Topos theory and the Copenhagen Interpretation of quantum mechanics

Klaas Landsman, Radboud University Nijmegen
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Literature

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Paths that cross

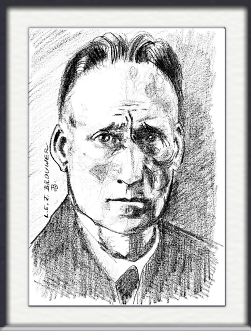
Intuitionistic logic and constructive analysis

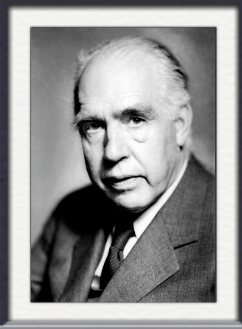
Copenhagen Interpretation à la Bohr

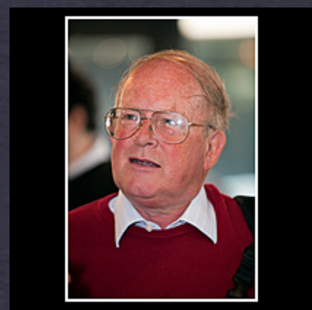
Algebraic quantum theory: C*-algebra of observables

Topos theory as a foundation of (quantum) physics

Constructive (hence topos-valid) Gelfand duality







“Conventional” approach

Model physical system by C*-algebra of observables

Physical theories: state-observable duality, but here: observables ⇝ states

Classical physics: commutative algebra $C_0(M)$

Phase space $M \cong \Sigma(C_0(M))$ recovered as Gelfand spectrum of C*-algebra

Propositions correspond to (Borel) subsets of Gelfand spectrum M

Propositions form Boolean lattice ☞ classical propositional logic ☺

Quantum physics: noncommutative algebra B(H)

Von Neumann: (projective) Hilbert space H plays role of phase space

Propositions correspond to (closed) subspaces of H

Propositions form orthomodular lattice ☞ quantum ‘logic’ ☹

Bohrification

Find <u>good</u> version of "phase space" in quantum theory

"subsets" define "propositions" which form "reasonable" (i.e. distributive) logic

Quantum theory: *noncommutative* C*-algebra $A \subseteq B(H)$

Jordan-von Neumann-Wigner (1934), Segal (1947), Haag & Kastler (1964), ...

Bohr's *doctrine of classical concepts*

C*-algebra A empirically accessible through its commutative C*-subalgebras C

Bohr's *doctrine of complementarity*

All such C together with inclusion information define commutative C*-algebra $\underline{A}$

Gelfand spectrum $\underline{\Sigma}(\underline{A})$ is "quantum phase space"

Defined because $\underline{A}$ is commutative, unlike original algebra of observables A

Lattice of open subsets of $\underline{\Sigma}(\underline{A})$ "is" *intuitionistic* logic

$O(\underline{\Sigma}(\underline{A}))$ is *Heyting algebra* i.e. (distributive) lattice with $\to$ s.t. $x \leq (y \to z)$ iff $(x \wedge y) \leq z$

Complication

- $\underline{A}$ inhabits mathematical "universe" **Q** ≠ **Sets**

  Gelfand spectrum $\underline{\Sigma}(\underline{A})$ lives in same "universe" as $\underline{A}$

- This "universe" is a *topos* i.e. category with exactly the right structure to interpret ***constructive mathematics:***

  no TND, no AC → avoid points → space X replaced by topology O(X) *as lattice* (complete Heyting algebra)

- "Internally": $\underline{\Sigma}(\underline{A})$ is "space" (*locale*) within topos **Q** ☹

  "Externally": $\underline{\Sigma}(\underline{A})$ yields quantum phase space in **Sets** ☺

Technically, ...

- ***Noncommutative*** unital C*-algebra A (in Sets)
- Poset C(A) of all unital ***commutative*** C*-subalgebras
- Functor topos **Q** = [C(A), Sets] ≅ Sh(C(A)) Alexandrov topology
- Internal ***commutative*** C*-algebra $\underline{A}$: C ↦ C in **Q**
- Internal Gelfand spectrum $\underline{\Sigma}(\underline{A})$ in Sh(C(A)) has external description π: Σ(A) → C(A) as map in **Sets**
- Heyting algebra O(Σ(A)) is ***intuitionistic*** quantum logic

Example: n×n matrices

- $A = M_n(\mathbb{C}) \Rightarrow C(A) \cong$ poset of Boolean sublattices of lattice $L(\mathbb{C}^n)$

of projections in $M_n(\mathbb{C})$ i.e. on $\mathbb{C}^n$ (or of linear subspaces of $\mathbb{C}^n$)

$$\boxed{O(\Sigma(A)) = \{P: C(A) \to L(\mathbb{C}^n) \mid P(B) \in B,\ P(C) \leq P(D) \text{ if } C \subseteq D\}}$$

- $P \leq Q$ iff $P(B) \subseteq Q(B)$ for all $B \in C(A) \Rightarrow (P \wedge Q)(B) = P(B) \wedge Q(B)$ etc.

- Each value $P(B) \in L(\mathbb{C}^n)$ is proposition in the sense of ***von Neumann***
- Each element P of $O(\Sigma(A))$ is proposition in the sense of "***Bohr***":

  to each *classical* context, i.e. to each *Boolean* sublattice B of $L(\mathbb{C}^n)$

  P assigns a projection/subspace $P(B) \in B$ *pertinent to that context*

- $O(\Sigma(A))$ is non-Boolean Heyting algebra: ***quantum logic is intuitionistic***

The missing link

"Bohrification" $\underline{A}$ of (unital) ***noncommutative*** C*-algebra A is built from (unital) ***commutative*** C*-subalgebras of A:

- ***Can we recover A from*** $\underline{A}$ or: does Bohr's doctrine of classical concepts fully capture quantum theory?

  Harding & Döring (based on Harding & Navara), 2010:

  for von Neumann algebras: ***yes but only as Jordan algebra*** (this even recovers all of A for finite-dimensional C*-algebras)

- Missing link in doctrine of classical concepts: *what, beyond commutative information C(A), characterizes A?*

- Similar problem: *does state space S(A) characterize A?* Solved by Alfsen & Shultz (1982-2007): need "orientation" of S(A)

Epilogue

'Hence it is interesting to compare the modifications which [the models for propositional calculi given by lattices of closed subspaces of Hilbert space] introduce into Boolean algebra, with those which logicians on "intuitionistic" and related grounds have tried introducing. The main difference seems to be that whereas logicians have usually assumed that properties [law of excluded middle] of negation were the ones least able to withstand a critical analysis, the study of mechanics points to the distributive identities as the weakest link in the algebra of logic' (Birkhoff and von Neumann, 1936)

'All departures from common language and ordinary logic are entirely avoided by reserving the word "phenomenon" solely for reference to unambiguously communicable information, in the account of which the word "measurement" is used in its plain meaning of standardized comparison.' (Bohr, 1958)

'It is of interest that the kind of change in classical logic which would fit what Birkhoff and von Neumann suggest [...] would be the rejection of the law of the excluded middle, as proposed by Brouwer, but rejected by Birkhoff and von Neumann' (Popper, 1968)