The computational content of classical arithmetic

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Epistemological concerns

The question of the *consistency* of modern mathematical methods was salient in the early twentieth century "crisis of foundations."

But lurking beneath the surface was an even deeper concern, namely, the extent to which modern methods are *meaningful*, and *appropriate* to mathematics.

After all, if mathematics is "about" symbolic calculation and explicitly computable results, then "nonconstructive mathematics" is a contradiction in terms.

So it is not surprising that the question as to the "computational content" of classical methods has long been central to proof theory.

Peano arithmetic

Classical first-order (Peano) arithmetic at first seems like a toy theory.

But it interprets a surprisingly large portion of ordinary mathematical reasoning fairly directly, and even more by various proof-theoretic reductions.

Question: suppose PA proves $\forall x \; \exists y \; R(x, y)$, where R is decidable. Is there necessarily an algorithm to compute such a y?

Answer: 0f course. Just search.

The computational content of arithmetic

Better question: do proofs in *PA* provide more explicit algorithms?

Answer:

- (Gentzen) $\prec\varepsilon_0$-recursive functions
  - Cut elimination or normalization, with ordinal analysis
- (Gödel) Primitive recursive functionals of finite type
  - Double-negation translation following by the Dialectica interpretation.
  - Double-negation translation followed by the Friedman-Dragalin translation and realizability.

These give two different characterizations of the algorithmic content.

The computational content of arithmetic

In comparison:

- Ordinal analysis is most easily applied directly to classical theories.
- Functional interpretation passes through intuitionistic arithmetic (HA).

Also:

- Proofs in intuitionistic arithmetic yield canonical results (confluence, E-stability).
- Proofs in classical arithmetic are "nondeterministic."

I will focus on interpretations that involve a double-negation translation.

Outline

Topics I will discuss:

- A particularly efficient double-negation translation.
- An interesting, but awkward, double-negation translation.
- Nondeterminism in classical logic.

Double-negation translations

The Gödel-Gentzen translation maps formulas φ to φ^N:

$$
\begin{aligned}
\bot^N &\equiv \bot \\
\theta^N &\equiv \neg\neg\theta, \text{ if } \theta \text{ is atomic} \\
(\varphi \wedge \psi)^N &\equiv \varphi^N \wedge \psi^N \\
(\varphi \vee \psi)^N &\equiv \neg(\neg\varphi^N \wedge \neg\psi^N) \\
(\varphi \to \psi)^N &\equiv \varphi^N \to \psi^N \\
(\forall x\, \varphi)^N &\equiv \forall x\, \varphi^N \\
(\exists x\, \varphi)^N &\equiv \neg\forall x\, \neg\varphi^N
\end{aligned}
$$

Theorem

1. *Classical logic proves* $\varphi \leftrightarrow \varphi^N$
2. *If* Γ *proves* φ *in classical logic, then* Γ^N *proves* φ^N *in minimal logic.*
3. *Ditto for PA and HA.*

Representing classical logic

Use negation normal form:

- Start with literals, A, $\bar{A}$.
- Close under $\varphi \wedge \psi$, $\varphi \vee \psi$, $\forall x\ \varphi$, $\exists x\ \varphi$.

Take classical negation $\sim\varphi$ to be a defined operation.

A one-sided calculus:

$$\Gamma, A, \bar{A}$$

$$\frac{\Gamma, \varphi \qquad \Gamma, \psi}{\Gamma, \varphi \wedge \psi} \qquad\qquad \frac{\Gamma, \varphi_i}{\Gamma, \varphi_0 \vee \varphi_1}$$

$$\frac{\Gamma, \varphi}{\Gamma, \forall x\ \varphi} \qquad\qquad \frac{\Gamma, \varphi[t/x]}{\Gamma, \exists x\ \varphi}$$

$$\frac{\Gamma, \varphi \qquad \Gamma, \neg\varphi}{\Gamma}$$

Another double-negation translation

This variant applies to negation-normal form formulas:
Define φ^M to be $\neg(\sim\varphi)_M$, where:

$$\begin{aligned}
A_M &\equiv A \\
\bar{A}_M &\equiv \neg A \\
(\varphi \vee \psi)_M &\equiv \varphi_M \vee \psi_M \\
(\varphi \wedge \psi)_M &\equiv \neg(\sim\varphi \vee \sim\psi)_M \\
(\exists x\ \varphi)_M &\equiv \exists x\ \varphi_M \\
(\forall x\ \varphi)_M &\equiv \neg(\exists x \sim\varphi)_M.
\end{aligned}$$

Note that of any two formulas, φ^M and $(\sim\varphi)^M$, one is the negation of the other.

Another double-negation translation

Theorem

For every formula φ in negation-normal form, $\varphi^M \leftrightarrow \varphi^N$ is provable in minimal logic. Hence PA proves φ if and only if HA proves φ^M.

In fact, it is easy to translate proofs: if

$$\{\varphi_1, \ldots, \varphi_n\}$$

is provably classically (resp. in PA), then

$$(\sim\varphi_1)_M, \ldots, (\sim\varphi_n)_M \Rightarrow \bot$$

is provable in intuitionistic logic (resp. in HA).

Another double-negation translation

If Γ is $\{\psi_1, \ldots, \psi_n\}$, write $(\sim\Gamma)_M$ for $\{(\sim\psi_1)_M, \ldots, (\sim\psi_n)_M\}$.

The cut rule

$$\frac{\Gamma, \varphi \qquad \Gamma, \sim\varphi}{\Gamma}$$

translates to

$$\dfrac{\dfrac{(\sim\Gamma)_M, (\sim\varphi)_M \Rightarrow \bot}{(\sim\Gamma)_M \Rightarrow \neg(\sim\varphi)_M} \qquad \dfrac{(\sim\Gamma)_M, \varphi_M \Rightarrow \bot}{(\sim\Gamma)_M \Rightarrow \neg\varphi_M}}{(\sim\Gamma)_M \Rightarrow \bot}$$

Another double-negation translation

The $\wedge$ rule,

$$\frac{\Gamma, \varphi \qquad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}$$

translates to

$$\frac{(\sim\Gamma)_M, (\sim\varphi)_M \Rightarrow \bot \qquad (\sim\Gamma)_M, (\sim\psi)_M \Rightarrow \bot}{(\sim\Gamma)_M, (\sim\varphi)_M \vee (\sim\psi)_M \Rightarrow \bot}$$

and the $\vee$ rule,

$$\frac{\Gamma, \varphi}{\Gamma, \varphi \vee \psi}$$

translates to

$$\frac{\dfrac{(\sim\Gamma)_M, (\sim\varphi)_M \Rightarrow \bot}{(\sim\Gamma)_M \Rightarrow \neg(\sim\varphi)_M} \qquad \neg(\varphi_M \vee \psi_M) \Rightarrow \neg\varphi_M}{(\sim\Gamma)_M, \neg(\varphi_M \vee \psi_M) \Rightarrow \bot}$$

Another double-negation translation

Two variations:

- Translate $(\varphi \wedge \psi)_M$ to $\varphi_M \wedge \psi_M$ instead of $\neg(\sim\varphi \vee \sim\psi)_M$.
- In PA/HA, use an atomic equivalent $\bar{A}$ for $\neg A$.

The result maps φ to $\varphi^M = \neg(\sim\varphi)_M$, where

$$
\begin{aligned}
\theta_M &\equiv \theta, \text{ if } \theta \text{ is atomic or negated atomic} \\
(\varphi \vee \psi)_M &\equiv \varphi_M \vee \psi_M \\
(\varphi \wedge \psi)_M &\equiv \varphi_M \wedge \psi_M \\
(\exists x\, \varphi)_M &\equiv \exists x\, \varphi_M \\
(\forall x\, \varphi)_M &\equiv \neg\exists x\, (\sim\varphi)_M.
\end{aligned}
$$

This is very sparing with intuitionistic negations, which is important to the realizing terms.

Classical realizability

Fix a predicate $A(x)$.

$$
\begin{aligned}
a \ \textit{realizes} \ \theta &\equiv \theta, \text{ if } \theta \text{ is atomic} \\
a \ \textit{realizes} \ \varphi \wedge \psi &\equiv ((a)_0 \ \textit{realizes} \ \varphi) \wedge ((a)_1 \ \textit{realizes} \ \varphi) \\
a \ \textit{realizes} \ \varphi \vee \psi &\equiv ((\mathrm{isleft}(a) \wedge \mathrm{left}(a) \ \textit{realizes} \ \varphi) \vee \\
&\qquad (\mathrm{isright}(a) \wedge \mathrm{right}(a) \ \textit{realizes} \ \psi)) \\
a \ \textit{realizes} \ \exists x \, \varphi(x) &\equiv (a)_1 \ \textit{realizes} \ \varphi((a)_0)
\end{aligned}
$$

Take a *refutes* φ to be the formula $\forall b \, (b \ \textit{realizes} \ \varphi \to A(a(b)))$.

$$
a \ \textit{realizes} \ \forall x \, \varphi(x) \equiv a \ \textit{refutes} \ \exists x \sim\varphi(x)
$$

From a proof of $\exists y \, A(y)$, we obtain a term a that refutes $\forall y \, \bar{A}(y)$. But the identity function realizes $\forall y \, \bar{A}(y)$. So $A(a(id))$.

A classical Dialectica interpretation

Maps each formula φ to a formula φ^D of the form $\forall x\, \exists y\, \varphi_D(x, y)$.

Assuming ψ^D is $\forall u\, \exists v\, \psi_D(u, v)$:

$$\begin{aligned}
\theta^D &\equiv \theta, \text{ if } \theta \text{ is atomic} \\
(\varphi \wedge \psi)^D &\equiv \forall x, u\, \exists y, v\, (\varphi_D(x, y) \wedge \psi_D(u, v)) \\
(\varphi \vee \psi)^D &\equiv \forall x, u\, \exists y, v\, (\varphi_D(x, y) \vee \psi_D(u, v)) \\
(\forall z\, \varphi)^D &\equiv \forall z, x\, \exists y\, \varphi_D(x, y)
\end{aligned}$$

If $(\sim\varphi(z))^D$ is $\forall r\, \exists s\, (\sim\varphi)_D(z, r, s)$, define

$$(\exists z\, \varphi)^D \equiv \forall S\, \exists z, r\, \neg(\sim\varphi)_D(z, r, S(z, r)).$$

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- An interesting, but awkward, double-negation translation.
- Nondeterminism in classical logic.

An awkward translation

Here is a surprisingly simple double-negation translation: let φ^{awk} denote $\neg(\sim\varphi)$.

Theorem

$\varphi^N \to \varphi^{awk}$ is provable in minimal logic. Hence, PA proves φ if and only if HA proves φ^{awk}.

Proof.

If ψ is in negation-normal form, minimal logic proves $\psi \to \psi^N$.

So minimal logic proves $\sim\varphi \to (\sim\varphi)^N$, and hence $\neg(\sim\varphi)^N \to \varphi^{awk}$.

But in minimal logic, we have $\neg(\sim\varphi)^N \leftrightarrow \neg\neg\varphi^N \leftrightarrow \varphi^N$.

$\square$

Explaining the awkwardness

The difference between $\cdot^N$ and $\cdot^{awk}$: intuitionistic logic proves $\varphi^N \to \varphi^{awk}$ but not the converse.

So even though they are classically equivalent, from the point of view of intuitionistic logic, φ^{awk} is weaker.

In particular, the $\cdot^{awk}$-translation behaves poorly with respect to modus ponens: φ^{awk} and $(\varphi \to \psi)^{awk}$ do not always entail ψ^{awk} intuitionistically.

The no-counterexample interpretation

Let φ be a formula in prenex form, for example,

$$\exists x \, \forall y \, \exists z \, \forall w \, \theta(x, y, z, w).$$

Compute the *Herbrand normal form* φ^H:

$$\exists x, z \, \theta(x, f(x), z, g(x, z)).$$

The methods above make it possible to extract terms $F_1(f, g)$ and $F_2(f, g)$ satisfying

$$\forall f, g \, \theta(F_1(f, g), f(F_1(f, g)), F_2(f, g), g(F_1(f, g), F_2(f, g))).$$

This is the no-counterexample interpretation.

A result due to Kohlenbach shows that this is weaker than the Dialectica interpretation.

Explaining the awkwardness

Theorem (Kohlenbach)

For every n there are sentences φ and ψ of arithmetic such that:

1. *φ is prenex.*
2. *ψ is a Π_2 sentence, that is, of the form $\forall x\, \exists y\; R(x, y)$ for some primitive recursive relation R.*
3. *Primitive recursive arithmetic proves φ.*
4. *PA proves $\varphi \to \psi$.*
5. *φ and every prenexation of $\varphi \to \psi$ has a no-counterexample interpretation with functionals in PR_0^ω.*

But:

6. *There is no term F of PR_n^ω which satisfies the no-counterexample interpretation of ψ; that is, there is no term F such that $\forall x\; R(x, F(x))$ is true in the standard model of arithmetic.*

Explaining the awkwardness

But now notice that if φ is prenex, the Dialectica interpretation of φ^{awk} is exactly the no-counterexample interpretation.

This yields:

Theorem

For any fragment T of HA, there are formulas φ and ψ such that the following hold:

1. *PA proves φ and $\varphi \to \psi$, but*
2. *T together with φ^{awk} and $(\varphi \to \psi)^{awk}$ does not prove ψ^{awk}.*

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Nondeterminism in classical logic

Classical logic is often held to be "nondeterministic."

There is a well-known example due to Yves Lafont:

$$\dfrac{\dfrac{\varphi, 0 = 0}{\varphi, \exists x\,(x = x)} \qquad \dfrac{\sim\varphi, 1 = 1}{\sim\varphi, \exists x\,(x = x)}}{\exists x\,(x = x)}$$

Which is the "intended" witness to the conclusion?

Nondeterminism in classical logic

Note that the judgment of "nondeterminism" applies to the *proof system*, and computational interpretations thereof.

For example, if $\varphi \wedge \varphi \rightarrow \varphi$ were an axiom in an intuitionistic proof system, it would lead to nondeterminism:

$$\exists x\ A(x) \wedge \exists x\ A(x) \rightarrow \exists x\ A(x).$$

But, typically, it isn't.

Nondeterminism in classical logic

What goes wrong with classical logic?

- Classical logic uses disjunctive sequents.
- (Girard) There are two ways of proving $\neg\neg(\varphi \wedge \psi)$ from $\neg\neg\varphi$ and $\neg\neg\psi$, and this is needed to prove the double-negation translation of $\neg\neg\theta \rightarrow \theta$.
- (Girard) $\neg\neg\varphi$ gets identified with φ.

Two responses:

- (Urban and Bierman) Classical logic corresponds to nondeterministic algorithms.
- (Girard) Repair classical calculi to make nondeterministic choices deterministic.

But where do nondeterministic choices come into the M-translation?

Nondeterminism in classical logic

They don't!

In Lafont's example, the cut formula determines the choice.

- In the simplest version of the M-translation, the outermost connective settles it.
- In the variant, if φ is atomic, its truth value determines the choice (in the natural way).

So the translation makes choices for us, in a principled way, yielding a third response:

- Use the M-translation.

Back to epistemology

Let's reconsider the original question: what is the computational content of classical mathematics?

Consider applications:

- Extracting programs automatically from classical proofs.
- Designing programming languages and verification systems inspired by classical constructs.
- Proof mining: extract mathematically useful information from classical proofs.

More refined analysis is needed here:

- Remember that ordinary proofs are very different from formal axiomatic derivations.
- The context matters.

Back to epistemology

But let's construe the question in a broad philosophical sense.

We we have a satisfying answer:

- Classical proofs typically do have computational content.
- But there are various incompatible ways of extracting such content.
- By suppressing computational details, classical methods leave some of the computational details unspecified.

Proof-theoretic results provide an answer that:

- is clear, interesting, informative, and to the point; and
- provides a sound theoretical basis for the applications.