Constructing Categories of Corelations

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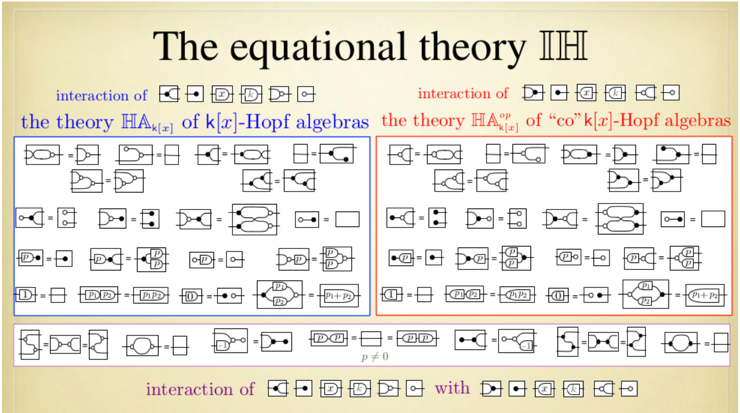
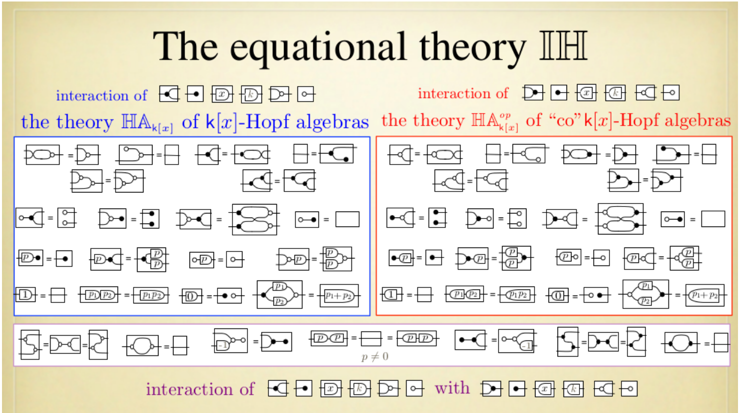
Octoberfest
Carnegie Mellon University
28 October 2017

I. Motivation



The equational theory $\mathbb{IH}$
interaction of
the theory $\mathbb{HA}_{\mathsf{k}[x]}$ of $\mathsf{k}[x]$-Hopf algebras
interaction of
the theory $\mathbb{HA}^{op}_{\mathsf{k}[x]}$ of “co” $\mathsf{k}[x]$-Hopf algebras
$p \neq 0$
interaction of
with



This is a presentation of the category $\mathrm{LinRel}_{\mathsf{k}(x)}$.

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$$\begin{tikzcd}
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Linear relations interpret diagrams of linear maps

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This leads to a presentation of LinRel.

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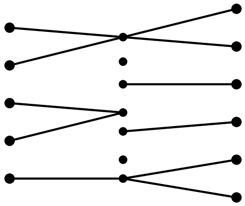
However, combining systems using colimits indiscriminately accumulates information.

Consider cospans in FinSet.



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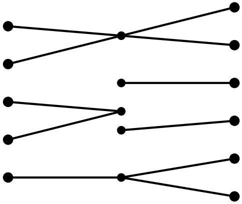

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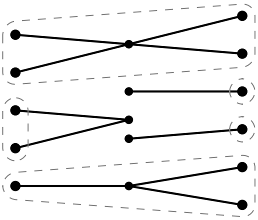
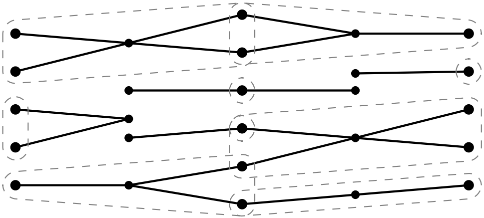


$$X \longrightarrow N \longleftarrow Y$$

If we think about these as circuits, all we care about is the induced equivalence relation on $X + Y$.



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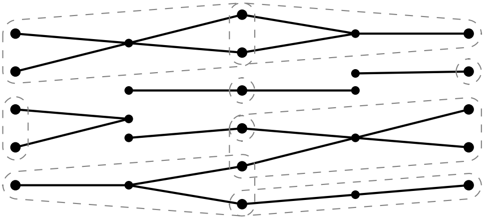

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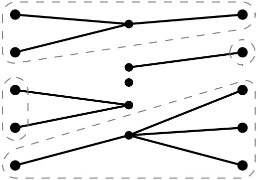
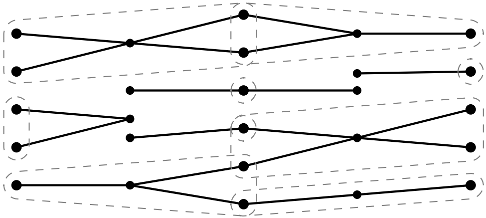
$= \quad X$



Z



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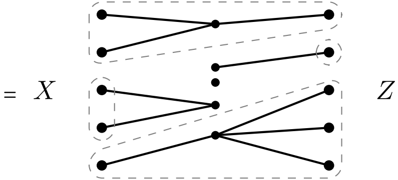
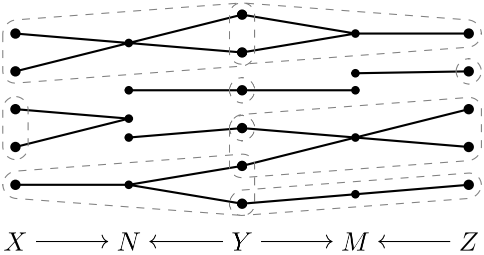
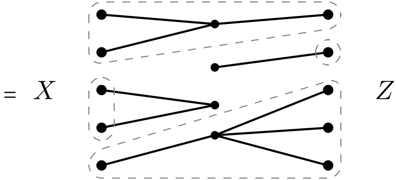


Cospans accumulate internal structure (witnesses for ‘empty equivalence classes’).





Cospans accumulate internal structure (witnesses for ‘empty equivalence classes’).
Corelations forget this.

Factorisation hides internal structure.

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A **factorisation system** $(\mathcal{E}, \mathcal{M})$ comprises subcategories $\mathcal{E}$, $\mathcal{M}$ such that

- $\mathcal{E}$ and $\mathcal{M}$ contain all isomorphisms
- every f admits factorisation $f = m \circ e$.
- we have the universal property:

$$\begin{tikzcd} {} \arrow[r, "e"] \arrow[d, "u"'] & {} \arrow[r, "m"] \arrow[d, dashed, "\exists! s"] & {} \arrow[d, "v"] \\ {} \arrow[r, "e'"'] & {} \arrow[r, "m'"'] & {} \end{tikzcd}$$

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For example, epi–mono factorisation systems (like in FinSet).

A **corelation** is an equivalence class of cospans, where two cospans are equivalent if

$$\begin{tikzcd}
 & N & \\
 & N' \arrow[u, "m"] & \\
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When $\mathcal{M}$ is stable under pushout, composition by pushout defines a category $\mathrm{Corel}(\mathcal{C})$.

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So we claim:

I. Corelations model system interconnection and

II. A universal property is useful for computing presentations.

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So we claim:

I. Corelations model system interconnection and

II. A universal property is useful for computing presentations.

Does this universal construction generalise to other corelation categories?

II. A Universal Property for Corelations

$$\mathrm{Cospan}(\mathcal{C}) \longrightarrow \mathrm{Corel}(\mathcal{C})$$

$$
\begin{tikzcd}
? + ?^{\mathrm{op}} \arrow[r] \arrow[d] & \mathrm{Span}(?) \arrow[d] \\
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A functor $\mathrm{Span}(\mathcal{C}) \to \mathrm{Corel}(\mathcal{C})$ does not in general exist. Under what conditions might it exist?

Define a map $\mathrm{Span}_{\mathcal{C}}(\mathcal{A}) \to \mathrm{Corel}(\mathcal{C})$ by taking pushouts.

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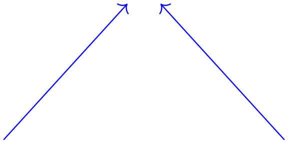

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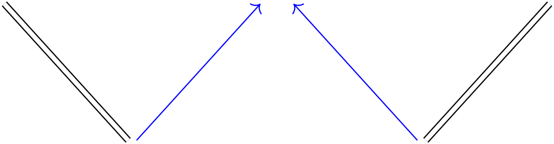

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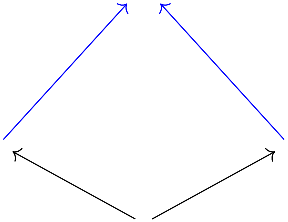

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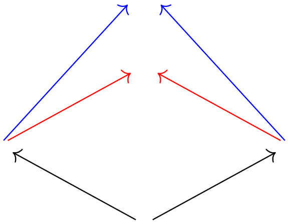

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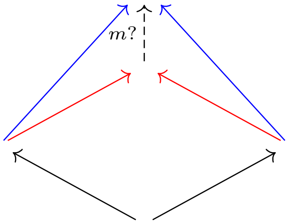

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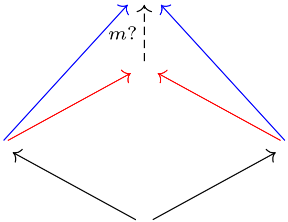

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Call this the **pullback–pushout property** (with respect to $\mathcal{M}$).

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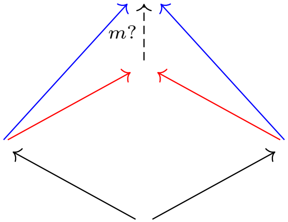

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> When $\mathcal{A}$ obeys the pullback–pushout property, then there exists a functor $\mathrm{Span}_{\mathcal{C}}(\mathcal{A}) \to \mathrm{Corel}(\mathcal{C})$.

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Then we have a pushout square in $\mathbf{Cat}$:

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\mathcal{M} +_{|\mathcal{M}|} \mathcal{M}^{\mathrm{op}} \arrow[r] \arrow[d] & \mathrm{Span}_{\mathcal{C}}(\mathcal{M}) \arrow[d] \\
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Theorem: generalising $\mathcal{M}$

Suppose $\mathcal{C}$ has

- pushouts and pullbacks
- a factorisation system with $\mathcal{M} \subseteq$ monos, stable under pushout
- a subcategory $\mathcal{A} \supseteq \mathcal{M}$, stable under pullback, obeying the pullback–pushout property.

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Corollary: abelian case

Let $\mathcal{C}$ be an abelian category. This has a (co)stable epi–mono factorisation system.

The theorem can also be extended to monoidal categories, by requiring that the monoidal product preserve pushouts in $\mathcal{C}$ and pullbacks in $\mathcal{A}$, and that $\mathcal{M}$ and $\mathcal{A}$ are closed under the monoidal product.

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Examples

Corelations (Equivalence relations):

$$\begin{array}{ccc} \mathrm{Inj} +_\bullet \mathrm{Inj}^{\mathrm{op}} & \longrightarrow & \mathrm{Span}(\mathrm{Inj}) \\ \downarrow & & \downarrow \\ \mathrm{Cospan}(\mathrm{FinSet}) & \longrightarrow & \mathrm{Corel}(\mathrm{FinSet}) \end{array}$$

Partial equivalence relations:

$$\begin{array}{ccc} \mathrm{PInj} +_\bullet \mathrm{PInj}^{\mathrm{op}} & \longrightarrow & \mathrm{Span}(\mathrm{PInj}) \\ \downarrow & & \downarrow \\ \mathrm{Cospan}(\mathrm{ParFunc}) & \longrightarrow & \mathrm{PER} \cong \mathrm{Corel}(\mathrm{ParFunc}) \end{array}$$

Examples

Linear relations:

$$\begin{tikzcd} \mathrm{Vect} +_\bullet \mathrm{Vect}^{\mathrm{op}} \arrow[r] \arrow[d] & \mathrm{Span}(\mathrm{Vect}) \arrow[d] \\ \mathrm{Cospan}(\mathrm{Vect}) \arrow[r] & \mathrm{Corel}(\mathrm{Vect}) \cong \mathrm{LinRel} \end{tikzcd}$$

Discrete time,
linear,
time-invariant,
dynamical
systems over k:

$$\begin{tikzcd} \mathrm{SpltM} +_\bullet \mathrm{SpltM}^{\mathrm{op}} \arrow[r] \arrow[d] & \mathrm{Span}(\mathrm{SpltM}) \arrow[d] \\ \mathrm{Cospan}(\mathrm{Mat}_{\mathsf{k}[s,s^{-1}]}) \arrow[r] & \mathrm{Corel}(\mathrm{Mat}_{\mathsf{k}[s,s^{-1}]}) \end{tikzcd}$$

Examples

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This property is obeyed by the cofree comonad on the double finite power set functor, which has been used to model logic programs.

Theorem: dual case

Suppose a category $\mathcal{C}$ has

- pushouts and pullbacks
- a factorisation system $(\mathcal{E}, \mathcal{M})$ with $\mathcal{E} \subseteq$ epis, stable under pullback
- such that $\mathcal{E}$ obeys the pullback–pushout property.

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\end{tikzcd}$$

Non-example: Relations

Surj does not obey pushout–pullback property.

$$\begin{tikzcd}
\mathrm{Surj} + \mathrm{Surj}^{\mathrm{op}} \arrow[r] \arrow[d] & \mathrm{Span}(\mathrm{FinSet}) \arrow[d] \\
\mathrm{Cospan}(\mathrm{Surj}) \arrow[r, squiggly] & \mathrm{Rel}(\mathrm{FinSet})
\end{tikzcd}$$

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$$\begin{tikzcd}
\mathrm{Surj} + \mathrm{Surj}^{\mathrm{op}} \arrow[r] \arrow[d] & \mathrm{Span}(\mathrm{FinSet}) \arrow[d] \\
\mathrm{Cospan}(\mathrm{Surj}) \arrow[r, squiggly, red] & \mathrm{Rel}(\mathrm{FinSet})
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$$\begin{tikzcd}
& 3 \arrow[dl] \arrow[dr] & \\
2 & & 2
\end{tikzcd}$$

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Not an epi!

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 & 4 \arrow[dl] \arrow[dr] & \\
2 \arrow[dr] & & 2 \arrow[dl] \\
 & 1 &
\end{tikzcd}$$

Not an epi! (We cannot construct $\mathrm{Rel} = \mathrm{Rel}(\mathrm{FinSet})$ as a pushout.)

To recap:

I. Corelations model system interconnection

II. Categories of corelations can be constructed as a pushout of span and cospan categories.

III. This helps derive presentations.

I thank **Fabio Zanasi** for collaborating on this work.
Thank you for listening.

For more: http://www.brendanfong.com/