Update procedures and the 1-consistency of arithmetic

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The ordinal ε_0

ε_0 is defined to be the limit of the sequence

$$\omega, \omega^\omega, \omega^{(\omega^\omega)}, \ldots$$

Every nonzero $\alpha < \varepsilon_0$ can be written in *Cantor normal form*:

$$\alpha = \omega^{\alpha_1} \cdot n_1 + \ldots + \omega^{\alpha_k} \cdot n_k$$

where $\alpha > \alpha_1 > \ldots > \alpha_k$ and $n_i \in \mathbb{N}$.

So there is an effective (primitive recursive, or even elementary recursive) set of notations, such that the associated ordering is also effective.

Ordinal recursive functions

An α-*recursive function* is given by elementary functions $start(\vec{x})$, $next(q)$, $norm(q)$, $result(q)$.

These data define a function $F(\vec{x})$:

$$
\begin{array}{l}
clock \leftarrow \alpha \\
state \leftarrow start(\vec{x}) \\
\textsf{while } norm(state) \prec clock \textsf{ do} \\
\quad clock \leftarrow norm(state) \\
\quad state \leftarrow next(state) \\
\textsf{return } result(state)
\end{array}
$$

Ordinal recursive functionals

The previous definition relativizes well.

A α-*recursive functional* $F(\vec{x}, f_1, \ldots, f_k)$ is given by elementary functions $start(\vec{x})$, $next(q, u_1, \ldots, u_k)$, $query_1(q)$, $\ldots$, $query_k(q)$, $norm(q)$, and $result(q)$.

These define a functional $F(\vec{x}, f_1, \ldots, f_k)$:

$$
\begin{array}{l}
clock \leftarrow \alpha \\
state \leftarrow start(\vec{x}) \\
\textsf{while } norm(state) \prec clock \textsf{ do} \\
\quad clock \leftarrow norm(state) \\
\quad state \leftarrow next(state, f_1(query_1(state)), \ldots, \\
\qquad f_k(query_k(state))) \\
\textsf{return } result(state)
\end{array}
$$

The ordinal analysis of PA

Theorem. Suppose Peano Arithmetic proves $\forall x \, \exists y \, \varphi(x, y)$, for a Σ_1 formula φ. Then there is a $<\varepsilon_0$-recursive function F such that for every x, $\varphi(x, F(x))$.

Notes:

- The statement can be relativized to a function parameter.
- A suitable formalization can be proved in primitive recursive arithmetic.
- This yields the usual results of the ordinal analysis of PA.

Embedding PA in a quantifier-free calculus

Iteratively introduce Skolem functions for quantifier-free formulae:

$$\varphi(\vec{x}, y) \to \varphi(\vec{x}, \mu_\varphi(\vec{x})) \wedge \mu_\varphi(\vec{x}) \leq y$$

In other words, $\mu_\varphi(\vec{x})$ returns a least y satisfying $\varphi(\vec{x}, y)$, if there is one.

Proposition. If PA proves $\forall x \, \exists y \, \psi(x, y)$ with ψ quantifier-free, then for each n there is a *propositional* proof of $\varphi(\bar{n}, \mu_\varphi(\bar{n}))$ from closed instances of μ axioms, the definitions of the initial functions and relations $(0, 1, +, \times, <, \ldots)$, and axioms of equality.

The task

Given a finite set S of closed instances of μ axioms, finite a finite arithmetic interpretation of the μ symbols.

Idea: assume everything returns 0 by default. Suppose the following instance of an axiom fails under this interpretation:

$$\varphi(\vec{s}, t) \to \varphi(\vec{s}, \mu_\varphi(\vec{s})) \land \mu_\varphi(\vec{s}) \leq t.$$

Correct it by mapping $\mu_\varphi(\vec{s}) = t$. Iterate.

The difficulty:

- Can order $\mu_1, \mu_2, \ldots, \mu_n$ so that the definition of μ_i involves only μ_j with $j < i$.
- But the terms $\vec{s}$ and t above can involve any μ_i.

Overview

1. We will define the notion of a “nested system of update procedures.”
2. By general continuity considerations, these always have solutions.
3. The task on the previous slide amounts to finding solutions to elementary systems of equations.
4. Ordinals can be used in place of continuity.

This is essentially a repackaging of Ackermann’s proof, using the Hilbert substitution method. The emphasis on continuity dates back to Tait ’65.

Background definitions

Let $\rho, \sigma, \tau, \ldots$ range over finite partial functions from $\mathbb{N}$ to $\mathbb{N}$.

Let $\hat{\sigma}$ to denote the extension to a total function:

$$\hat{\sigma}(x) = \begin{cases} \sigma(x) & x \in dom(\sigma) \\ 0 & \text{otherwise.} \end{cases}$$

Let $\sigma \oplus \langle u, v \rangle$ to denote the modification of σ that maps u to v:

$$(\sigma \oplus \langle u, v \rangle)(x) = \begin{cases} \sigma(x) & \text{if } x \in dom(\sigma), x \neq u \\ v & \text{if } x = u \\ \text{undef.} & \text{otherwise} \end{cases}$$

Define $\sigma \oplus \emptyset$ to be σ.

Update procedures

A functional $F(f_1, \ldots, f_k)$ is *continuous* if its value depends on only finitely many values of $f_1, \ldots, f_k$.

Suppose $F(g, f_1, \ldots, f_k)$ is continuous with range $\mathbb{N} \times \mathbb{N} \cup \{\emptyset\}$. Consider the map

$$\sigma \mapsto \sigma \oplus F(\hat{\sigma}, f_1, \ldots, f_k).$$

F is an *update procedure* in g if the following holds: whenever

- $F(\hat{\sigma}, f_1, \ldots, f_k) = \langle a, b \rangle$,
- τ extends $\sigma \oplus \langle a, b \rangle$, and
- $F(\hat{\tau}, h_1, \ldots, h_k) = \langle a, c \rangle$,

then $b = c$.

In other words, once F "sets" $\sigma(a)$ to b, it does not change it, no matter how the other arguments vary.

Fixed points

If $F(g)$ is a unary update procedure, a *finite fixed point* of F is a σ such that

$$\sigma = \sigma \oplus F(\hat{\sigma}).$$

Lemma. $F(g)$ has a finite fixed point.

Proof. Let $\sigma^0 = \emptyset$, and for each i, let $\sigma^{i+1} = \sigma^i \oplus F(\hat{\sigma}^i)$. Let $g = \bigcup_{i \in \mathbb{N}} \sigma^i$. By continuity, for some i we have $F(\hat{g}) = F(\hat{\sigma}_i)$. $\square$

The proof shows that if $F(g, \vec{h})$ is continuous, and for each $\vec{h}$

$$g \mapsto F(g, \vec{h})$$

is an update procedure, there is a continous functional $G(\vec{h})$ returning fixed points.

Nested update procedures

A *system of nested update procedures* is a sequence of continuous functionals $F_1(f_1, \ldots, f_n), \ldots, F_n(f_1, \ldots, f_n)$ such that for each i and fixed $f_1, \ldots, f_{i-1}$, the functional

$$f_i, f_{i+1}, \ldots, f_n \mapsto F_i(f_1, \ldots, f_n)$$

is an update procedure for f_i.

A *finite fixed point* of such a system is a sequence of finite partial functions $\sigma_1, \ldots, \sigma_n$ such that the equations

$$\begin{array}{ccc} \sigma_1 & = & \sigma_1 \oplus F_1(\hat{\sigma}_1, \ldots, \hat{\sigma}_n) \\ \vdots & & \vdots \\ \sigma_n & = & \sigma_n \oplus F_n(\hat{\sigma}_1, \ldots, \hat{\sigma}_n) \end{array}$$

are all satisfied.

Finding fixed points

Theorem. Every system of nested update procedures has a finite fixed point.

Proof. Use induction on n. We have already taken care of $n = 1$.

For the induction step, given $F_1, \ldots, F_{n+1}$, let $G(f_1, \ldots, f_n)$ be a continuous functional returning finite fixed points of the functional

$$f_{n+1} \mapsto F_{n+1}(f_1, \ldots, f_{n+1}).$$

Then

$$f_1, \ldots, f_n \mapsto F_i(f_1, \ldots, f_n, G(f_1, \ldots, f_n))$$

for $i = 1, \ldots, n$ is a system of nested update procedures of size n.

By the IH, the smaller system has a finite fixed point, $\sigma_1, \ldots, \sigma_n$. Let $\sigma_{n+1} = G(\hat{\sigma}_1, \ldots, \hat{\sigma}_n)$. □

The main theorem

Now restrict to systems of nested update procedures given by *elementary* functions $F_1, \ldots, F_n$.

Theorem. The following are pairwise equivalent:

1. Every $\prec\varepsilon_0$-recursive function is total.
2. Every system of nested elementary update procedures has a finite fixed point.
3. Every Π_2 sentence provable in PA is true.

This theorem is provable in PRA.

Fixed-point equations and arithmetic

Remember that Peano arithmetic can be embedded in a q.f. theory based on μ axioms:

$$\varphi(\vec{x}, y) \rightarrow \varphi(\vec{x}, \mu_\varphi(\vec{x})) \land \mu_\varphi(\vec{x}) \leq y$$

Lemma. Suppose every system of nested elementary update procedures has a finite fixed point. Then every set of closed instances of μ-axioms has a finite partial model.

The conclusion implies (and is in fact equivalent to) the Π_2 soundness of arithmetic.

Finding finite partial models

Let S be a finite set of closed instances of μ-axioms. Let $\mu_1, \ldots, \mu_n$ be the μ-symbols mentioned in S, such that if the definition of μ_i involves μ_j, then $i > j$.

For each i, let $F_i(f_1, \ldots, f_n)$ find an instance of the defining axiom for μ_i in S,

$$\theta(\vec{s}, t) \rightarrow \theta(\vec{s}, \mu_i(\vec{s})) \land \mu_i(\vec{s}) \leq t.$$

that is falsified under the assignment

$$\mu_1, \ldots, \mu_n \mapsto f_1, \ldots, f_n.$$

Update $\mu_i(\vec{s}) \mapsto \mu m \leq t^{\vec{f}} \; (\theta(\vec{s}, m))^{\vec{f}}$.

Then:

- This is a system of nested update procedures.
- A fixed point is a finite interpretation of $\mu_1, \ldots, \mu_n$ satisfying S.

Finding a fixed-point

We only need to show that one can find fixed-points using ordinal recursion (instead of continuity).

Lemma. Suppose $F(g, \vec{h})$ is α-recursive, and for each $\vec{h}$, $g \mapsto F(g, \vec{h})$ is an update procedure. Then there is an ω^α-recursive functional $G(\vec{h})$ that returns finite fixed points.

Idea: start with $i = 0, \sigma^0 = \emptyset$. Then

- Compute $F(\hat{\sigma}^i, \vec{h})$.
- Update: $\sigma^{i+1} = \sigma^i \oplus F(\hat{\sigma}^i, \vec{h})$.
- If the computation sequence for $F(\hat{\sigma}^i, \vec{h})$ is no longer valid, revise it, and compute $F(\hat{\sigma}^{i+1}, \vec{h})$.
- Repeat.

To show this converges, it suffices to assign ordinals to steps.

Assigning ordinals

Consider a partial computation sequence

$$s_0, s_1, \ldots, s_m$$

of $F(\hat{\sigma}^i, \vec{h})$, with norms

$$\alpha_0, \alpha_1, \ldots, \alpha_m.$$

Assign to this the ordinal

$$\omega^{\alpha_0} \cdot \delta_0 + \ldots + \omega^{\alpha_{m-1}} \cdot \delta_{m-1} + \omega^{\alpha_m} \cdot (\delta_m + 1),$$

where

$$\delta_i = \begin{cases} 2 & \text{if } \mathit{query}_{H,1}(s_i) \text{ is not } \mathit{dom}(\sigma) \\ 0 & \text{otherwise.} \end{cases}$$

Two cases:

1. Computation of F is not done. Take the next step.
2. Computation of F is done. Update σ^i and revise computation.

First case

If the computation of F is not done,

- let s_{m+1} the next state in the computation of $F(\sigma^i, \vec{h})$, and
- let α_{m+1} be the corresponding norm.

Then the ordinal drops from

$$\omega^{\alpha_0} \cdot \delta_0 + \ldots + \omega^{\alpha_{m-1}} \cdot \delta_{m-1} + \omega^{\alpha_m} \cdot (\delta_m + 1),$$

to

$$\omega^{\alpha_0} \cdot \delta_0 + \ldots + \omega^{\alpha_{m-1}} \cdot \delta_{m-1} + \omega^{\alpha_m} \cdot \delta_m + \omega^{\alpha_{m+1}} \cdot (\delta_{m+1} + 1).$$

Second case

Otherwise, let $\sigma^{i+1} = \sigma^i \oplus F(\hat{\sigma}^i, \vec{h})$.

If this invalidates the computation sequence for $F(\hat{\sigma}^{i+1}, \vec{h})$, let j be the first point at which the new value is queried.

In other words, the computation sequence is cut back from

$$s_0, \ldots, s_j, \ldots, s_m$$

to

$$s_0, \ldots, s_j.$$

The norm drops from

$$\omega^{\alpha_0} \cdot \delta_0 + \ldots + \omega^{\alpha_{j-1}} \cdot \delta_{j-1} + \omega^{\alpha_j} \cdot 2 + \ldots \\ \omega^{\alpha_{m-1}} \cdot \delta_{m-1} + \omega^{\alpha_m} \cdot (\delta_m + 1),$$

to

$$\omega^{\alpha_0} \cdot \delta_0 + \ldots + \omega^{\alpha_{j-1}} \cdot \delta_{j-1} + \omega^{\alpha_j} \cdot 1$$

since δ_j has dropped from 2 to 0.

Finishing it off

Theorem. Suppose every $\prec\varepsilon_0$-recursive function is total. Then every system of nested elementary update procedures has a finite fixed point.

Proof. Let $F_1, \ldots, F_n$ be a system of nested update procedures.

As before, use induction (recursion) up to n; the previous lemma handles the induction step. □

Final remarks

The approach

- Yields sharp bounds for fragments of arithmetic.
- Works for transfinite induction.

Related approaches to the OA of

- Systems with transfinite jump hierarchies (predicative analysis)
- Admissible set theory

can be put in this framework.

Questions:

- Can one eliminate the nesting?
- Will this lead to interesting computational or combinatorial principles?