# Formalizing a sophisticated definition

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joint work with

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# Extending functions

### Theorem

*Let $A \subset \mathbb{R}^p$ be a dense subset. Every uniformly continuous function $f\colon A \to \mathbb{R}^q$ extends to a (uniformly) continuous function $\bar{f}\colon \mathbb{R}^p \to \mathbb{R}^q$.*

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For every $x \in \mathbb{R}^p$, choose a sequence $a : \mathbb{N} \to A$ converging to $x$. Uniform continuity of $f$ ensures $f \circ a$ is Cauchy, completeness of $\mathbb{R}^q$ gives a limit $y$. Set $\bar{f}(x) = y$. Then prove continuity of $\bar{f}$.

Example: $(+) : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q} \subset \mathbb{R}$ extends to $(+) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

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But multiplication or inversion are *not* uniformly continuous.

### Theorem

*$A \subset \mathbb{R}^p$ dense subset. If $f : A \to \mathbb{R}^q$ is continuous and*

$$\forall x \in \mathbb{R}^p, \exists y \in \mathbb{R}^q, \forall u : \mathbb{N} \to A, u_n \longrightarrow x \Rightarrow f(u_n) \longrightarrow y$$

*then $f$ extends to a continuous function $\bar{f}\colon \mathbb{R}^p \to \mathbb{R}^q$.*

This applies to multiplication $\mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$.

# A better framework?

In order to handle inversion $\mathbb{Q}^* \to \mathbb{R}^*$ and more general spaces, we want a version where $\mathbb{R}^p$ and $\mathbb{R}^q$ are replaced by general topological spaces $X$ and $Y$.

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We can still say that $f(x)$ converges to $y$ when $x$ tends to $x_0$ while remaining in $A$:

$$\forall W \in \mathcal{N}_y, \exists V \in \mathcal{N}_x, \forall a \in A \cap V, f(a) \in W.$$

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## Theorem

*Let $X$ be a topological space, $A$ a dense subset of $X$, and $f : A \to Y$ a continuous mapping of $A$ into a regular space $Y$. If, for each $x_0 \in X$, $f(x)$ tends to a limit in $Y$ when $x$ tends to $x_0$ while remaining in $A$ then $f$ extends to a continuous map $\bar{f} : X \to Y$*

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Better framework:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, hook, "i"'] & Y \\
X \arrow[ur, dashed, "\exists ? \bar{f}"'] &
\end{tikzcd}$$

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Issue: will we need discussions of

$$\begin{tikzcd}
\mathbb{Q} \arrow[d, hook] & \mathbb{Q}^* \arrow[l, hook'] \arrow[r, "inv"] \arrow[d, hook] & \mathbb{Q}^* \arrow[r, hook] \arrow[d, hook] & \mathbb{Q} \arrow[d, hook] \\
\mathbb{R} & \mathbb{R}^* \arrow[l, hook'] \arrow[r, "inv"] & \mathbb{R}^* \arrow[r, hook] & \mathbb{R}
\end{tikzcd}$$

# Side issue: how to formally refer to $\bar{f}$?

`extend f i de h` where `de` is a proof that $i$ is a dense topological embedding, and `h` is a proof that $f$ admits a limit...?

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A better solution is to define an extension operator $E_i$ by:

$$E_i(f)(x) = \begin{cases} \text{some } y \text{ such that } f(a) \text{ tends to } y \text{ when } a \text{ tends to } x \\ \text{some junk value if no such } y \text{ exists} \end{cases}$$

Density of image of $i$ is used only to ensure $Y$ is non-empty!
Then use `de.extend f`

## Separation issues

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The $i$ map is *not* injective if $\{0\}$ is not closed in $R$.

# Standard solution

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Note: Even in ZFC, if $R$ is already separated, $R' \neq R$.

# Final extension theorem

Return to
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & Y \\ X \arrow[ur, dashed, "\exists ?\bar{f}"'] & \end{tikzcd}$$
Assume $Y$ is not empty so we can define $E_i$ without any assumption on $i$.

## Theorem

*Fix $x_0 \in X$. If, for every $x_1$ in a neighborhood of $x_0$, $f(a)$ tends to a limit in $Y$ when $i(a)$ tends to $x_1$ then $E_i f$ is continuous at $x_0$.*

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*If in addition $x_0 = i(a)$, $f$ is continuous at $a$ and $i$ pulls back the topology of $X$ to the topology of $A$ then $E_i(f)(i(a)) = f(a)$.*

# Side propaganda

The assumption in the first part of the previous theorem can be written as

$$\exists U \in \mathcal{N}_{x_0}, \forall x \in U, \exists y \in Y, \forall W \in \mathcal{N}_y, \exists V \in \mathcal{N}_x,$$
$$\forall a \in A, i(a) \in V \Rightarrow f(a) \in W.$$

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$$\forall a \in A, i(a) \in V \Rightarrow f(a) \in W.$$

In mathlib, the assumption is written as

$$\{x \mid \exists y, f_* i^* \mathcal{N}_x \leq \mathcal{N}_y\} \in \mathcal{N}_{x_0}.$$

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- Big projects are good. Next one?