FROM LAWS OF MOTION TO VACUUM FIELD EQUATIONS

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<u>Sachs/Wu</u> $\ddot{y} = -\nabla\varphi(y)$ Newton's law of motion

$$\begin{tikzcd}
\ddot{y} = -\nabla\varphi(y) \arrow[d, squiggly, "\text{mechanism}"'] \arrow[r, squiggly] & \Psi = \left(\dfrac{\partial^2\varphi}{\partial x^\alpha \partial x^\beta}\right)_{\alpha\beta} \arrow[dl, squiggly, "\text{trace}"] \\
\nabla^2\varphi \rightsquigarrow \nabla^2\varphi = 0 &
\end{tikzcd}$$

Laplace field equation

Not a logical deduction, but mechanism in terms of acceleration of a "swarm" of particles whose trajectories are "very close" to each other

<u>SDG</u> (Synthetic Differential Geometry)

"very close" : <u>infinitely</u> close

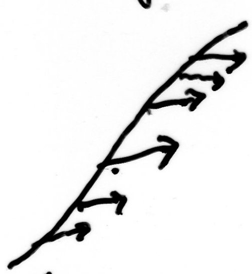
trajectories of the swarm

$$(\gamma_h : I \longrightarrow E)_{h \in D} \qquad\qquad D = \{x \in R \mid x^2 = 0\}$$

$$\begin{tikzcd}
 & E^D \arrow[d, "\pi_E"] \\
I \arrow[ur, "W"] \arrow[r, "\gamma"'] & E
\end{tikzcd}$$

$W(t)(h) = \gamma_h(t)$
$\pi_E(v) = v(0)$
$\gamma \approx \gamma_0$

<u>Picture</u>



$W(t)(h) = \gamma_h(t) = \gamma(t) + h\delta(t)$

Vector field along γ

Can define $\begin{cases} \dot{W}(t)(h) = \dot\gamma(t) + h\,\delta'(t) \\ \ddot{W}(t)(h) = \ddot\gamma(t) + h\,\delta''(t) \end{cases}$

Trouble Gravitational "field" is not a vector field in the sense of Differential Geometry

Recall

$$\begin{array}{c} M^D \\ Q \uparrow \downarrow \pi_M \\ M \end{array}$$

A vector field Q is a section of π_M, i.e.

$$\pi_M \circ Q = \mathrm{id}_M \quad (\pi_M(v) = v(0))$$

From now on : vector fields defined only on infinitesimal portions of M, curves defined on $D_\infty = \mathrm{Nil}(R)$ etc.

vector field $\sim$ autonomous first order DE

But equation of motion is a 2d order DE

Theorem (Reduction theorem)

An autonomous 2d order DE G is equivalent to a family $(G^u)_{u \in M_x}$, $u \neq 0$, $M_x = \pi_M^{-1}(x)$ in the following sense :

γ sol of $G \Rightarrow \gamma$ sol of $G^{\dot\gamma(u)}$ (we assume $\dot\gamma(u) \neq 0$)

γ sol of $G^v \Rightarrow \gamma$ sol of G

NB $\dot\gamma$ is the vector field along γ defined by $\dot\gamma(t)(h) = \gamma(t+h)$

R-linear map $M_x \xrightarrow{\varphi^u} M_x$ (for $u \neq 0$)

$v \in M_x$. Let W^v be the swarm $(\gamma_h)_{h \in D}$ (i.e., $W^v(t)(h) = \gamma_h(t)$) where $\gamma_h : D_\infty \to M$ is the unique integral curve of G^u (i.e., $\dot\gamma_h(t) = G^u(\gamma_h(t))$ with $\gamma_h(0) = v(h)$

<u>Define</u> $\Psi^u(v) = W^v(0) = [h \mapsto \gamma(0) + h\delta''(0)]$

<u>Example 1</u> : Laplace equation

M 3-dimensional manifold

Take $u \in M_x \simeq R^3$ $u \neq 0$

Let γ_h be the unique integral curve of G^u with $\gamma_h(0) = v(h)$

$$\begin{cases} \ddot\gamma_h(t) = -\nabla\varphi(\gamma_h(t)) & \text{since } \gamma_h \text{ sol of } G \text{ (Reduction Theorem)} \\ \ddot\gamma(t) = -\nabla\varphi(\gamma(t)) \end{cases}$$

$h\,\ddot\delta(t) = -\left(\nabla\varphi(\gamma(t) + h\delta(t)) - \nabla\varphi(\gamma(t))\right)$

Developping in Taylor (first order since $h^2 = 0$)

$\ddot\delta(0) = -Mv$ with $M = \left(\frac{\partial^2\varphi}{\partial x^\alpha \partial x^\beta}\right)_{\alpha\beta}$ $\delta(0) = v$

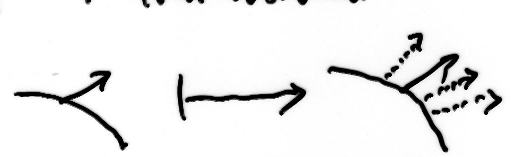
$\Psi^u = -\left(\frac{\partial^2\varphi}{\partial x^\alpha \partial x^\beta}\right)_{\alpha\beta}$

trace $\Psi^u = -\nabla^2\varphi$

vacuum field equation : trace$(\Psi^u) = 0 \;\forall u \neq 0$ i.e. $\boxed{\nabla^2\varphi = 0}$

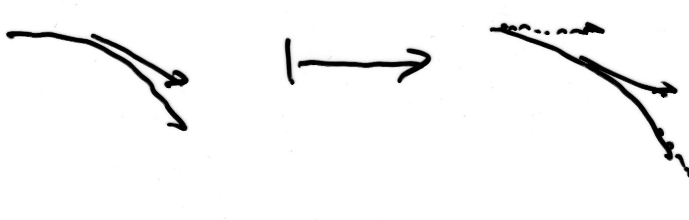
<u>Example 2</u> : Einstein's vacuum field equation

M 4-dimensional manifold with connection ∇

∇:  with linearity conditions

equation of motion $\ddot y = \nabla(\dot y, \dot y)$

Solutions are geodesics



tangency is preserved

$$\begin{tikzcd}
\ddot{y} = D(y, \dot{y}) \arrow[d, rightsquigarrow] \arrow[r, rightsquigarrow] & \psi^u(v) = R_{uv}u \arrow[dl, rightsquigarrow, "\text{trace}"] \\
\mathrm{Ric}(u,u) &
\end{tikzcd}$$

Riemann-Christoffel

Einstein vacuum field equation: $\mathrm{Ric}(u,u) = 0 \ \forall u \neq 0$

This implies (little algebra) $\boxed{\mathrm{Ric} = 0}$

(Details in my paper)

<u>Example</u> 3: Falling of a projectil with air resistance

$$\ddot{y} = -g - k\dot{y}$$

$$\big\downarrow$$

$$\boxed{0 = 1}$$

<u>Inverse Problem</u>

Are there other "laws of motion" leading to a given "field equation"?

More reasonable problem: are there other laws of motion leading to a given family $(\psi_u)_{u \in M_x}$ of linear transformation?

I don't know the answer even for the Laplace equation. Only a partial result: only the Newton equation

$$\ddot{y} = -\nabla(\varphi(y)) + C \quad (C \text{ constant vector})$$

leads to the constant family $\left(\frac{\partial^2 \varphi}{\partial x^\alpha \partial x^\beta}\right)_{\alpha\beta}$ of linear transformations among laws of motion of the form $\ddot{y} = f(y)$

REFERENCES

"A derivation of Einstein's vacuum field equation" to appear in the CRM volume in honor of Makkai

Sachs/Wu: "General Relativity for Mathematicians" Springer-Verlag 1977