Gromov hyperbolic spaces in proof assistants

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S. Gouëzel and A. Karlsson, Subadditive and multiplicative ergodic theorems, *Journal of the European Mathematical Society*, to appear.

Theorem 1.1. *Let $a(n, \omega)$ be an integrable and subadditive cocycle relative to the ergodic system (Ω, μ, T) as above, with finite asymptotic average A. Then for almost every ω there are integers $n_i := n_i(\omega) \to \infty$ and positive real numbers $\delta_\ell := \delta_\ell(\omega) \to 0$ such that for every i and every $\ell \leq n_i$,*

$$-\ell\delta_\ell(\omega) \leq a(n_i, \omega) - a(n_i - \ell, T^\ell \omega) - A\ell \leq \ell\delta_\ell(\omega). \tag{1.1}$$

Remark 1.3. As a test case for the usability of proof assistants for current mathematical research, Theorem 1.1 and its proof given below have been completely formalized and checked in the proof assistant Isabelle/HOL, see the file `Gouezel_Karlsson.thy` in [Go15]. In particular, the correctness of this theorem is certified.

S. Gouëzel, Growth of normalizing sequences in limit theorems for conservative maps, *Electron. Commun. Probab.* **23** (2018), no. 99, 1–11.

```
locale conservative_limit =
  conservative M + PS: prob_space P + PZ: real_distribution Z
    for M::"'a measure" and P::"'a measure" and Z::"real measure" +
  fixes f g::"'a ⇒ real" and B::"nat ⇒ real"
  assumes PabsM: "absolutely_continuous M P"
      and Bpos: "⋀n. B n > 0"
      and M [measurable]: "f ∈ borel_measurable M" "g ∈ borel_measurable M" "sets P = sets M"
      and non_trivial: "PZ.prob {0} < 1"
      and conv: "weak_conv_m (λn. distr P borel (λx. (g x + birkhoff_sum f n x) / B n)) Z"
```

```
theorem subexponential_growth:
  "(λn. max 0 (ln (B n) /n)) ⟶ 0"
```

Theorem (SG, 2020?)

In a Gromov-hyperbolic group, excursions of length n of a random walk converge in distribution, as metric spaces, towards the continuous random tree.

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No hope to formalize the proof in a proof assistant. What about the statement? Still very far.

Definition

A metric space is Gromov-hyperbolic if there exists $\delta \geq 0$ such that, for all x, y, z, w,

$$d(x, y) + d(z, w) \leqslant \max(d(x, z) + d(y, w), d(x, w) + d(y, z)) + \delta.$$

Captures the notion of negative curvature on large scale.

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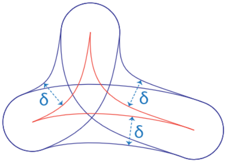
Geometric intuition when the space is geodesic (i.e., any two points can be joined by a geodesic): triangles are thin.



Wikimedia Commons

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Lemma

Assume that X is δ-hyperbolic. Let $x, y \in X$. If there is no midpoint between x and y, one can add one while retaining δ-hyperbolicity.

Proof.

Set $d(m, z) = d(x, y)/2 + \sup_w(d(z, w) - \max(d(a, w), d(b, w)))$.
It works. $\square$

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Proof of Bonk-Schramm Theorem.

Enumerate all pairs of points. Add middles, then complete, and do it all over again until it stops by transfinite induction. $\square$

```
instantiation Bonk_Schramm_extension :: (Gromov_hyperbolic_space) Gromov_hyperbolic_space_geodesic
begin
definition deltaG_Bonk_Schramm_extension::"('a Bonk_Schramm_extension) itself ⇒ real" where
  "deltaG_Bonk_Schramm_extension _ = deltaG(TYPE('a))"

instance apply standard
unfolding deltaG_Bonk_Schramm_extension_def using Bonk_Schramm_extension_hyperbolic by auto
end (* of instantiation proof *)
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Key point: use an inductive type to model both the middle construction and the completion:

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datatype 'a Bonk_Schramm_extension_unfolded =
  basepoint 'a
  | middle "'a Bonk_Schramm_extension_unfolded" "'a Bonk_Schramm_extension_unfolded"
  | would_be_Cauchy "nat ⇒ 'a Bonk_Schramm_extension_unfolded"
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Lesson 1

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Lesson 1

Inductive types are useful (even for mathematicians)

Lesson 1'

Computer scientists are useful (even for mathematicians)

(datatype package in Isabelle/HOL, by Blanchette and al.)

Definition

Let $\lambda \geqslant 1$ and $C \geqslant 0$. A (λ, C)-quasigeodesic is a map $f : [a, b] \to X$ such that, for all $s, t \in [a, b]$,

$$\lambda^{-1}|t - s| - C \leqslant d(f(s), f(t)) \leqslant \lambda|t - s| + C.$$

Theorem (Morse Lemma)

Let $f : [a, b] \to X$ be a (λ, C)-quasigeodesic, where X is δ-hyperbolic. Then there exists $A = A(\lambda, C, \delta)$ such that $f[a, b]$ and a geodesic from $f(a)$ to $f(b)$ are at distance at most A.

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Theorem (Shchur, 2013)

One can take $A(\lambda, C, \delta) = 37723\lambda^2(C + \delta)$.

Optimal, up to the constant 37723.

and because the function e^{-X} is decreasing for $X \geqslant 0$, we can use the estimate

$$\sum_{i=1}^{n} e^{-X_i}(X_{i-1} - X_i) \leqslant \int_0^{\infty} e^{-X}\, dX = -e^{-x}\big|_0^{\infty} = 1.$$

Summarizing all the facts, returning to the initial notation, and recalling that $K = \ln 2/19$, we finally obtain the claimed result

$$H = 4\lambda^2\left(78c + \left(78 + \frac{133}{\ln 2} e^{157\ln 2/38}\right)\delta\right). \qquad \square$$

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Lesson 2

Mathematicians (as a community) can be wrong, and proof assistants can already help.

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lemma ineq:
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by (approximation 13)
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by (approximation 98)
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Lesson 2'

Computer scientists are useful

(approximation package in Isabelle/HOL, by Hölzl, while an undergrad)

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Definition

Gromov-Hausdorff space: space of all nonempty compact metric spaces up to isometry, with the Gromov-Hausdorff distance.

Theorem

The Gromov-Hausdorff space is a complete second-countable metric space (a.k.a. Polish space).

One can do probability theory on the Gromov-Hausdorff space.

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I formalized the proof of this theorem, but not in Isabelle/HOL because I can not make sense of the sentence "a sequence of compact metric types converges to a compact metric type there". I formalized it in Lean 3.

```
/-- The Gromov-Hausdorff space is second countable. -/
instance second_countable : second_countable_topology GH_space :=
/-- The Gromov-Hausdorff space is complete. -/
instance : complete_space (GH_space) :=
```

Lesson 3

Dependent types are useful (especially to mathematicians)

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(Lean 3, developed by de Moura et al.)