Cut elimination revisited

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Contents

A nonconstructive proof

Theorem. Anything provable in the classical sequent calculus with the cut rule is also provable without the cut rule.

Lemma. The sequent calculus with cut is sound.

Lemma. The sequent calculus without cut is complete.

Question: where is the algorithm to eliminate cuts?

The classical sequent calculus

Language: $A, \neg A, \wedge, \vee, \forall, \exists$

(Formulae are in negation normal form)

$\sim\varphi$ is defined using DeMorgan equivalences

Prove sequents $\{\varphi_1, \ldots, \varphi_k\}$, read disjunctively

$$\Gamma, A, \neg A$$

$$\frac{\Gamma, \varphi \qquad \Gamma, \psi}{\Gamma, \varphi \wedge \psi} \qquad\qquad \frac{\Gamma, \varphi}{\Gamma, \varphi \vee \psi} \quad \frac{\Gamma, \psi}{\Gamma, \varphi \vee \psi}$$

$$\frac{\Gamma, \varphi(x)}{\Gamma, \forall x\ \varphi(x)} \qquad\qquad \frac{\Gamma, \varphi(t)}{\Gamma, \exists x\ \varphi(x)}$$

$$\frac{\Gamma, \varphi \qquad \Gamma, \sim\varphi}{\Gamma}$$

The "tableau" proof of completeness

Try to build a model of $p = {\sim}\Gamma = \{{\sim}\varphi_1, \ldots, {\sim}\varphi_k\}$.

- To build a model of $q, \varphi \wedge \psi$, build a model of q, φ, ψ.
- To build a model of $q, \varphi \vee \psi$, branch, and build a model of q, φ or q, ψ.
- To build a model of $q, \forall x\ \varphi(x)$, build a model of $q, \forall x\ \varphi(x), \varphi(t)$.
- To build a model of $q, \exists y\ \varphi(x)$, build a model of $q, \varphi(c)$.
- If you see $q, A, \neg A$, terminate along that branch.

Case 1: every attempt fails; Γ has a proof

Case 2: there is an infinite branch: ${\sim}\Gamma$ has a model

Making it constructive

Main ideas:

- Build a tree *generically*
- Reason about what is *forced* to be true in any model built from a tree rooted at p

Definition: Say $p \preceq q$ (p is stronger than q) if there is a cut-free proof of $\sim p$ from $\sim q$ using rules for $\vee$, $\exists$, and weakening.

Intuition: $p \preceq q$ means any model of p is also a model of q.

Definition: Define $p \Vdash \varphi$ for φ in the language with $\forall$, $\wedge$, $\rightarrow$, $\bot$, as follows:

1. $p \Vdash \bot$ if and only if there is a cut-free proof of $\sim p$.
2. If A is atomic, $p \Vdash A$ if and only if there is a cut-free proof of $\sim p, A$.
3. $p \Vdash \theta \wedge \eta$ if and only if $p \Vdash \theta$ and $p \Vdash \eta$.
4. $p \Vdash \theta \rightarrow \eta$ if and only if for every $q \preceq p$, if $q \Vdash \theta$ then $q \Vdash \eta$.
5. $p \Vdash \forall x\, \theta(x)$ if and only if for every term t, $p \Vdash \theta(t)$.

Abbreviation: $\Vdash \varphi$ means $\forall p\; p \Vdash \varphi$

Some lemmata

Lemma (monotonicity): If $q \preceq p$ and $p \Vdash \varphi$, then $q \Vdash \varphi$.

Lemma: If φ is provable intuitionistically, $\Vdash \varphi$.

These hold for any forcing relation like ours.

Lemma: If $\{\varphi_1, \ldots, \varphi_k\}$ is provable classically, then $(\sim\varphi_1)^N \wedge \ldots \wedge (\sim\varphi_k)^N \rightarrow \bot$ is provable intuitionistically.

Here $\cdot^N$ is just the Gödel-Gentzen double-negation translation.

Lemma: For each formula φ in the language of the classical sequent calculus, $\{\varphi\} \Vdash \varphi^N$.

The proof is by induction on φ.

Putting it all together

Theorem: If $\{\varphi_1, \ldots, \varphi_k\}$ is provable in the classical sequent calculus, then it has a cut-free proof.

Proof. Suppose $\{\varphi_1, \ldots, \varphi_k\}$ is provable classically.

Then $({\sim}\varphi_1)^N \wedge \ldots \wedge ({\sim}\varphi_k)^N \rightarrow \perp$ is provable intuitionistically.

Hence it is forced.

By the lemma, for each i, $\{{\sim}\varphi_i\} \Vdash ({\sim}\varphi_i)^N$.

By monotonicity,
$\{{\sim}\varphi_1, \ldots, {\sim}\varphi_k\} \Vdash ({\sim}\varphi_1)^N \wedge \ldots \wedge ({\sim}\varphi_k)^N$.

Hence, $\{{\sim}\varphi_1, \ldots, {\sim}\varphi_k\} \Vdash \perp$.

By definition, this means that there is a cut-free proof of $\{\varphi_1, \ldots, \varphi_k\}$.

Notes

Some further details:

1. Mapping φ to $[\![\varphi]\!] = \{p \mid p \Vdash \varphi\}$ gives the proof an algebraic character.
2. The proof is constructive: for each fixed classical derivation d of Γ, one obtains a proof that "there is a cut-free proof of Γ" in intuitionistic first-order logic plus some syntax.
3. Realizing this proof yields a typed lambda term.
4. With a notion of "covering," can extend the proof to intuitionistic logic.
5. Both proofs extend to higher-order logic.
6. A version of the double-negation translation shows that the classical cut-elimination theorem can be viewed as a special case of the intuitionistic one.

Algebraic proofs of cut elimination for intuitionistic higher-order logic are well-known (e.g. Buchholz '75).

A version of the double-negation translation

Theorem. Suppose φ is provable classically. Then $\neg(\neg\varphi)^{nnf}$ is provable in minimal logic, where $\cdot^{nnf}$ denotes negation-normal form.

Proof #1: If θ is in negation normal form, $\theta^{nnf} \to \theta^N$ is provable intuitionistically. So we have

$$\begin{aligned} \varphi^N &\to \neg\neg\varphi^N \\ &\to \neg(\neg\varphi)^{nnf} \end{aligned}$$

This provides efficient translations between proofs with cut or modus ponens.

Proof #2: A cut-free classical proof of

$$\{\varphi_1, \ldots, \varphi_k\}$$

corresponds to a cut-free minimal proof of

$$\{{\sim}\varphi_1, \ldots, {\sim}\varphi_k\} \Rightarrow \bot.$$

This provides efficient translations between cut-free proofs.