = \langle 0, x\rangle \\ gx & \text{if } u = \langle 1, x\rangle \ . \end{cases}$$

Exercise 1.6.10 Suppose A and B are Abelian groups.[10] What is the difference between their coproduct in the category Group of groups, and their coproduct in the category $\mathsf{AbGroup}$ of Abelian groups?

1.6.8 The initial object

An *initial object* in a category $\mathcal{C}$ is an object $\mathsf{0} \in \mathcal{C}$ such that for every $A \in \mathcal{C}$ there exists a *unique* morphism $\mathsf{o}_A : \mathsf{0} \to A$.

An initial object is the colimit of the empty diagram.

In Set, the initial object is the empty set.

Exercise 1.6.11 What is the initial and what is the terminal object in the category of groups?

A *zero object* is an object that is both initial and terminal.

Exercise 1.6.12 Show that in the category of Abelian groups finite products and coproducts agree, that is $\mathsf{0} \cong \mathsf{1}$ and $A \times B \cong A + B$.

1.6.9 Coequalizers

Given objects and morphisms

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] & Q \end{tikzcd}$$

we say that q *coequalizes* f and g when $e \circ f = e \circ g$. A *coequalizer* of f and g is a *universal* coequalizing morphism; thus $q : B \to Q$ is a coequalizer of f and g when it coequalizes them and, for all $s : B \to S$, if $s \circ f = s \circ g$ then there exists a *unique* morphism $r : Q \to S$ such that $s = r \circ q$:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "s"'] & Q \arrow[d, dotted, "r"] \\ & & S \end{tikzcd}$$

[10] An Abelian group is one that satisfies the commutative law $x \cdot y = y \cdot x$.

In Set the coequalizer of parallel functions $f : A \to B$ and $g : A \to B$ is the quotient set $Q = B/{\sim}$ where $\sim$ is the least equivalence relation on B satisfying

$$fx = gy \Longrightarrow x \sim y \;.$$

The function $q : B \to Q$ is the canonical quotient map which assigns to each element $x \in B$ its equivalence class $[x] \in B/{\sim}$. In general, coequalizers can be thought of as quotients of those equivalence relations that that can be defined (generated) by a single equation.

Exercise 1.6.13 Show that a coequalizer is an epimorphism, i.e., if $q : B \to Q$ is a coequalizer of f and g, then, for all $u, v : Q \to T$, $u \circ q = v \circ q$ implies $u = v$. [Hint: use the duality between limits and colimits and Exercise 1.6.3.]

Definition 1.6.14 A morphism is a *regular epi* if it is a coequalizer.

The difference between epis and regular epis is best illustrated in the category Top: a continuous map $f : X \to Y$ is epi when it is surjective, whereas it is a regular epi when it is a topological quotient map.[11]

1.6.10 Pushouts

A *pushout* of $f : A \to B$ and $g : A \to C$ is an object Q with morphisms $q_0 : B \to Q$ and $q_1 : C \to Q$ such that $q_0 \circ f = q_1 \circ g$, and whenever $r_0 : B \to R$, $r_1 : C \to R$ are such that $r_0 \circ f = r_1 \circ g$, then there exists a unique $s : Q \to R$ such that $r_0 = s \circ q_0$ and $r_1 = s \circ q_1$:

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "q_1"] \arrow[ddr, bend left, "r_1"] & \\
B \arrow[r, "q_0"'] \arrow[drr, bend right, "r_0"'] & Q \arrow[dr, dotted, "s"] \arrow[ul, phantom, very near start, "\ulcorner"] & \\
 & & R
\end{tikzcd}$$

We indicate that Q is a pushout by drawing a square corner next to it, as in the above diagram. The above pushout Q is sometimes denotes by $B +_A C$.

A pushout, as in the above diagram, is the colimit of the diagram $D : \mathcal{I} \to \mathcal{C}$ where the index category $\mathcal{I}$ is

$$\begin{tikzcd}
\bullet \arrow[r, "2"] \arrow[d, "1"'] & \bullet \\
\bullet &
\end{tikzcd}$$

[11] A continuous map $f : X \to Y$ is a topological quotient map when it is surjective and, for every $U \subseteq Y$, U is open if, and only if, f^*U is open.

and $D1 = f$, $D2 = g$.

In Set, the pushout of $f : A \to C$ and $g : B \to C$ is the quotient set

$$Q = (B + C)/\sim$$

where $B + C$ is the disjoint union of B and C, and $\sim$ is the least equivalence relation on $B + C$ such that, for all $x \in A$,

$$fx \sim gx \ .$$

The functions $q_0 : B \to Q$, $q_1 : C \to Q$ are the injections, $q_0 x = [x]$, $q_1 y = [y]$, where $[x]$ is the equivalence class of x.

1.6.11 Limits and Colimits as Adjoints

An object $A \in \mathcal{C}$ can be viewed as a functor from the terminal category $\mathsf{1}$ to $\mathcal{C}$, namely the functor which maps the only object $\star$ of $\mathsf{1}$ to A and the only morphism $1_\star$ to 1_A.

Now if $\mathcal{C}$ has a terminal object $\mathsf{1}_\mathcal{C}$ we can ask whether the corresponding functor $\mathsf{1}_\mathcal{C} : \mathsf{1} \to \mathcal{C}$ has any adjoints. Since $\mathsf{1}$ is the terminal object in Cat, there exists a unique functor $!_\mathcal{C} : \mathcal{C} \to \mathsf{1}$, which maps every object of $\mathcal{C}$ to $\star$. This functor is indeed adjoint to $\mathsf{1}_\mathcal{C}$ because, for every $A \in \mathcal{C}$ we have a (trivially natural) bijective correspondence

$$\frac{!_A : A \to \mathsf{1}_\mathcal{C}}{\overline{\overline{1_\star : !_\mathcal{C} A \to \star}}}$$

Similarly, an initial object is left adjoint to $!_\mathcal{C}$:

$$\mathsf{0}_\mathcal{C} \dashv \, !_\mathcal{C} \dashv \mathsf{1}_\mathcal{C} \ .$$

If $\mathcal{C}$ has binary products then they can be viewed as a functor

$$- \times - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

which maps $\langle A, B \rangle$ to $A \times B$ and a pair of morphisms $\langle f : A \to A', g : B \to B' \rangle$ to the unique morphism $f \times g : A \times B \to A' \times B'$ for which $\pi_0 \circ (f \times g) = f \circ \pi_0$ and $\pi_1 \circ (f \times g) = g \circ \pi_1$,

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times B \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] \arrow[d, "f \times g"] & B \arrow[d, "g"] \\
A' & A' \times B' \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & B'
\end{tikzcd}$$

The binary product functor has a left adjoint, namely the diagonal diagram functor

$$\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$$

defined by $\Delta A = \langle A, A\rangle$, $\Delta f = \langle f, f\rangle$. Indeed, there is a natural bijective correspondence

$$\frac{\langle f, g\rangle : \langle A, A\rangle \to \langle B, C\rangle}{\overline{f \times g : A \to B \times C}}$$

Similarly, binary coproducts are left adjoint to the diagonal functor:

$$(- + -) \dashv \Delta \dashv (- \times -) \; .$$

In general, suppose $\mathcal{C}$ has limits of shape $\mathcal{I}$. Then the limit construction is a functor

$$\lim : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$$

that maps each diagram $D \in \mathcal{C}^{\mathcal{I}}$ to its limit $\lim D$. In the opposite direction there is the constant diagram functor

$$\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$$

that maps $A \in \mathcal{C}$ to the constant diagram $\Delta_A : \mathcal{I} \to \mathcal{C}$. These two are adjoint because there is a natural bijective correspondence between cones $\alpha : \Delta_A \Longrightarrow D$ on D, and their factorizations through the limit of D,

$$\frac{\alpha : \Delta_A \Longrightarrow D}{\overline{A \to \lim D}}$$

An analogous correspondence holds for colimits so that we obtain a pair of adjunctions

$$\operatorname{colim} \dashv \Delta \dashv \lim \; .$$

Exercise 1.6.15 How are the functors $\lim : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$, $\operatorname{colim} : \mathcal{C}^{\mathcal{I}} \to \mathcal{C}$, and $\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{I}}$ defined on morphisms?

1.6.12 Preservation of Limits and Colimits by Functors

We say that a functor $F : \mathcal{C} \to \mathcal{D}$ *preserves products* when, given a product

$$A \xleftarrow{\pi_0} A \times B \xrightarrow{\pi_1} B$$

its image in $\mathcal{D}$,

$$FA \xleftarrow{F\pi_0} F(A \times B) \xrightarrow{F\pi_1} FB$$

is a product of FA and FB. If $\mathcal{D}$ has chosen binary products, F preserves binary products if, and only if, the unique morphism $f : F(A \times B) \to FA \times FB$

which makes the following diagram commutative is an isomorphism: [12]

$$\begin{tikzcd}
 & F(A \times B) \arrow[dl, "F\pi_0"'] \arrow[d, "f"] \arrow[dr, "F\pi_1"] & \\
FA & FA \times FB \arrow[l, "\pi_0"] \arrow[r, "\pi_1"'] & FB
\end{tikzcd}$$

In general, a functor $F : \mathcal{C} \to \mathcal{D}$ is said to *preserve limits* of shape $\mathcal{I}$ when it maps limit cones to limit cones: if (L, λ) is a limit of $D : \mathcal{I} \to \mathcal{C}$ then $(FL, F \circ \lambda)$ is a limit of $F \circ D : \mathcal{I} \to \mathcal{D}$.

Analogously, a functor $F : \mathcal{C} \to \mathcal{D}$ is said to *preserve colimits* of shape $\mathcal{I}$ when it maps colimit cocones to colimit cocones: if (C, ζ) is a colimit of $D : \mathcal{I} \to \mathcal{C}$ then $(FC, F \circ \zeta)$ is a colimit of $F \circ D : \mathcal{I} \to \mathcal{D}$.

Proposition 1.6.16 *(a) A functor preserves finite (small) limits if, and only if, it preserves equalizers and finite (small) products. (b) A functor preserves finite (small) colimits if, and only if, it preserves coequalizers and finite (small) coproducts.*

Proof. This follows from the fact that limits are constructed from equalizers and products, cf. Proposition 1.6.8, and that colimits are constructed from coequalizers and coproducts, cf. Exercise 1.6.9. ∎

Proposition 1.6.17 *For a locally small category $\mathcal{C}$, the Yoneda embedding $\mathsf{y} : \mathcal{C} \to \widehat{\mathcal{C}}$ preserves all limits that exist in $\mathcal{C}$.*

Proof. Suppose (L, λ) is a limit of $D : \mathcal{I} \to \mathcal{C}$. The Yoneda embedding maps D to the diagram $\mathsf{y} \circ D : \mathcal{I} \to \widehat{\mathcal{C}}$, defined by

$$(\mathsf{y} \circ D)_i = \mathsf{y}D_i = \mathcal{C}(-, D_i)\ .$$

and it maps the limit cone (L, λ) to the cone $(\mathsf{y}L, \mathsf{y} \circ \lambda)$ on $\mathsf{y} \circ D$, defined by

$$(\mathsf{y} \circ \lambda)_i = \mathsf{y}\lambda_i = \mathcal{C}(-, \lambda_i)\ .$$

To see that $(\mathsf{y}L, \mathsf{y} \circ \lambda)$ is a limit cone on $\mathsf{y} \circ D$, consider a cone (M, μ) on $\mathsf{y} \circ D$. Then $\mu : \Delta_M \Longrightarrow D$ consists of a family of functions, one for each $i \in \mathcal{I}$ and $A \in \mathcal{C}$,

$$(\mu_i)_A : MA \to \mathcal{C}(A, D_i)\ .$$

For every $A \in \mathcal{C}$ and $m \in MA$ we get a cone on D consisting of morphisms

$$(\mu_i)_A m : A \to D_i\ . \qquad (i \in \mathcal{I})$$

[12]Products are determined up to isomorphism only, so it would be too restrictive to require $F(A \times B) = FA \times FB$. When that is the case, however, we say that the functor F *strictly* preserves products.

There exists a unique morphism $\phi_A m : A \to L$ such that $(\mu_i)_A m = \lambda_i \circ \phi_A m$. The family of functions

$$\phi_A : MA \to \mathcal{C}(A, L) = (\mathsf{y} \circ L)A \qquad (A \in \mathcal{C})$$

forms a factorization $\phi : M \Longrightarrow \mathsf{y}L$ of the cone (M, μ) through the cone (L, λ). This factorization is unique because each $\phi_A m$ is unique. ∎

In effect we showed that a covariant representable functor $\mathcal{C}(A, -) : \mathcal{C} \to \mathsf{Set}$ preserves existing limits,

$$\mathcal{C}(A, \lim_{i \in \mathcal{I}} D_i) \cong \lim_{i \in \mathcal{I}} \mathcal{C}(A, D_i) \ .$$

By duality, the contravariant representable functor $\mathcal{C}(-, A) : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ maps existing colimits to limits,

$$\mathcal{C}(\operatorname*{colim}_{i \in \mathcal{I}} D_i, A) \cong \lim_{i \in \mathcal{I}} \mathcal{C}(D_i, A) \ .$$

Exercise 1.6.18 Prove the above claim that a contravariant representable functor $\mathcal{C}(-, A) : \mathcal{C}^{\mathsf{op}} \to \mathsf{Set}$ maps existing colimits to limits. Use duality between limits and colimits. Does it also follow *by a simple duality argument* that a contravariant representable functor $\mathcal{C}(-, A)$ maps existing limits to colimits? How about a covariant representable functor $\mathcal{C}(A, -)$ mapping existing colimits to limits?

Exercise 1.6.19 Prove that a functor $F : \mathcal{C} \to \mathcal{D}$ preserves monos if it preserves limits. In particular, the Yoneda embedding preserves monos. Hint: Exercise 1.6.5.

Proposition 1.6.20 *Right adjoints preserve limits, and left adjoints preserve colimits.*

Proof. Suppose we have adjoint functors

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd}$$

and a diagram $D : \mathcal{I} \to \mathcal{D}$ whose limit exists in $\mathcal{D}$. We would like to use the following slick application of Yoneda Lemma to show that G preserves limits: for every $A \in \mathcal{C}$,

$$\mathcal{C}(A, G(\lim D)) \cong \mathcal{D}(FA, \lim D) \cong \lim_{i \in \mathcal{I}} \mathcal{D}(FA, D_i) \cong \\ \lim_{i \in \mathcal{I}} \mathcal{C}(A, GD_i) \cong \mathcal{C}(A, \lim(G \circ D)) \ ,$$

therefore $G(\lim D) \cong \lim(G \circ D)$. However, this argument only works if we already know that the limit of $G \circ D$ exists.

We can also prove the stronger claim that whenever the limit of $D : \mathcal{I} \to \mathcal{D}$ exists then the limit of $G \circ D$ exists in $\mathcal{C}$ and its limit is $G(\lim D)$. So suppose (L, λ) is a limit cone of D. Then $(GL, G \circ \lambda)$ is a cone on $G \circ D$. If (A, α) is another cone on $G \circ D$, we have by adjunction a cone (FA, γ) on D,

$$\frac{\alpha_i : A \to GD_i}{\gamma_i : FA \to D_i}$$

There exists a unique factorization $f : FA \to L$ of this cone through (L, λ). Again by adjunction, we obtain a unique factorization $g : A \to GL$ of the cone (A, α) through the cone $(GL, G \circ \lambda)$:

$$\frac{f : FA \to L}{g : A \to GL}$$

The factorization g is unique because γ is uniquely determined from α, f uniquely from α, and g uniquely from f.

By a dual argument, a left adjoint preserves colimits. ∎