Formalizing o-minimality

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FoMM / Lean Together

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- Virtually no mathematical prerequisites
- Probably constructive
- I claim it is basically infeasible to formalize without some specialized automation.

Paths in classical algebraic topology

Let X be a topological space and a and b points of X.

Definition

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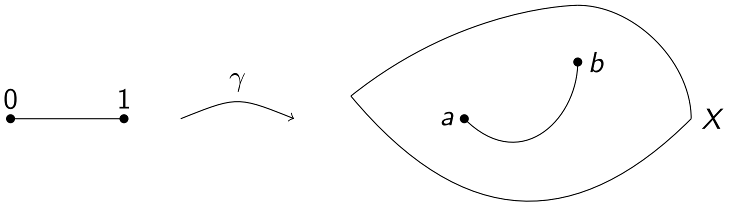

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- Suppose X is the union of two closed subsets A and B.
  A continuous map $\gamma : [0,1] \to X$ might "enter and leave" A and B infinitely many times.
  For example, take $X = \mathbb{R}$, $A = (-\infty, 0]$, $B = [0, \infty)$,
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Furthermore, X itself might be "pathological" from the standpoint of homotopy theory. For example, $X = \mathbb{Z}_p$ (topologically a Cantor set) has no nonconstant paths and so might as well be discrete, but it has a nontrivial topology.

Grothendieck on topology

After some ten years, I would now say, with hindsight, that "general topology" was developed (during the thirties and forties) by analysts and in order to meet the needs of analysts, not for topology per se, i.e. the study of the topological properties of the various geometrical shapes. That the foundations of topology are inadequate is manifest from the very beginning, in the form of "false problems" (at least from the point of view of the topological intuition of shapes) such as the "invariance of domains", even if the solution to this problem by Brouwer led him to introduce new geometrical ideas.

— Grothendieck, *Esquisse d'un Programme* (1984)
(translated by Schneps and Lochak)

Tame topology

Objective: Develop a setting for the homotopy theory of spaces which is flexible enough to allow the usual sorts of constructions but also “tame” enough to rule out the pathologies we saw earlier.

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Definition

A *semialgebraic set* in R^n is a finite union of sets of the form

$$\{ x \in R^n \mid f_1(x) = 0, \ldots, f_k(x) = 0, g_1(x) > 0, \ldots, g_l(x) > 0 \}$$

for polynomials $f_1, \ldots, f_k, g_1, \ldots, g_l$ in the coordinates of x.

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for polynomials f_1, ..., f_k, g_1, ..., g_l in the coordinates of x.

Definition

Let $X \subset R^m$ and $Y \subset R^n$ be semialgebraic sets. A function $f : X \to Y$ is *semialgebraic* if its graph

$$\Gamma(f) = \{ (x, y) \mid y = f(x) \} \subset X \times Y \subset R^{m+n}$$

is semialgebraic.

Tameness of semialgebraic functions

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A semialgebraic function $\gamma : [0, 1] \to R^n$ *is differentiable at all but finitely many points.*

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Theorem

If $X = A \cup B$ is the union of two closed semialgebraic subsets then for any continuous semialgebraic function $\gamma : [0,1] \to X$, the domain $[0,1]$ can be decomposed into finitely many closed intervals each of which is mapped by γ into either A or B.

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There is a more sophisticated notion of *weakly semialgebraic space*; these model all homotopy types.

O-minimal structures

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More specifically, semialgebraic sets are an example of an *o-minimal structure* and the preceding theorems are valid for any "o-minimal expansion of a real closed field".

Structures

Fix any set R.

Definition

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A *structure* consists of, for each $n \geq 0$, a family of subsets of R^n called the *definable* subsets such that:

- For each $n \geq 0$, the definable subsets of R^n form a boolean algebra of subsets (the empty set is definable, and the definable sets are closed under union and complementation).
- For each $n \geq 0$, if $A \subset R^n$ is definable, then $R \times A \subset R^{n+1}$ and $A \times R \subset R^{n+1}$ are definable.
- For each $n \geq 2$, the set $\{ (x_1, \ldots, x_n) \in R^n \mid x_1 = x_n \}$ is definable.
- For each $n \geq 0$, writing $\pi : R^{n+1} = R^n \times R \to R^n$ for the projection, if $A \subset R^{n+1}$ is definable, then $\pi(A) \subset R^n$ is definable.

O-minimal structures

Now suppose R is an ordered field.

Definition

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- (Extension) The sets

$$\{(x, y) \mid x < y\} \subset R^2,$$

$$\{(x, y, z) \mid x + y = z\} \subset R^3,$$

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  are definable.
- (Minimality) Any definable set in R is a finite union of singletons and open intervals.

Examples of o-minimal structures

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Example

Wilkie's theorem: The smallest structure containing $\mathbb{R}_{\mathrm{sa}}$ and the graph of $\exp : \mathbb{R} \to \mathbb{R}$ is o-minimal.

Definable functions

Fix a set R and a structure on R.

Definition

Suppose $X \subset R^m$ and $Y \subset R^n$ are definable sets. A function $f : X \to Y$ is *definable* if its graph

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Proving definability

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Proof.

For simplicity assume $f : R \to R$ and $g : R \to R$ are definable functions. Let $\pi : R \times R \times R \to R \times R$ project out the second coordinate. Then

$$\begin{aligned}\Gamma(g \circ f) &= \{\, (x, z) \mid z = g(f(x)) \,\} \\ &= \pi(\{\, (x, y, z) \mid y = f(x), z = g(y) \,\}) \\ &= \pi(\{\, (x, y, z) \mid y = f(x) \,\} \cap \{\, (x, y, z) \mid z = g(y) \,\}) \\ &= \pi((\Gamma(f) \times R) \cap (R \times \Gamma(g)))\end{aligned}$$

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This style of proof is not sustainable.

Definability of the interior

Suppose R is totally ordered by a definable relation $<$. Equip R with the order topology and R^n with the product topology.

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Proof.

We have $(x_1, \ldots, x_n) \in \operatorname{int} A$ if and only if

$$\exists l_1, \ldots, l_n, u_1, \ldots, u_n, l_1 < x_1 < u_1 \wedge \cdots \wedge l_n < x_n < u_n \wedge$$
$$(\forall y_1, \ldots, y_n, l_1 < y_1 < u_1 \wedge \cdots \wedge l_n < y_n < u_n \implies$$
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$$\exists l_1, \ldots, l_n, u_1, \ldots, u_n, l_1 < x_1 < u_1 \wedge \cdots \wedge l_n < x_n < u_n \wedge \\ (\forall y_1, \ldots, y_n, l_1 < y_1 < u_1 \wedge \cdots \wedge l_n < y_n < u_n \implies \\ (y_1, \ldots, y_n) \in A).$$

Therefore

$$\operatorname{int} A = [\text{some large expression involving } A \text{ and } <]$$

is definable. □

Definability by formulas

Theorem

Let $\varphi(x_1, \ldots, x_n)$ be any formula of first-order logic using relation symbols r_i and function symbols f_j and suppose each relation and function symbol is given an interpretation in R which is a definable set. Then the interpretation of φ is a definable set in R^n.

This theorem completes the previous proof.

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But maybe automation is overkill—we just prove a few dozen lemmas about definable sets and definable functions, and we're done?

The beginning of tameness

Lemma

Let $f : (a, b) \to R$ be a definable function. Then there exists an open interval contained in (a, b) on which f is either injective or constant.

Proof.

Two cases.

- Suppose $f^{-1}(\{y\})$ is infinite for some $y \in R$. Then it contains an interval, and so f is constant on this interval.
- Otherwise, $f^{-1}(\{y\})$ is finite for every $y \in R$. Define

$$K = \{ x \in (a, b) \mid \forall x' \in (a, b), f(x) = f(x') \implies x \leq x' \}.$$

Then $f(K) = f((a, b))$ and so K is infinite, and therefore contains an interval. By definition, f is injective on K. □

Is this argument constructive?

In the constructive setting, the "minimality" axiom should take the form of a function which takes a definable set in R and outputs a description of that set as a finite union of singletons and open intervals. (For this to be possible, `definable A` should not be a `Prop` but should contain data.)

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In the previous proof, we need to decide the formula

$$\exists a', b', y, x, a' < x < b' \implies f(x) = y.$$