The join construction

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Table: The homotopy interpretation (Awodey and Warren, Voevodsky)

Type theory	*Homotopy theory*
Types	Spaces
Dependent types	Fibrations
Terms	Points
Dependent pair type	Total space
Identity type	Path fibration

Why Homotopy Type Theory?

- The univalence axiom reinforces the connection between dependent type theory and homotopy theory.
- Isomorphisms between structures identify them, and all constructions have to respect those.
- Constructions in homotopy type theory apply to all its models.
- Paths are primitive in HoTT, which allows for a 'synthetic' approach to homotopy theory that makes many constructions more elegant.
- Homotopy Type Theory is constructive, admits a computational interpretation, but is still compatible with classical reasoning.
- Large scale computer formalization of mathematics becomes feasible with HoTT... **and it is indispensable!**

What are some challenges in HoTT?

- All constructions have to be homotopy invariant.
- Some spaces (like the spheres, the real and complex projective spaces) have been defined as higher inductive types, while for many familiar spaces (e.g. the Grassmannians) it is an open problem to define them in HoTT.
- We only know how to interpret some higher inductive types (e.g. pushouts) in some models.

Therefore it would be interesting to know what we get from just pushouts:

- Can we quotient a type by an equivalence relation?
- Can we define the homotopy image of a map?
- Can we 'truncate' types, so that all homotopy groups above level n become trivial?

- Dependent types

$$x : A \vdash P(x) : \mathsf{Type}$$

- Dependent functions

$$\frac{x : A \vdash f(x) : P(x)}{\vdash \lambda x.\, f(x) : \prod_{(x:A)} P(x)}$$

- Dependent pairs

$$\frac{\vdash a : A \qquad \vdash p : P(a)}{\vdash (a, p) : \sum_{(x:A)} P(x)}$$

- The universe is a type $\mathcal{U}$ with a type family

$$X : \mathcal{U} \vdash X : \mathsf{Type}$$

that contains $\mathbf{0}$, $\mathbf{1}$, and $\mathbb{N}$, and is closed under the type-forming operations Π, Σ, and Id.

For each type X there is an family of types

$$x, y : X \vdash \mathsf{Id}_X(x, y)$$

with constructor

$$\frac{}{x : X \vdash \mathsf{refl}_x : \mathsf{Id}_X(x, y)}$$

The elimination principle for Id states that for any

$$x, y : X, p : \mathsf{Id}_X(x, y) \vdash P(x, y, p) : \mathsf{Type}$$

one has:

$$\frac{x : X \vdash t : P(x, x, \mathsf{refl}_x)}{x, y : X, p : \mathsf{Id}_X(x, y) \vdash \mathsf{ind}_{\mathsf{Id}_X}(x, y, p, t) : P(x, y, p)}$$

Definition (Voevodsky)

A type X is said to be contractible if there is a term of type

$$\mathrm{isContr}(X) :\equiv \sum_{(x:X)} \prod_{(y:X)} \mathrm{Id}_X(x, y).$$

Theorem

For any type X and any $x : X$, the type

$$\sum_{(y:X)} \mathrm{Id}_X(x, y)$$

is contractible.

Definition (Voevodsky)

A map $f : X \to Y$ is said to an equivalence if there is a term of type

$$\mathsf{isEquiv}(f) :\equiv \prod_{(y:Y)} \mathsf{isContr}\Big(\sum_{(x:X)} \mathsf{Id}_Y(f(x), y)\Big)$$

We write $X \simeq Y$ for the type $\sum_{(f:X\to Y)} \mathsf{isContr}(f)$.

Definition

Let $f, g : \prod_{(x:X)} P(x)$ be two dependent functions. We define the type of homotopies from f to g

$$f \sim g :\equiv \prod_{(x:X)} \mathsf{Id}_{P(x)}(f(x), g(x)).$$

Theorem

A map is an equivalence if and only if it has both a left and a right inverse (up to homotopy).

Definition

A type P is said to be a proposition if its identity types are contractible.

Theorem

For any map $f : A \to B$ the following are equivalent:

1. *The fibers of f are mere propositions*
2. *The canonical map*

$$\mathsf{ap}_f : \prod_{(x,y:A)} \mathsf{Id}_A(x,y) \to \mathsf{Id}_B(f(x), f(y))$$

   is an equivalence.

We call such maps embeddings.

Definition

We say that a type X is a set if its identity types are propositions

Theorem

The type of natural numbers is a set.

Definition (Voevodsky)

- We say that a type is (-2)-truncated if it is contractible.
- We say that a type is $(n+1)$-truncated if its identity types are n-truncated.

The univalence axiom (Voevodsky)

The canonical map

$$\mathsf{Id}_{\mathcal{U}}(X, Y) \to (X \simeq Y)$$

is an equivalence for every $X, Y : \mathcal{U}$.

Consequences of the univalence axiom

- It challenges us to rethink what equality is!
- Function extensionality: for any two dependent functions $f, g : \prod_{(x:X)} P(x)$ the canonical map

  $$\mathrm{Id}(f, g) \to (f \sim g)$$

  is an equivalence.
- Being contractible, an equivalence, and being *n*-truncated are all properties, not structure.
- The universe is closed under limits.
- The universe classifies all maps between small types.
- Isomorphic structures (sets, groups, modules, ...) can be identified.
- Descent for higher inductive types.
- ...

We work in MLTT with a univalent universe $\mathcal{U}$ that is closed under (homotopy) pushouts.

- for every span $A \xleftarrow{f} S \xrightarrow{g} B$ in $\mathcal{U}$ we can form the higher inductive type $A \sqcup^S B : \mathcal{U}$ with constructors

$$\begin{aligned} \mathsf{inl} &: A \to A \sqcup^S B \\ \mathsf{inr} &: B \to A \sqcup^S B \\ \mathsf{glue} &: \textstyle\prod_{(x:S)} \mathsf{inl}(f(x)) = \mathsf{inr}(g(x)) \end{aligned}$$

with the according induction principle.

Can we define the image of a map?

Can we define the n-truncations for any truncation level n and construct their corresponding stable OFSs?

Can we characterize in the language of HoTT those types that are a loop space?

Can we define a notion of ∞-equivalence relation, and an effective quotient operation?

Definition

Let $f : A \to X$ and $g : B \to X$ be maps with a common codomain. The join $f * g$ of maps is defined by first pulling back, and then pushing out the pullback span:

$$
\begin{tikzcd}
\sum_{(a:A)} \sum_{(b:B)} f(a) = g(b) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "\mathrm{inr}"] \arrow[ddr, bend left, "g"] & \\
A \arrow[r, "\mathrm{inl}"'] \arrow[drr, bend right, "f"'] & A *_X B \arrow[dr, dotted, "f * g"'] & \\
& & X
\end{tikzcd}
$$

The join construction

Consider the finite join-powers f^{*n} of maps. This gives rise to a sequence

$$\begin{tikzcd}
A \arrow[r, "\mathsf{inr}"] \arrow[dr, "f"'] & A *_X A \arrow[r, "\mathsf{inr}"] \arrow[d, "f*f"'] & A *_X (A *_X A) \arrow[r, "\mathsf{inr}"] \arrow[dl, "f^{*3}"'] & \cdots \arrow[dll, bend left, "f^{*4}"] \\
& X. & &
\end{tikzcd}$$

Theorem

The colimit $f^{*\infty}$ is an embedding that satisfies the universal property of the image inclusion of f. We get the desired stable orthogonal factorization system.

Theorem (R)

Let $A : \mathcal{U}$ *and let* X *be locally small with respect to* $\mathcal{U}$*, in the sense that for any* $x, y : X$ *there is a type* $x =' y : \mathcal{U}$ *and an equivalence*

$$(x = y) \simeq (x =' y).$$

Then we can construct

- *a type* $\mathrm{im}'(f) : \mathcal{U}$ *and an embedding* $i'_f : \mathrm{im}'(f) \to X$
- $i_f : \mathrm{im}'(f) \to X$ *satisfies the universal property of the image inclusion of* f*.*

Corollary

Any connected component of the universe is essentially small

Theorem (R)

We can define, for every $n \geq -2$, an n-truncation operation

$$\|\_\|_n : \mathcal{U} \to \mathcal{U}$$

and for every $A : \mathcal{U}$ a map

$$\eta^A : A \to \|A\|_n,$$

such that

- *for each $A : \mathcal{U}$ the type $\|A\|_n$ is an n-truncated type*
- *for every n-truncated type B the canonical map*

  $$\_ \circ \eta^A : (\|A\|_n \to B) \to (A \to B)$$

  is an equivalence.

Proof.

By induction on $n : \mathbb{N}_{-2}$.

- In the base case we simply take $A \mapsto \mathbf{1}$.
- Given an n-truncation operation with the universal property, define
$$\|A\|_{n+1} :\equiv \mathrm{im}\Big(x \mapsto (\lambda y.\, \|x = y\|_n)\Big).$$

More details in ArXiv:1701.07538 □