A Higher Structure Identity Principle

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(cww B. Ahrens, P. North, M. Shulman)

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Main Idea

Theorem (HoTT Book, Theorem 9.4.16)

For any univalent precategories (=categories) $\mathcal{C}$ and $\mathcal{D}$, the type of categorical equivalences $\mathcal{C} \simeq_{\mathit{precat}} \mathcal{D}$ is equivalent to $\mathcal{C} =_{\mathbf{UniCat}} \mathcal{D}$

$$\left(\mathcal{C} \simeq_{\mathit{precat}} \mathcal{D}\right) \quad \simeq \quad \left(\mathcal{C} =_{\mathbf{UniCat}} \mathcal{D}\right)$$

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Pre-Theorem

For any univalent models $\mathcal{M}$ and $\mathcal{N}$ of an $\mathcal{L}$-theory $\mathbb{T}$, the type of $\mathcal{L}$-equivalences $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$ is equivalent to $\mathcal{M} =_{\mathbf{UniMod}(\mathbb{T})} \mathcal{N}$

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The Setting: Two-Level Type Theory (2LTT)

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One level of 2LTT is a **fibrant** fragment of **fibrant types** which consists of $\Pi, \Sigma, +, \mathbf{1}, \mathbf{0}, \mathbb{N}$, intensional $=$, propositional truncation $|| - ||$ and a hierarchy of univalent universes $\mathcal{U}$.

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The other level of 2LTT is the **strict** fragment of **pretypes** which consists of $+^s, \mathbf{0}^s, \mathbb{N}^s$, a strict equality $\equiv$ with UIP and function extensionality, a hierarchy of strict universes $\mathcal{U}^s$. It shares the type constructors $\Pi, \Sigma, \mathbf{1}$ with the fibrant fragment.

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The rules for the type constructors are the usual ones, and we also have a rule that allows us to consider any fibrant type as a pretype, i.e. the fibrant universes $\mathcal{U}$ can be thought of as subuniverses of $\mathcal{U}^s$, as well as rules that ensure that Σ and Π preserve fibrancy, and that the fibrant universes are closed under strict isomorphism.

s-categories

For a pretype X, we can write isfibrant(X) for the pretype $\sum_{Y:\mathcal{U}}(Y \equiv X)$.

Definition (Definition 27, 2LTT)

A pretype A is **cofibrant** if for any fibration $p : X \to Y$, the induced map $(A \to X) \to (A \to Y)$ is a fibration.

Definition (Definition 7, 2LTT)

A **s-category** is given by the following data

1. A pretype $\mathcal{C}$ of *objects*
2. For each $x, y : \mathcal{C}$ a pretype $\mathcal{C}(x, y)$ of *arrows*
3. For each $x : \mathcal{C}$ an arrow $1 : \mathcal{C}(x, x)$
4. A *composition* operation $\circ : \mathcal{C}(y, z) \to \mathcal{C}(x, y) \to \mathcal{C}(x, z)$ that is strictly associative and for which 1_x is a strict left and right unit.

A s-category **cofibrant** if its pretypes of objects and arrows are cofibrant.

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An $\mathcal{L}$-**theory** $\mathbb{T}$ is a pretype of $\mathcal{L}$-sentences.

An example

$$\begin{tikzcd}
 & \mathcal{L}_{\mathrm{rg}} \arrow[rrrrr, "\Gamma"] & & & & & \mathbf{Set} \\
2 & I \arrow[d, "i"'] & \color{blue}{\tau} \arrow[d, mapsto] & & \color{red}{\tau} & & \{\tau\} \arrow[d] \\
1 & A \arrow[d, bend right, "c"'] \arrow[d, bend left, "d"] & g \arrow[d, mapsto, bend right] \arrow[d, mapsto, bend left] & & f \arrow[dl, mapsto] \arrow[dr, mapsto] & & \{f, g\} \arrow[d, bend right, "c"'] \arrow[d, bend left, "d"] \\
0 & O & z & y & & x & \{x, y, z\}
\end{tikzcd}$$

$$di = ci$$

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$$\Gamma = x, y, z \colon O, f \colon A(x, y), g \colon A(z, z), \tau \colon I(g, z)$$

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$$\mathbf{Form}(x \colon O) \quad \forall g \colon A(z, z).\exists {\color{blue}\tau} \colon I(g, z).\top \quad \sim \quad \forall g \colon A(z, z).I(g, z)$$

Some terminology and notation

$r(K)$ $\qquad \mathcal{L} \longleftarrow K//\mathcal{L}$ $\qquad \partial K = \mathcal{L}(K, -)$

$$\begin{tikzcd}
n = H(\mathcal{L}) & R \arrow[d, "i"'] & & \\
n-1 & A \arrow[d, dashed, bend right] \arrow[d, dashed, bend left] & & \\
m & K \arrow[d, dashed] & & \mathcal{L}^{\leq K}, \mathcal{L}^{<K}, \ldots \\
1 & X \arrow[d] & X \leq K & \\
0 & O & &
\end{tikzcd}$$

Semantics of FOLDS in 2LTT

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We want to define a type of $\mathcal{L}$-structures $\mathbf{Struc}(\mathcal{L})$.

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$$\sum_{O : \mathcal{U}} \dots$$

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$\mathfrak{D}(\mathcal{L}_{\mathrm{rg}})$

$$\ldots \sum_{A\colon O\times O\to\mathcal{U}} \ldots$$

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\end{tikzcd}$$

We would like $\mathfrak{D}(\mathcal{L}) \equiv \mathfrak{R}(\mathcal{L})$ but the situation is not that simple.

Semantics of FOLDS in 2LTT

$$\begin{tikzcd}
F_K \arrow[d, two heads] & \\
M^F_A \arrow[r, equal] \arrow[d, "K"'] & \lim\left(A//\mathcal{L} \xrightarrow{\mathsf{cod}} \mathcal{L} \xrightarrow{F} \mathcal{U}\right) \\
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Theorem

$\mathfrak{D}(\mathcal{L}) \simeq \mathfrak{R}(\mathcal{L})$ *as s-categories.*

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We define the type of $\mathcal{L}$-structures as $\mathbf{Struc}(\mathcal{L}) = \mathfrak{D}(\mathcal{L})$ but we will use the equivalence of the above theorem to transfer constructions from $\mathfrak{R}(\mathcal{L})$.

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Similarly, we denote by $\mathbf{Mod}(\mathbb{T})$ the type of $\mathcal{L}$-structures satisfying all the sentences of $\mathbb{T}$.

The $\mathcal{L}_{\mathsf{cat}}$-theory $\mathbb{T}_{\mathsf{cat}}$

$$\begin{tikzcd}
 & 2 & \circ \arrow[dr, bend left, "t_0"] \arrow[dr, "t_1"'] \arrow[dr, bend right=60, "t_2"'] & I \arrow[d, "i"] & E_A \arrow[dl, bend right, "t"'] \arrow[dl, bend left, "s"] & dt_0 = dt_2 \;\; ct_1 = ct_2 \;\; dt_1 = ct_0 \\
\mathcal{L}_{\mathsf{cat}} & 1 & & A \arrow[d, bend right, "c"'] \arrow[d, bend left, "d"] & & ds = dt \;\; cs = ct \\
 & 0 & & O & & ci = di
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Theorem

If E_A is interpreted as the identity type on A then

$$\mathbf{Mod}(\mathbb{T}_{cat}) \simeq \sum_{O\,:\,\mathcal{U}} \; \sum_{A\,:\; O \to O \to \mathcal{U}} \; \sum_{\substack{\circ\,:\; \prod_{x,y,z\,:\,O} A(x,y) \to A(y,z) \to A(x,z) \\ I\,:\; \prod_{x\,:\,O} A(x,x)}} (\ldots\ldots)$$

Generalized Isomorphism?

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Generalized Isomorphism?

Let $\mathcal{M} : \mathbf{Mod}(\mathbb{T})$

$$\begin{tikzcd}
\mathcal{M}_Q \arrow[dr] & \mathcal{M}_R \arrow[d] & \\
& \mathcal{M}_A \arrow[dl, bend right] \arrow[dl, bend left] \arrow[dd] \arrow[dr] & \\
\mathcal{M}_K \arrow[dr, dashed] & & \mathcal{M}_{K'} \arrow[dl, dashed] \\
& \mathcal{M}_O &
\end{tikzcd}$$

$$\begin{tikzcd}
q & r & ??? \\
a \arrow[d, mapsto] & b \arrow[dl, mapsto] & a \cong b? \\
x & & \partial a = \partial b
\end{tikzcd}$$

FOLDS $\mathcal{L}$-equivalence

$$\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N} \qquad =_{\text{df}} \qquad \sum_{\langle P,m,n\rangle} \left(\begin{tikzcd} & \mathcal{P} \arrow[dl, two heads, "m"', "\text{v.s.}"] \arrow[dr, two heads, "n", "\text{v.s.}"'] & \\ \mathcal{M} & & \mathcal{N} \end{tikzcd} \right)$$

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Definition

$\mathrm{isverysurjective}(m) =_{\mathrm{df}} \prod_{K : \mathcal{L}} \mathrm{issurjective}(\mathcal{P}_K \to M_K^{\mathcal{P}} \times_{M_K^{\mathcal{M}}} \mathcal{M}_K)$

Theorem (Makkai, 1995)

If $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$ *then* $\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models \phi$

FOLDS pre-isomorphism

Fix $\mathcal{M} : \mathbf{Mod}(\mathbb{T})$ and $K : \mathcal{L}$. Let $a, b : \mathcal{M}_K$.

Definition (Pre-isomorphism)

A **pre-isomorphism** from a to b is given by the following cospan of spans

$$\begin{tikzcd}
 & \mathcal{M} \arrow[dl, equal] \arrow[d, "s"] \arrow[dr, equal] & \\
\mathcal{M} & \mathcal{P} \arrow[l, "m"'] \arrow[r, "n"'] & \mathcal{M} \\
 & \partial K \arrow[ul, "a"] \arrow[u, "t"'] \arrow[ur, "b"'] &
\end{tikzcd}$$

where $\langle m, \mathcal{P}, n \rangle$ is a FOLDS equivalence.

Theorem (Makkai, 1995)

If a is pre-isomorphic to b then $\mathcal{M} \models \phi[a] \Leftrightarrow \mathcal{M} \models \phi[b]$

FOLDS isomorphism

Fix $\mathcal{M} : \mathbf{Mod}(\mathbb{T})$ and $K : \mathcal{L}$. Let $a, b : \mathcal{M}_K$.

Definition (FOLDS isomorphism)

A **FOLDS isomorphism** is a pre-isomorphism $\langle m, \mathcal{P}, n \rangle$ such that:

1. For any $f : K \to A$ we have
$$\begin{tikzcd}
\mathcal{M} & \mathcal{P} \arrow[l, "m"'] \arrow[r, "n"] & \mathcal{M} \\
& \mathcal{M} \amalg \partial A \arrow[ul, "{[\mathrm{id}, \mathcal{M}_f(a)]}"] \arrow[u, "\sim"] \arrow[ur, "{[\mathrm{id}, \mathcal{M}_f(b)]}"'] &
\end{tikzcd}$$
2. For any $A > K$ we have $\mathcal{M}_A \times_{M_A^{\mathcal{M}}} M_A^{\mathcal{P}} \xleftarrow{\sim} \mathcal{P}_A \xrightarrow{\sim} \mathcal{M}_A \times_{M_A^{\mathcal{M}}} M_A^{\mathcal{P}}$

We write $a \cong b$ for the type of FOLDS isomorphisms.

FOLDS isomorphism

Fix $\mathcal{M} : \mathbf{Mod}(\mathbb{T})$ and $K : \mathcal{L}$. Let $a, b : \mathcal{M}_K$.

Definition (FOLDS isomorphism)

A **FOLDS isomorphism** is a pre-isomorphism $\langle m, \mathcal{P}, n \rangle$ such that:

1. For any $f : K \to A$ we have

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\mathcal{M} & \mathcal{P} \arrow[l, "m"'] \arrow[r, "n"] & \mathcal{M} \\
& \mathcal{M} \amalg \partial A \arrow[ul, "{[\mathrm{id}, \mathcal{M}_f(a)]}"] \arrow[u, "\sim"] \arrow[ur, "{[\mathrm{id}, \mathcal{M}_f(b)]}"'] &
\end{tikzcd}$$

2. For any $A > K$ we have $\mathcal{M}_A \times_{M_A^{\mathcal{M}}} M_A^{\mathcal{P}} \xleftarrow{\sim} \mathcal{P}_A \xrightarrow{\sim} \mathcal{M}_A \times_{M_A^{\mathcal{M}}} M_A^{\mathcal{P}}$

We write $a \cong b$ for the type of FOLDS isomorphisms.

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Corollary

$\mathrm{idtoiso}_{a,b} : a =_{\mathcal{M}_K} b \to a \cong b$

Univalent Models

Fix $\mathcal{M} : \mathbf{Mod}(\mathbb{T})$

Definition (Univalence for $\mathcal{M}$)

K-univalent $\quad \mathrm{univ}_K(\mathcal{M}) =_{\mathrm{df}} \prod_{a,b : \mathcal{M}_K} \mathrm{isequiv}(\mathrm{idtoiso}_{a,b})$

m-univalent $\quad \mathrm{univ}_m(\mathcal{M}) =_{\mathrm{df}} \prod_{K : \mathcal{L}^{\geq m}} \mathrm{univ}_K(\mathcal{M})$

univalent $\quad \mathrm{univ}(\mathcal{M}) =_{\mathrm{df}} \prod_{K : \mathcal{L}} \mathrm{univ}_K(\mathcal{M})$

Definition (Type of Univalent Models)

$$\mathbf{UniMod}_m(\mathbb{T}) =_{\mathrm{df}} \sum_{\mathcal{M} : \mathbf{Mod}(\mathbb{T})} \mathrm{univ}_m(\mathcal{M})$$

$$\mathbf{UniMod}(\mathbb{T}) =_{\mathrm{df}} \sum_{\mathcal{M} : \mathbf{Mod}(\mathbb{T})} \mathrm{univ}(\mathcal{M})$$

Some Results

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Theorem

Let $H(\mathcal{L}) \geq n \geq m$, $K : \mathcal{L}^{=n}$ *and* $\mathcal{M} : \mathbf{UniMod}_m(\mathbb{T})$. *Then* $\mathcal{M}_K$ *is an* m*-type.*

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Theorem

Let $\mathbb{T}_{cat}$ *be the* $\mathcal{L}_{cat}$*-theory of categories. Then we have:*

$$\mathbf{UniMod}_1(\mathbb{T}_{cat}) \simeq \mathbf{PreCat}$$
$$\mathbf{UniMod}(\mathbb{T}_{cat}) \simeq \mathbf{UniCat}$$

A Higher Structure Identity Principle

We began with:

Theorem

For any univalent models $\mathcal{M}$ and $\mathcal{N}$ of an $\mathcal{L}$-theory $\mathbb{T}$, the type of $\mathcal{L}$-equivalences $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$ is equivalent to $\mathcal{M} =_{\mathbf{UniMod}(\mathbb{T})} \mathcal{N}$.

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And now we can obtain the precise version:

Theorem (A Higher Structure Identity Principle, in progress)

For any $\mathcal{M}, \mathcal{N} : \mathbf{UniMod}(\mathbb{T})$ *for a FOLDS* $\mathcal{L}$*-theory* $\mathbb{T}$ *we have*

$$\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N} \quad \leftrightarrow \quad \mathcal{M} =_{\mathbf{UniMod}(\mathbb{T})} \mathcal{N}$$

Thank you