Generating Mathematical Structure Hierarchies using Coq-ELPI

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Structures in Mathematics

- A **carrier** in Set / Type,
- A set of **constants** in the carrier, and **operations**,
- Proofs of the **axioms** of the structure

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```
Record is_ring A := mk_ring {
   zero : A; add : A -> A -> A; opp : A -> A;
   one : A;  mul : A -> A -> A;
   addrA : associative add;
   addrC : commutative add;
   add0r : left_id zero add;
   addNr : left_inverse zero opp add;
   mulrA : associative mul;
   mul1r : left_id one mul;
   mulr1 : right_id one mul;
   mulrDl : left_distributive mul add;
   mulrDr : right_distributive mul add;
 }.
```

Structures in formalization

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- factor theorems across instances, using the **theory** of each structure,
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- **predictability** of inferred instance,
- **robustness** of user code with regard to *new declarations*.x

Structures relating to each other

Examples:

- Monoid $\leftarrow$ Group $\leftarrow$ Ring $\leftarrow$ Field $\leftarrow$ ...
- Euclidean Spaces $\rightarrow$ Normed Spaces $\rightarrow$ Complete Space $\rightarrow$ Metric Spaces $\rightarrow$ Topological Spaces $\rightarrow$ ...

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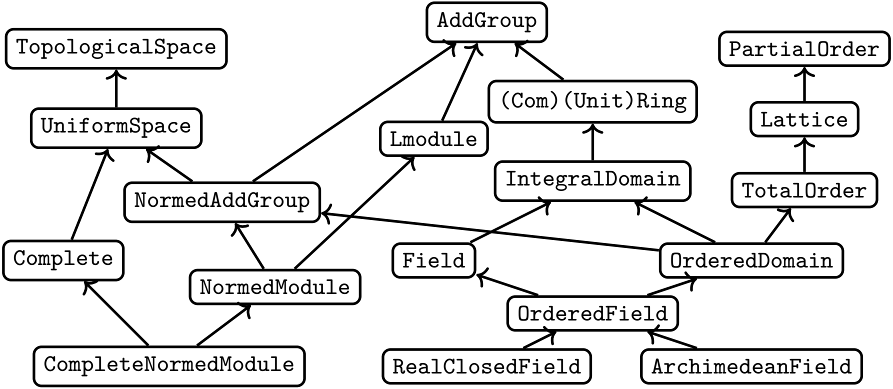
Two kinds arrows:

- Extensions: add operations, axioms or combine structures
- Entailment/Induction/Deduction/Generalization.

More examples



"Calculus" structures

"Algebraic" structures

Structure extension

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- **Not all arrows! Really?**

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- Major breakage when arbitrary entailment is automatic; e.g. given two normed spaces, and making their Cartesian product, in order to obtain the resulting topology, one can either:
  - first consider the normed space product, then derive the corresponding topological space, or
  - first derive the topological spaces and then consider the topological space product.

Our Design

The best of two the worlds:

- **Extension**, through *mixins* for **internal declaration** and **automatic inference**
- **Entailment**, through *factory* for any other use.
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We follow a fully bundled approach, where carriers are packaged together with their axiomatic.

Hiearchy builder at work

Five commands:

- `declare_mixin FactoryModuleName TypeName Factories.`
- `declare_factory FactoryModuleName TypeName Factories.`
- `end Functions.`
- `structure ModuleName Factories.`
- `instance Carrier Factories.`

Demo

https://github.com/math-comp/hierarchy-builder

Conclusion

- High-level commands to declare structures and instances, easy to use.
- Predictable outcome of inference,
- Takes into account the evolution of knowledge
  - which is formalized, and
  - which the user has.

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Also, Coq-ELPI turned out to be a very comfortable meta-programming language for this.

Future work on Hierarchy Builder

- Adding support for parameters.
- Generating hierarchies of morphisms from structures.
- Generating hierarchies of subobjects from structures.
- Supporting multiple instances on the same carrier.
- Replacing all uses in math-comp and extensions.
- Get better error messages.