

# A Higher Structure Identity Principle

Dimitris Tsementzis

(cww B. Ahrens, P. North, M. Shulman)

October 28, 2017

# Main Idea

## Theorem (HoTT Book, Theorem 9.4.16)

*For any univalent precategories (=categories)  $\mathcal{C}$  and  $\mathcal{D}$ , the type of categorical equivalences  $\mathcal{C} \simeq_{\text{precat}} \mathcal{D}$  is equivalent to  $\mathcal{C} =_{\text{UniCat}} \mathcal{D}$*

$$(\mathcal{C} \simeq_{\text{precat}} \mathcal{D}) \simeq (\mathcal{C} =_{\text{UniCat}} \mathcal{D})$$

# Main Idea

## Theorem (HoTT Book, Theorem 9.4.16)

*For any univalent precategories (=categories)  $\mathcal{C}$  and  $\mathcal{D}$ , the type of categorical equivalences  $\mathcal{C} \simeq_{\text{precat}} \mathcal{D}$  is equivalent to  $\mathcal{C} =_{\mathbf{UniCat}} \mathcal{D}$*

$$(\mathcal{C} \simeq_{\text{precat}} \mathcal{D}) \simeq (\mathcal{C} =_{\mathbf{UniCat}} \mathcal{D})$$

## Pre-Theorem

*For any univalent models  $\mathcal{M}$  and  $\mathcal{N}$  of an  $\mathcal{L}$ -theory  $\mathbb{T}$ , the type of  $\mathcal{L}$ -equivalences  $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$  is equivalent to  $\mathcal{M} =_{\mathbf{UniMod}(\mathbb{T})} \mathcal{N}$*

$$(\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}) \simeq (\mathcal{M} =_{\mathbf{UniMod}(\mathbb{T})} \mathcal{N})$$

# Main Idea

## Pre-Theorem

For any *univalent models*  $\mathcal{M}$  and  $\mathcal{N}$  of an  $\mathcal{L}$ -theory  $\mathbb{T}$ , the type of  $\mathcal{L}$ -equivalences  $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$  is *equivalent* to  $\mathcal{M} =_{\mathbf{UniMod}(\mathbb{T})} \mathcal{N}$ .

# Main Idea

## Pre-Theorem

For any *univalent models*  $\mathcal{M}$  and  $\mathcal{N}$  of an  $\mathcal{L}$ -theory  $\mathbb{T}$ , the type of  $\mathcal{L}$ -equivalences  $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$  is *equivalent* to  $\mathcal{M} =_{\mathbf{UniMod}(\mathbb{T})} \mathcal{N}$ .

$\mathcal{L}$ -theory  $\mathbb{T}$  = A theory  $\mathbb{T}$  over a **FOLDS** signature  $\mathcal{L}$

# Main Idea

## Pre-Theorem

For any *univalent models*  $\mathcal{M}$  and  $\mathcal{N}$  of an  $\mathcal{L}$ -theory  $\mathbb{T}$ , the type of  $\mathcal{L}$ -equivalences  $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$  is *equivalent* to  $\mathcal{M} =_{\mathbf{UniMod}(\mathbb{T})} \mathcal{N}$ .

$\mathcal{L}$ -theory  $\mathbb{T}$  = A theory  $\mathbb{T}$  over a **FOLDS** signature  $\mathcal{L}$

$\mathcal{L}$ -equivalence = **FOLDS**  $\mathcal{L}$ -equivalence

# Main Idea

## Pre-Theorem

For any *univalent models*  $\mathcal{M}$  and  $\mathcal{N}$  of an  $\mathcal{L}$ -theory  $\mathbb{T}$ , the type of  $\mathcal{L}$ -equivalences  $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$  is *equivalent* to  $\mathcal{M} =_{\mathbf{UniMod}(\mathbb{T})} \mathcal{N}$ .

$\mathcal{L}$ -theory  $\mathbb{T}$  = A theory  $\mathbb{T}$  over a **FOLDS** signature  $\mathcal{L}$

$\mathcal{L}$ -equivalence = **FOLDS**  $\mathcal{L}$ -equivalence

**univalent model** = Model of  $\mathbb{T}$  where **FOLDS isomorphism** is equivalent to identity

# Main Idea

## Pre-Theorem

For any *univalent models*  $\mathcal{M}$  and  $\mathcal{N}$  of an  $\mathcal{L}$ -theory  $\mathbb{T}$ , the type of  $\mathcal{L}$ -equivalences  $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$  is *equivalent* to  $\mathcal{M} =_{\mathbf{UniMod}(\mathbb{T})} \mathcal{N}$ .

$\mathcal{L}$ -theory  $\mathbb{T}$  = A theory  $\mathbb{T}$  over a **FOLDS** signature  $\mathcal{L}$

$\mathcal{L}$ -equivalence = **FOLDS**  $\mathcal{L}$ -equivalence

**univalent model** = Model of  $\mathbb{T}$  where **FOLDS isomorphism** is equivalent to identity

**UniMod**( $\mathbb{T}$ ) = The type of univalent models



# Main Idea

## Pre-Theorem

For any *univalent models*  $\mathcal{M}$  and  $\mathcal{N}$  of an  $\mathcal{L}$ -theory  $\mathbb{T}$ , the type of  $\mathcal{L}$ -equivalences  $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$  is *equivalent* to  $\mathcal{M} =_{\mathbf{UniMod}(\mathbb{T})} \mathcal{N}$ .

$\mathcal{L}$ -theory  $\mathbb{T}$  = A theory  $\mathbb{T}$  over a **FOLDS** signature  $\mathcal{L}$

$\mathcal{L}$ -equivalence = **FOLDS**  $\mathcal{L}$ -equivalence

**univalent model** = Model of  $\mathbb{T}$  where **FOLDS isomorphism** is equivalent to identity

**UniMod**( $\mathbb{T}$ ) = The type of univalent models

**The Setting:** Two-Level Type Theory (2LTT)

## 2LTT (Annenkov, Capriotti, Kraus, 2017)

2LTT internalizes the set-theoretic semantics of HoTT.

## 2LTT (Annenkov, Capriotti, Kraus, 2017)

2LTT internalizes the set-theoretic semantics of HoTT.

One level of 2LTT is a **fibrant** fragment of **fibrant types** which consists of  $\Pi, \Sigma, +, \mathbf{1}, \mathbf{0}, \mathbb{N}$ , intensional  $=$ , propositional truncation  $\| - \|$  and a hierarchy of univalent universes  $\mathcal{U}$ .

## 2LTT (Annenkov, Capriotti, Kraus, 2017)

2LTT internalizes the set-theoretic semantics of HoTT.

One level of 2LTT is a **fibrant** fragment of **fibrant types** which consists of  $\Pi, \Sigma, +, \mathbf{1}, \mathbf{0}, \mathbb{N}$ , intensional  $=$ , propositional truncation  $\| - \|$  and a hierarchy of univalent universes  $\mathcal{U}$ .

The other level of 2LTT is the **strict** fragment of **pretypes** which consists of  $+^s, \mathbf{0}^s, \mathbb{N}^s$ , a strict equality  $\equiv$  with UIP and function extensionality, a hierarchy of strict universes  $\mathcal{U}^s$ . It shares the type constructors  $\Pi, \Sigma, \mathbf{1}$  with the fibrant fragment.

## 2LTT (Annenkov, Capriotti, Kraus, 2017)

2LTT internalizes the set-theoretic semantics of HoTT.

One level of 2LTT is a **fibrant** fragment of **fibrant types** which consists of  $\Pi, \Sigma, +, \mathbf{1}, \mathbf{0}, \mathbb{N}$ , intensional  $=$ , propositional truncation  $\| - \|$  and a hierarchy of univalent universes  $\mathcal{U}$ .

The other level of 2LTT is the **strict** fragment of **pretypes** which consists of  $+^s, \mathbf{0}^s, \mathbb{N}^s$ , a strict equality  $\equiv$  with UIP and function extensionality, a hierarchy of strict universes  $\mathcal{U}^s$ . It shares the type constructors  $\Pi, \Sigma, \mathbf{1}$  with the fibrant fragment.

The rules for the type constructors are the usual ones, and we also have a rule that allows us to consider any fibrant type as a pretype, i.e. the fibrant universes  $\mathcal{U}$  can be thought of as subuniverses of  $\mathcal{U}^s$ , as well as rules that ensure that  $\Sigma$  and  $\Pi$  preserve fibrancy, and that the fibrant universes are closed under strict isomorphism.

## s-categories

For a pretype  $X$ , we can write  $\text{isfibrant}(X)$  for the pretype  $\sum_{Y:\mathcal{U}} (Y \equiv X)$ .

### Definition (Definition 27, 2LTT)

A pretype  $A$  is **cofibrant** if for any fibration  $p : X \rightarrow Y$ , the induced map  $(A \rightarrow X) \rightarrow (A \rightarrow Y)$  is a fibration.

### Definition (Definition 7, 2LTT)

A **s-category** is given by the following data

- 1 A pretype  $\mathcal{C}$  of *objects*
- 2 For each  $x, y : \mathcal{C}$  a pretype  $\mathcal{C}(x, y)$  of *arrows*
- 3 For each  $x : \mathcal{C}$  an arrow  $1 : \mathcal{C}(x, x)$
- 4 A *composition* operation  $\circ : \mathcal{C}(y, z) \rightarrow \mathcal{C}(x, y) \rightarrow \mathcal{C}(x, z)$  that is strictly associative and for which  $1_x$  is a strict left and right unit.

A s-category **cofibrant** if its pretypes of objects and arrows are cofibrant.

# FOLDS (First-Order Logic with Dependent Sorts)

Invented by Makkai in his 1995 paper.

# FOLDS (First-Order Logic with Dependent Sorts)

Invented by Makkai in his 1995 paper.

The **signatures**  $\mathcal{L}$  of FOLDS are (cofibrant) inverse categories with finite fan-out and of finite height.



# FOLDS (First-Order Logic with Dependent Sorts)

Invented by Makkai in his 1995 paper.

The **signatures**  $\mathcal{L}$  of FOLDS are (cofibrant) inverse categories with finite fan-out and of finite height.

The **contexts** are finite functors  $\Gamma : \mathcal{L} \rightarrow \mathbf{Set}$  and formulas, sentences, sequents etc. in context are defined inductively in the usual way, taking a bit of care with the binding of variables.

# FOLDS (First-Order Logic with Dependent Sorts)

Invented by Makkai in his 1995 paper.

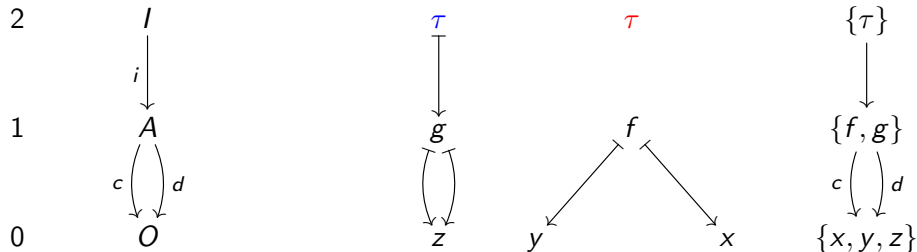
The **signatures**  $\mathcal{L}$  of FOLDS are (cofibrant) inverse categories with finite fan-out and of finite height.

The **contexts** are finite functors  $\Gamma : \mathcal{L} \rightarrow \mathbf{Set}$  and formulas, sentences, sequents etc. in context are defined inductively in the usual way, taking a bit of care with the binding of variables.

An  $\mathcal{L}$ -**theory**  $\mathbb{T}$  is a pretype of  $\mathcal{L}$ -sentences.

# An example

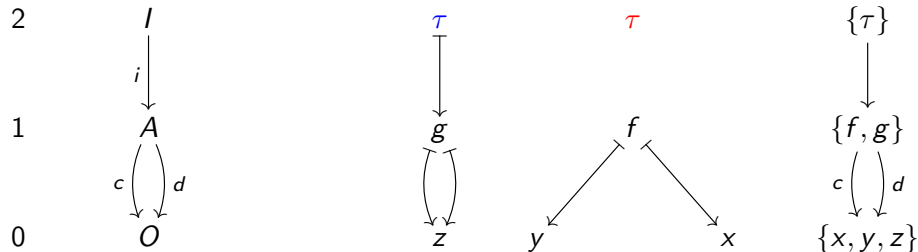
$\mathcal{L}_{rg} \xrightarrow{\Gamma} \mathbf{Set}$



$$di = ci$$

# An example

$\mathcal{L}_{rg} \xrightarrow{\Gamma} \mathbf{Set}$

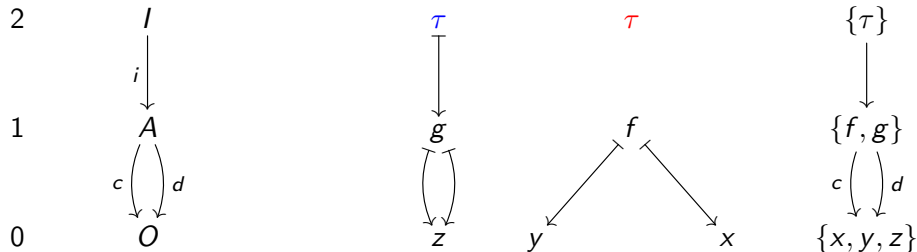


$$di = ci$$

$$\Gamma = x, y, z: O, f: A(x, y), g: A(z, z), \tau: I(g, z)$$

# An example

$\mathcal{L}_{\text{rg}}$   $\xrightarrow{\Gamma}$  **Set**



$$di = ci$$

$$\Gamma = x, y, z: O, f: A(x, y), g: A(z, z), \tau: I(g, z)$$

$$\mathbf{Form}(x: O) \quad \forall g: A(z, z). \exists \tau: I(g, z). \tau \sim \forall g: A(z, z). I(g, z)$$

## Some terminology and notation

$$r(K) \quad \mathcal{L} \longleftarrow K // \mathcal{L} \quad \partial K = \mathcal{L}(K, -)$$

$$n = H(\mathcal{L})$$

 $R$  $i \downarrow$ 

$n - 1$

 $A$  $\wedge$  $\vee$ 

$m$

 $K$  $\downarrow$  $\downarrow$ 

$1$

 $X$ 

$X \leq K$

$\mathcal{L} \leq^K, \mathcal{L} <^K, \dots$

$0$

 $O$

# Semantics of FOLDS in 2LTT

# Semantics of FOLDS in 2LTT

We want to define a type of  $\mathcal{L}$ -structures **Struc**( $\mathcal{L}$ ).



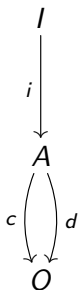
# Semantics of FOLDS in 2LTT

We want to define a type of  $\mathcal{L}$ -structures **Struc**( $\mathcal{L}$ ).

$$\mathcal{L}_{\text{rg}}$$


# Semantics of FOLDS in 2LTT

We want to define a type of  $\mathcal{L}$ -structures **Struc**( $\mathcal{L}$ ).

 $\mathcal{D}(\mathcal{L}_{rg})$  $\mathcal{L}_{rg}$ 

# Semantics of FOLDS in 2LTT

We want to define a type of  $\mathcal{L}$ -structures **Struc**( $\mathcal{L}$ ).

$\mathcal{D}(\mathcal{L}_{rg})$

$\mathcal{L}_{rg}$



$\sum_{O: \mathcal{U}} \dots$

# Semantics of FOLDS in 2LTT

We want to define a type of  $\mathcal{L}$ -structures **Struc**( $\mathcal{L}$ ).

$\mathcal{D}(\mathcal{L}_{rg})$

$\mathcal{L}_{rg}$

$\dots \sum_{A: O \times O \rightarrow \mathcal{U}} \dots$

$\sum_{O: \mathcal{U}} \dots$



# Semantics of FOLDS in 2LTT

We want to define a type of  $\mathcal{L}$ -structures **Struc**( $\mathcal{L}$ ).

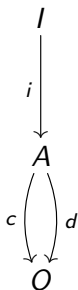
$\mathcal{D}(\mathcal{L}_{rg})$

$\mathcal{L}_{rg}$

$$\dots \left( \sum_{x: O} A(x, x) \right) \rightarrow U$$

$$\dots \sum_{A: O \times O \rightarrow U} \dots$$

$$\sum_{O: U} \dots$$



# Semantics of FOLDS in 2LTT

We want to define a type of  $\mathcal{L}$ -structures **Struc**( $\mathcal{L}$ ).

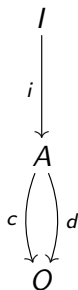
$\mathcal{D}(\mathcal{L}_{rg})$

$$\dots \left( \sum_{x: O} A(x, x) \right) \rightarrow \mathcal{U}$$

$$\dots \sum_{A: O \times O \rightarrow \mathcal{U}} \dots$$

$$\sum_{O: \mathcal{U}} \dots$$

$\mathcal{L}_{rg}$



$\mathfrak{R}(\mathcal{L}_{rg})$

# Semantics of FOLDS in 2LTT

We want to define a type of  $\mathcal{L}$ -structures **Struc**( $\mathcal{L}$ ).

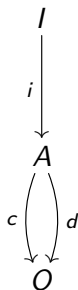
$\mathcal{D}(\mathcal{L}_{rg})$

$$\dots \left( \sum_{x: O} A(x, x) \right) \rightarrow \mathcal{U}$$

$$\dots \sum_{A: O \times O \rightarrow \mathcal{U}} \dots$$

$$\sum_{O: \mathcal{U}} \dots$$

$\mathcal{L}_{rg}$



$\mathfrak{R}(\mathcal{L}_{rg})$

**ReedyFib**( $\mathcal{L}_{rg}, \mathcal{U}$ )

# Semantics of FOLDS in 2LTT

We want to define a type of  $\mathcal{L}$ -structures **Struc**( $\mathcal{L}$ ).

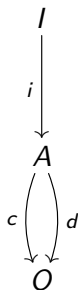
$\mathcal{D}(\mathcal{L}_{rg})$

$$\dots \left( \sum_{x: O} A(x, x) \right) \rightarrow \mathcal{U}$$

$$\dots \sum_{A: O \times O \rightarrow \mathcal{U}} \dots$$

$$\sum_{O: \mathcal{U}} \dots$$

$\mathcal{L}_{rg}$



$\mathfrak{R}(\mathcal{L}_{rg})$

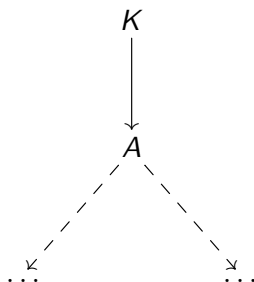
**ReedyFib**( $\mathcal{L}_{rg}, \mathcal{U}$ )

We would like  $\mathcal{D}(\mathcal{L}) \equiv \mathfrak{R}(\mathcal{L})$  but the situation is not that simple.



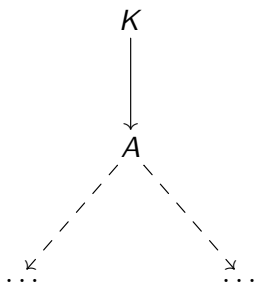
# Semantics of FOLDS in 2LTT

$$\begin{array}{c} F_K \\ \downarrow \\ M_A^F \\ \downarrow K \\ \mathcal{U} \end{array} \quad \equiv \quad \lim \left( A // \mathcal{L} \xrightarrow{\text{cod}} \mathcal{L} \xrightarrow{F} \mathcal{U} \right)$$



# Semantics of FOLDS in 2LTT

$$\begin{array}{c} F_K \\ \downarrow \\ M_A^F \\ \downarrow K \\ \mathcal{U} \end{array} \quad \equiv \quad \lim \left( A // \mathcal{L} \xrightarrow{\text{cod}} \mathcal{L} \xrightarrow{F} \mathcal{U} \right)$$

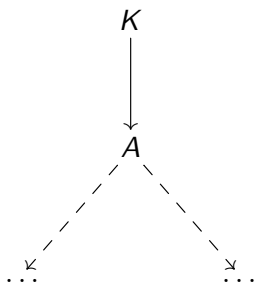


## Theorem

$\mathcal{D}(\mathcal{L}) \simeq \mathfrak{K}(\mathcal{L})$  as  $s$ -categories.

# Semantics of FOLDS in 2LTT

$$\begin{array}{c} F_K \\ \downarrow \\ M_A^F \\ \downarrow K \\ \mathcal{U} \end{array} \quad \equiv \quad \lim \left( A // \mathcal{L} \xrightarrow{\text{cod}} \mathcal{L} \xrightarrow{F} \mathcal{U} \right)$$



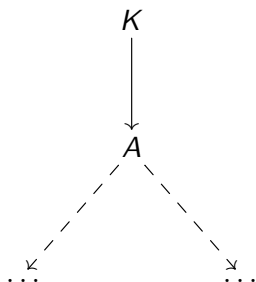
## Theorem

$\mathcal{D}(\mathcal{L}) \simeq \mathfrak{R}(\mathcal{L})$  as  $s$ -categories.

We define the type of  $\mathcal{L}$ -structures as  $\mathbf{Struc}(\mathcal{L}) = \mathcal{D}(\mathcal{L})$  but we will use the equivalence of the above theorem to transfer constructions from  $\mathfrak{R}(\mathcal{L})$ .

# Semantics of FOLDS in 2LTT

$$\begin{array}{c} F_K \\ \downarrow \\ M_A^F \\ \downarrow K \\ \mathcal{U} \end{array} \quad \equiv \quad \lim \left( A // \mathcal{L} \xrightarrow{\text{cod}} \mathcal{L} \xrightarrow{F} \mathcal{U} \right)$$



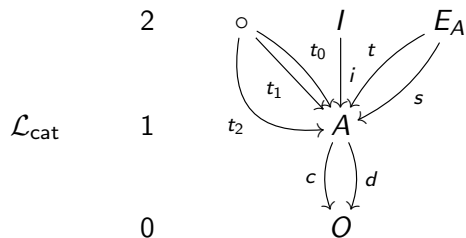
## Theorem

$\mathcal{D}(\mathcal{L}) \simeq \mathfrak{R}(\mathcal{L})$  as  $s$ -categories.

We define the type of  $\mathcal{L}$ -structures as  $\mathbf{Struc}(\mathcal{L}) = \mathcal{D}(\mathcal{L})$  but we will use the equivalence of the above theorem to transfer constructions from  $\mathfrak{R}(\mathcal{L})$ .

Similarly, we denote by  $\mathbf{Mod}(\mathbb{T})$  the type of  $\mathcal{L}$ -structures satisfying all the sentences of  $\mathbb{T}$ .

# The $\mathcal{L}_{\text{cat}}$ -theory $\mathbb{T}_{\text{cat}}$

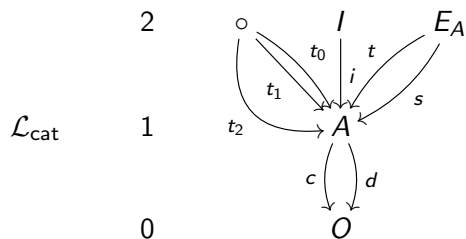


$$dt_0 = dt_2 \quad ct_1 = ct_2 \quad dt_1 = ct_0$$

$$ds = dt \quad cs = ct$$

$$ci = di$$

# The $\mathcal{L}_{\text{cat}}$ -theory $\mathbb{T}_{\text{cat}}$



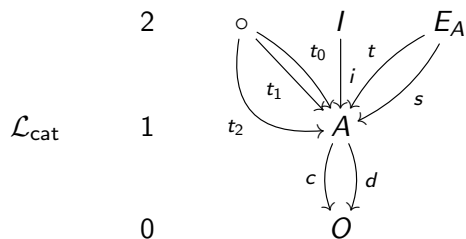
$$dt_0 = dt_2 \quad ct_1 = ct_2 \quad dt_1 = ct_0$$

$$ds = dt \quad cs = ct$$

$$ci = di$$

$\mathbb{T}_{\text{cat}}$  is the  $\mathcal{L}_{\text{cat}}$ -theory with the usual axioms of category theory expressed in relational form using  $E_A$  as the equality on arrows.

# The $\mathcal{L}_{\text{cat}}$ -theory $\mathbb{T}_{\text{cat}}$



$$dt_0 = dt_2 \quad ct_1 = ct_2 \quad dt_1 = ct_0$$

$$ds = dt \quad cs = ct$$

$$ci = di$$

$\mathbb{T}_{\text{cat}}$  is the  $\mathcal{L}_{\text{cat}}$ -theory with the usual axioms of category theory expressed in relational form using  $E_A$  as the equality on arrows.

## Theorem

If  $E_A$  is interpreted as the identity type on  $A$  then

$$\text{Mod}(\mathbb{T}_{\text{cat}}) \simeq \sum_{O: \mathcal{U}} \sum_{A: O \rightarrow O \rightarrow \mathcal{U}} \sum_{\begin{array}{l} o: \prod_{x,y,z: O} A(x,y) \rightarrow A(y,z) \rightarrow A(x,z) \\ l: \prod_{x: O} A(x,x) \end{array}} (\dots)$$

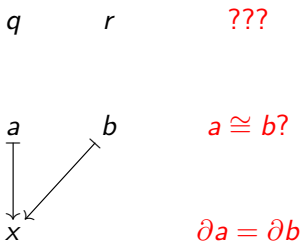
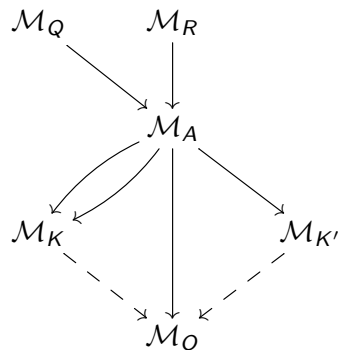
# Generalized Isomorphism?

Let  $\mathcal{M}: \mathbf{Mod}(\mathbb{T})$



# Generalized Isomorphism?

Let  $\mathcal{M}: \mathbf{Mod}(\mathbb{T})$



# FOLDS $\mathcal{L}$ -equivalence

$$\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N} \quad =_{\text{df}} \quad \sum_{\langle P, m, n \rangle} \left( \begin{array}{c} \mathcal{P} \\ \swarrow \quad \searrow \\ m \quad \text{v.s.} \quad \text{v.s.} \quad n \\ \swarrow \quad \searrow \\ \mathcal{M} \quad \quad \quad \mathcal{N} \end{array} \right)$$

# FOLDS $\mathcal{L}$ -equivalence

$$\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N} \quad =_{\text{df}} \quad \sum_{\langle P, m, n \rangle} \left( \begin{array}{c} P \\ \swarrow \quad \searrow \\ m \quad \text{v.s.} \quad n \\ \swarrow \quad \searrow \\ \mathcal{M} \quad \quad \mathcal{N} \end{array} \right)$$

## Definition

$\text{isvery surjective}(m) =_{\text{df}} \prod_{K: \mathcal{L}} \text{is surjective}(P_K \rightarrow M_K^P \times_{M_K^M} \mathcal{M}_K)$

## Theorem (Makkai, 1995)

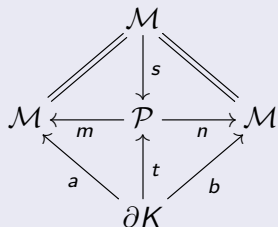
If  $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$  then  $\mathcal{M} \models \phi \Leftrightarrow \mathcal{N} \models \phi$

# FOLDS pre-isomorphism

Fix  $\mathcal{M} : \mathbf{Mod}(\mathbb{T})$  and  $K : \mathcal{L}$ . Let  $a, b : \mathcal{M}_K$ .

## Definition (Pre-isomorphism)

A **pre-isomorphism** from  $a$  to  $b$  is given by the following cospan of spans



where  $\langle m, \mathcal{P}, n \rangle$  is a FOLDS equivalence.

## Theorem (Makkai, 1995)

If  $a$  is pre-isomorphic to  $b$  then  $\mathcal{M} \models \phi[a] \Leftrightarrow \mathcal{M} \models \phi[b]$

# FOLDS isomorphism

Fix  $\mathcal{M} : \mathbf{Mod}(\mathbb{T})$  and  $K : \mathcal{L}$ . Let  $a, b : \mathcal{M}_K$ .

## Definition (FOLDS isomorphism)

A **FOLDS isomorphism** is a pre-isomorphism  $\langle m, \mathcal{P}, n \rangle$  such that:

- ① For any  $f : K \rightarrow A$  we have

$$\begin{array}{ccccc}
 \mathcal{M} & \xleftarrow{m} & \mathcal{P} & \xrightarrow{n} & \mathcal{M} \\
 & \swarrow & \uparrow & \searrow & \\
 & [\text{id}, \mathcal{M}_f(a)] & \mathcal{M} \amalg \partial A & [\text{id}, \mathcal{M}_f(b)] & \\
 & & \sim & & 
 \end{array}$$

- ② For any  $A > K$  we have  $\mathcal{M}_A \times_{M_A^{\mathcal{M}}} M_A^{\mathcal{P}} \xleftarrow{\sim} \mathcal{P}_A \xrightarrow{\sim} \mathcal{M}_A \times_{M_A^{\mathcal{M}}} M_A^{\mathcal{P}}$

We write  $a \cong b$  for the type of FOLDS isomorphisms.

# FOLDS isomorphism

Fix  $\mathcal{M} : \mathbf{Mod}(\mathbb{T})$  and  $K : \mathcal{L}$ . Let  $a, b : \mathcal{M}_K$ .

## Definition (FOLDS isomorphism)

A **FOLDS isomorphism** is a pre-isomorphism  $\langle m, \mathcal{P}, n \rangle$  such that:

- ① For any  $f : K \rightarrow A$  we have

$$\begin{array}{ccccc}
 \mathcal{M} & \xleftarrow{m} & \mathcal{P} & \xrightarrow{n} & \mathcal{M} \\
 & \swarrow [\text{id}, \mathcal{M}_f(a)] & \uparrow \sim & \searrow [\text{id}, \mathcal{M}_f(b)] & \\
 & & \mathcal{M} \amalg \partial A & & 
 \end{array}$$

- ② For any  $A > K$  we have  $\mathcal{M}_A \times_{M_A^{\mathcal{M}}} M_A^{\mathcal{P}} \xleftarrow{\sim} \mathcal{P}_A \xrightarrow{\sim} \mathcal{M}_A \times_{M_A^{\mathcal{M}}} M_A^{\mathcal{P}}$

We write  $a \cong b$  for the type of FOLDS isomorphisms.

## Lemma

$\cong : \mathcal{M}_K \rightarrow \mathcal{M}_K \rightarrow \mathcal{U}$  is reflexive.

# FOLDS isomorphism

Fix  $\mathcal{M} : \mathbf{Mod}(\mathbb{T})$  and  $K : \mathcal{L}$ . Let  $a, b : \mathcal{M}_K$ .

## Definition (FOLDS isomorphism)

A **FOLDS isomorphism** is a pre-isomorphism  $\langle m, \mathcal{P}, n \rangle$  such that:

- ① For any  $f : K \rightarrow A$  we have

$$\begin{array}{ccccc}
 \mathcal{M} & \xleftarrow{m} & \mathcal{P} & \xrightarrow{n} & \mathcal{M} \\
 & \swarrow [\text{id}, \mathcal{M}_f(a)] & \uparrow \sim & \searrow [\text{id}, \mathcal{M}_f(b)] & \\
 & & \mathcal{M} \amalg \partial A & & 
 \end{array}$$

- ② For any  $A > K$  we have  $\mathcal{M}_A \times_{M_A^{\mathcal{M}}} M_A^{\mathcal{P}} \xleftarrow{\sim} \mathcal{P}_A \xrightarrow{\sim} \mathcal{M}_A \times_{M_A^{\mathcal{M}}} M_A^{\mathcal{P}}$

We write  $a \cong b$  for the type of FOLDS isomorphisms.

## Lemma

$\cong : \mathcal{M}_K \rightarrow \mathcal{M}_K \rightarrow \mathcal{U}$  is reflexive.

## Corollary

$\text{idtoiso}_{a,b} : a =_{\mathcal{M}_K} b \rightarrow a \cong b$

# Univalent Models

Fix  $\mathcal{M}: \mathbf{Mod}(\mathbb{T})$

## Definition (Univalence for $\mathcal{M}$ )

<b><math>K</math>-univalent</b>	$\text{univ}_K(\mathcal{M}) =_{\text{df}} \prod_{a,b: \mathcal{M}_K} \text{isequiv}(\text{idtoiso}_{a,b})$
<b><math>m</math>-univalent</b>	$\text{univ}_m(\mathcal{M}) =_{\text{df}} \prod_{K: \mathcal{L}^{\geq m}} \text{univ}_K(\mathcal{M})$
<b>univalent</b>	$\text{univ}(\mathcal{M}) =_{\text{df}} \prod_{K: \mathcal{L}} \text{univ}_K(\mathcal{M})$

## Definition (Type of Univalent Models)

$$\mathbf{UniMod}_m(\mathbb{T}) =_{\text{df}} \sum_{\mathcal{M}: \mathbf{Mod}(\mathbb{T})} \text{univ}_m(\mathcal{M})$$

$$\mathbf{UniMod}(\mathbb{T}) =_{\text{df}} \sum_{\mathcal{M}: \mathbf{Mod}(\mathbb{T})} \text{univ}(\mathcal{M})$$



# Some Results

## Theorem

*If  $r(K) = H(\mathcal{L})$  then  $a \cong b \simeq \mathbf{1}$*

# Some Results

## Theorem

*If  $r(K) = H(\mathcal{L})$  then  $a \cong b \simeq \mathbf{1}$*

## Corollary

*If  $r(K) = H(\mathcal{L})$  then  $\text{univ}_K(\mathcal{M}) \simeq \text{isprop}(\mathcal{M}_K)$*

# Some Results

## Theorem

If  $r(K) = H(\mathcal{L})$  then  $a \cong b \simeq \mathbf{1}$

## Corollary

If  $r(K) = H(\mathcal{L})$  then  $\text{univ}_K(\mathcal{M}) \simeq \text{isprop}(\mathcal{M}_K)$

## Theorem

Let  $H(\mathcal{L}) \geq n \geq m$ ,  $K: \mathcal{L}^{\equiv n}$  and  $\mathcal{M}: \mathbf{UniMod}_m(\mathbb{T})$ . Then  $\mathcal{M}_K$  is an  $m$ -type.

## Some Results

### Theorem

If  $r(K) = H(\mathcal{L})$  then  $a \cong b \simeq \mathbf{1}$

### Corollary

If  $r(K) = H(\mathcal{L})$  then  $\text{univ}_K(\mathcal{M}) \simeq \text{isprop}(\mathcal{M}_K)$

### Theorem

Let  $H(\mathcal{L}) \geq n \geq m$ ,  $K: \mathcal{L}^{\leq n}$  and  $\mathcal{M}: \mathbf{UniMod}_m(\mathbb{T})$ . Then  $\mathcal{M}_K$  is an  $m$ -type.

### Theorem

Let  $\mathbb{T}_{\text{cat}}$  be the  $\mathcal{L}_{\text{cat}}$ -theory of categories. Then we have:

$$\mathbf{UniMod}_1(\mathbb{T}_{\text{cat}}) \simeq \mathbf{PreCat}$$

$$\mathbf{UniMod}(\mathbb{T}_{\text{cat}}) \simeq \mathbf{UniCat}$$

# A Higher Structure Identity Principle

We began with:

## Theorem

For any *univalent models*  $\mathcal{M}$  and  $\mathcal{N}$  of an  $\mathcal{L}$ -theory  $\mathbb{T}$ , the type of  $\mathcal{L}$ -equivalences  $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$  is equivalent to  $\mathcal{M} =_{\mathbf{UniMod}(\mathbb{T})} \mathcal{N}$ .

# A Higher Structure Identity Principle

We began with:

## Theorem

For any *univalent models*  $\mathcal{M}$  and  $\mathcal{N}$  of an  $\mathcal{L}$ -theory  $\mathbb{T}$ , the type of  $\mathcal{L}$ -equivalences  $\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N}$  is *equivalent* to  $\mathcal{M} =_{\mathbf{UniMod}(\mathbb{T})} \mathcal{N}$ .

And now we can obtain the precise version:

## Theorem (A Higher Structure Identity Principle, in progress)

For any  $\mathcal{M}, \mathcal{N} : \mathbf{UniMod}(\mathbb{T})$  for a *FOLDS*  $\mathcal{L}$ -theory  $\mathbb{T}$  we have

$$\mathcal{M} \simeq_{\mathcal{L}} \mathcal{N} \quad \leftrightarrow \quad \mathcal{M} =_{\mathbf{UniMod}(\mathbb{T})} \mathcal{N}$$

# Thank you