

Formalizing a sophisticated definition

PATRICK MASSOT (Orsay)

joint work with

KEVIN BUZZARD (IC London) and JOHAN COMMELIN (Freiburg)

Formal Methods in Mathematics – Lean Together
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Extending functions

Theorem

Let $A \subset \mathbb{R}^p$ be a dense subset. Every uniformly continuous function $f: A \rightarrow \mathbb{R}^q$ extends to a (uniformly) continuous function $\bar{f}: \mathbb{R}^p \rightarrow \mathbb{R}^q$.

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For every $x \in \mathbb{R}^p$, choose a sequence $a: \mathbb{N} \rightarrow A$ converging to x . Uniform continuity of f ensures $f \circ a$ is Cauchy, completeness of \mathbb{R}^q gives a limit y . Set $\bar{f}(x) = y$. Then prove continuity of \bar{f} .

Example: $(+): \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{Q} \subset \mathbb{R}$ extends to $(+): \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$.

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But multiplication or inversion are *not* uniformly continuous.

Theorem

$A \subset \mathbb{R}^p$ dense subset. If $f : A \rightarrow \mathbb{R}^q$ is continuous and

$$\forall x \in \mathbb{R}^p, \exists y \in \mathbb{R}^q, \forall u : \mathbb{N} \rightarrow A, u_n \rightarrow x \Rightarrow f(u_n) \rightarrow y$$

then f extends to a continuous function $\bar{f} : \mathbb{R}^p \rightarrow \mathbb{R}^q$.

This applies to multiplication $\mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{R}$.

A better framework?

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We can still say that $f(x)$ converges to y when x tends to x_0 while remaining in A :

$$\forall W \in \mathcal{N}_y, \exists V \in \mathcal{N}_x, \forall a \in A \cap V, f(a) \in W.$$

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Theorem

Let X be a topological space, A a dense subset of X , and $f : A \rightarrow Y$ a continuous mapping of A into a regular space Y . If, for each $x_0 \in X$, $f(x)$ tends to a limit in Y when x tends to x_0 while remaining in A then f extends to a continuous map $\bar{f} : X \rightarrow Y$

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$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ i \downarrow & \nearrow \exists? \bar{f} & \\ X & & \end{array}$$

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Issue: will we need discussions of

$$\begin{array}{ccccccc} \mathbb{Q} & \longleftrightarrow & \mathbb{Q}^* & \xrightarrow{inv} & \mathbb{Q}^* & \longleftrightarrow & \mathbb{Q} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbb{R} & \longleftrightarrow & \mathbb{R}^* & \xrightarrow{inv} & \mathbb{R}^* & \longleftrightarrow & \mathbb{R} \end{array}$$

Side issue: how to formally refer to \bar{f} ?

`extend f i de h` where `de` is a proof that i is a dense topological embedding, and `h` is a proof that f admits a limit...?

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A better solution is to define an extension operator E_i by:

$$E_i(f)(x) = \begin{cases} \text{some } y \text{ such that } f(a) \text{ tends to } y \text{ when } a \text{ tends to } x \\ \text{some junk value if no such } y \text{ exists} \end{cases}$$

Density of image of i is used only to ensure Y is non-empty!

Then use `de.extend f`

Separation issues

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The i map is *not* injective if $\{0\}$ is not closed in R .

Standard solution

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Note: Even in ZFC, if R is already separated, $R' \neq R$.

Final extension theorem

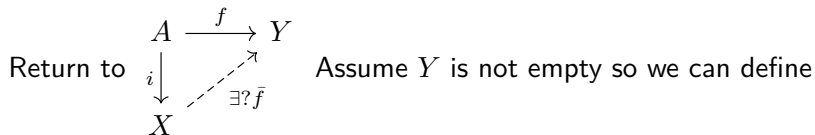
Return to
$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ i \downarrow & \nearrow \exists? \bar{f} & \\ X & & \end{array}$$
 Assume Y is not empty so we can define

E_i without any assumption on i .

Theorem

Fix $x_0 \in X$. If, for every x_1 in a neighborhood of x_0 , $f(a)$ tends to a limit in Y when $i(a)$ tends to x_1 then $E_i f$ is continuous at x_0 .

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If in addition $x_0 = i(a)$, f is continuous at a and i pulls back the topology of X to the topology of A then $E_i(f)(i(a)) = f(a)$.

Side propaganda

The assumption in the first part of the previous theorem can be written as

$$\begin{aligned} \exists U \in \mathcal{N}_{x_0}, \forall x \in U, \exists y \in Y, \forall W \in \mathcal{N}_y, \exists V \in \mathcal{N}_x, \\ \forall a \in A, i(a) \in V \Rightarrow f(a) \in W. \end{aligned}$$

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In mathlib, the assumption is written as

$$\{x \mid \exists y, f_* i^* \mathcal{N}_x \leq \mathcal{N}_y\} \in \mathcal{N}_{x_0}.$$

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- Big projects are good. Next one?