The Promise of Formal Mathematics

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Formalization of mathematics

"The development of mathematics toward greater precision has led, as is well known, to the formalization of large tracts of it, so one can prove any theorem using nothing but a few mechanical rules. The most comprehensive formal systems that have been set up hitherto are the system of *Principia Mathematica* on the one hand and the Zermelo-Fraenkel axiom system of set theory . . . on the other. These two systems are so comprehensive that in them all methods of proof used today in mathematics are formalized, that is, reduced to a few axioms and rules of inference. One might therefore conjecture that these axioms and rules of inference are sufficient to decide any mathematical question that can at all be formally expressed in these systems."

Formalization of mathematics

"It will be shown below that this is not the case. . . ."

(Kurt Gödel, *On formally undecidable propositions of Principia Mathematica and related systems*, 1931.)

The positive claim: most ordinary mathematics is formalizable, *in principle*.

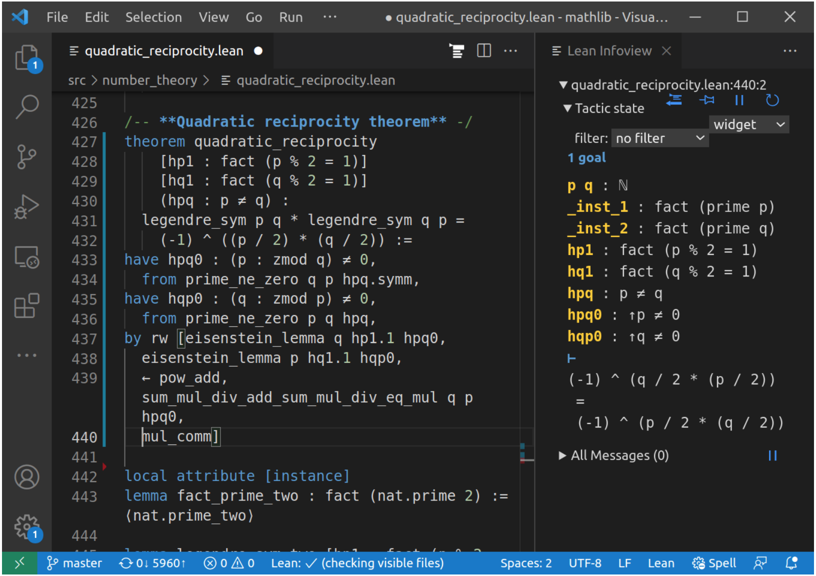
Formalization of mathematics

With the help of computational proof assistants, mathematics is formalizable *in practice*.

Working with such a proof assistant, users construct a formal axiomatic proof.

Systems with substantial mathematical libraries include Mizar, HOL, Isabelle, HOL Light, Coq, ACL2, PVS, Agda, Metamath, and Lean.

Formalization of mathematics



Formalization of mathematics

Most undergraduate mathematics, and a fair amount of graduate school mathematics, has been formalized.

A number of "big name" theorems have been formalized: the prime number theorem, the four color theorem, Dirichlet's theorem, the central limit theorem, the incompleteness theorems.

Gonthier et. al completed a verification of the Feit-Thompson theorem in 2012.

The *Flyspeck* project completed its verification of Hales' proof of the Kepler conjecture in 2014.

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- Lean and the Lean community
- Why formalize mathematics?
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- Resources

The Lean theorem prover

Lean is a new interactive theorem prover, developed principally by Leonardo de Moura at Microsoft Research, Redmond.

It is open source, with a permissive license, Apache 2.0.

Current developers: Leonardo de Moura and Sebastian Ullrich.

Current contributors: Wojciech Nawrocki, Daniel Fabian, Gabriel Ebner, Mario Carneiro, Marc Huisinga, Lewis “Mac” Malone, Daniel Selsam, . . .

Past contributors: Soonho Kong, Jared Roesch, . . .

The Lean theorem prover

A brief history:

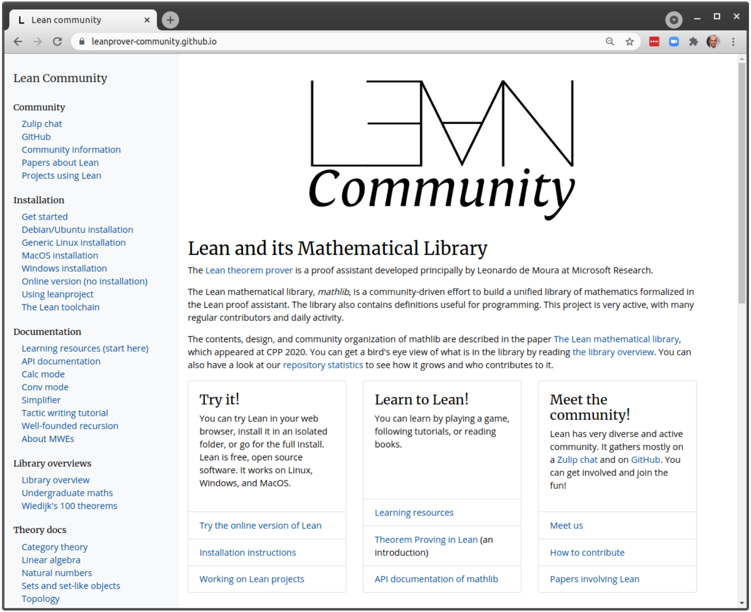
- 2013: the project begins
- 2014: Lean 2 released
- 2016: Lean 3 released
- 2021: prerelease of Lean 4

In 2017, Mario Carneiro split off the main library, *mathlib*. He revised and expanded it, and encouraged people to join in.

Core mathematicians like Kevin Buzzard, Johan Commelin, Patrick Massot, and Scott Morrison discovered Lean, and starting using it.

The *Lean community* was born.

The Lean community



The Lean community

The mathlib board of maintainers:

- Anne Baanen
- Reid Barton
- Mario Carneiro
- Bryan Gin-ge Chen
- Johan Commelin
- Rémy Degenne
- Floris van Doorn
- Gabriel Ebner
- Sébastien Gouëzel
- Markus Himmel
- Simon Hudon
- Chris Hughes
- Yury G. Kudryashov
- Robert Y. Lewis
- Heather Macbeth
- Patrick Massot
- Bhavik Mehta
- Scott Morrison
- Oliver Nash
- Adam Topaz
- Eric Wieser

The Lean community

There have been some notable successes.

Kevin Buzzard, Johan Commelin, and Patrick Massot formalized the notion of a *perfectoid space*.

Sander Dahmen, Johannes Hölzl, and Robert Lewis formalized a proof of the Ellenberg-Gijswijt theorem.

Jesse Han and Floris van Doorn formalized a proof of the independence of the continuum hypothesis.

Patrick Massot has launched a project to formalize sphere eversion.

The Lean community

On December 5, 2020, Peter Scholze challenged anyone to formally verify some of his recent work with Dustin Clausen.

Johan Commelin led the response from the Lean community. On June 5, 2021, Scholze acknowledged the achievement.

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"Exactly half a year ago I wrote the Liquid Tensor Experiment blog post, challenging the formalization of a difficult foundational theorem from my Analytic Geometry lecture notes on joint work with Dustin Clausen. While this challenge has not been completed yet, I am excited to announce that the Experiment has verified the entire part of the argument that I was unsure about. I find it absolutely insane that interactive proof assistants are now at the level that within a very reasonable time span they can formally verify difficult original research."

The Lean community

Formal mathematics is finally getting some recognition.

- The Lean Zulip channel is lively.
- Kevin Buzzard's blog posts and talks go viral.
- Lean and mathlib have been getting good press:
    - *Quanta:* "Building the mathematical library of the future"
    - *Quanta:* "At the Math Olympiad, computers prepare to go for the gold"
    - *Nature:* "Mathematicians welcome computer-assisted proof in 'grand unification' theory"
- Lean workshops are planned at ICERM (2022), MSRI (2023), and more.

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Why formalize mathematics?

Reason #5: Correctness.

Mathematics is about rigor and precison.

We want our proofs to be correct.

Formalization isn't a replacement for understanding, but the mathematics that we understand isn't meaningful if it isn't correct.

Why formalize mathematics?

Reason #4: Libraries.

Digital technology allows us to develop communal repositories of knowledge.

Every definition, theorem, and proof is recorded, and can be accessed.

Libraries support exploration and search.

Why formalize mathematics?

Reason #3: Education.

Formalism is demanding, and can be frustrating at times.

But it provides instant feedback, instant gratification, and fun.

Formal tools can be designed for different audiences, from elementary school students to PhD students.

Why formalize mathematics?

Reason #2: Discovery.

Symbolic AI and machine learning have had a profound impact on hardware and software verification, AI, planning, constraint solving, optimization, knowledge representation, expert systems, databases, language processing, . . .

But they have had almost no impact on pure mathematics.

We have no idea what the tools can do, and there is a lot we need to learn.

Formalization is a gateway to automation.

Why formalize mathematics?

Reason #1: Collaboration.

The Lean Zulip channel is remarkable. People ask questions, explain things, pose challenges, share results, discuss plans.

Newcomers are welcome. Each successive generation helps the next.

Of course, we can collaborate without formalism.

But contributing to a formal library is *transcendental*, and provides focus.

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Mission

"The mission of the center is to use formal methods to improve global access to mathematics and to assist in the dissemination, verification, and discovery of mathematics. The center will focus on the use of the *Lean* programming language and interactive proof system, with the following three goals:

- *Formalization*. The center will support the development of Lean's communal mathematics library, mathlib, and formalization projects of contemporary mathematical interest.
- *Infrastructure.* The center will support the development of better interfaces, automated reasoning tools, and formal infrastructure for collaboration, exploration, and discovery.
- *Education.* The center will support the development of resources for teaching mathematics, based on Lean and formal methods."

Mission

Imagine a world where:

- elementary school students can experiment interactively with Euclidean geometry;

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Formal mathematics and digital technology make that possible.

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Resources

Potential postdocs:

- Gabriel Ebner
- Mario Carneiro
- Edward Ayers

Also a joint postdoc with Mathematical Sciences.

Students: Seul Baek, Wojciech Nawrocki, Joshua Clune, Evan Lohn, Cayden Codel

Staff: Jackie DeFazio, Mary Grace Joseph

The center will have a web presence, visitors, meetings.

Resources

Marijn Heule and students (Emre Yolcu, Joseph Reeves, Evan Lohn, and Cayden Codel)

Thomas Hales, Reid Barton, and students (Jesse Han, Koundinya Vajjha, Luis Berlioz)

Colleagues in philosophy: Wilfried Sieg, Steve Awodey, Adam Bjorndahl, Francesca Zaffora-Blando, . . .

Colleagues in mathematics: Clinton Conley, James Cummings, Rami Grossberg, Ernest Schimmerling, Wesley Pegden, Po-Shen Loh. . .

Colleagues in computer science: Robert Harper, Karl Crary, Jan Hoffmann, Frank Pfenning, André Platzer, Ryan O'Donnell, . . .

Resources

The Department of Mathematical Sciences

The Computer Science Department

The Department of Philosophy

Resources

The Department of Mathematical Sciences

The Computer Science Department

The Department of Philosophy

The Machine Learning Department

Resources

The Department of Mathematical Sciences

The Computer Science Department

The Department of Philosophy

The Machine Learning Department

The Human-Computer Interaction Institute

Resources

The Department of Mathematical Sciences

The Computer Science Department

The Department of Philosophy

The Machine Learning Department

The Human-Computer Interaction Institute

The Language Technologies Institute

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The Computer Science Department

The Department of Philosophy

The Machine Learning Department

The Human-Computer Interaction Institute

The Language Technologies Institute

The Psychology Department and the Simon Initiative

Resources

We'll have the friendship and support of Leonardo de Moura and the Lean development team.

Resources

We'll have the friendship and support of Leonardo de Moura and the Lean development team.

And we'll have the most beautiful, wonderful piece of software ever written.

Resources

Lucas Allen, Ellen Arlt, Aaron Anderson, Edward Ayers, Anne Baanen, Seul Baek, Reid Barton, Tim Baumann, Alexander Bentkamp, Alex Best, Jasmin Blanchette, Riccardo Brasca, Thomas Browning, Aaron Bryce, Kevin Buzzard, Louis Carlin, Mario Carneiro, Nicoló Cavallieri, Cyril Cohen, Bryan Gin-ge Chen, Johan Commelin, Sander Dahmen, Benjamin Davidson, María Inés de Frutos-Fernéndez, Leonardo de Moura, Anatole Dedecker, Rémy Degenne, Yaël Dillies, Floris van Doorn, Gabriel Ebner, Daniel Fabian, Sébastien Gouëzel, Thomas Hales, Markus Himmel, Johannes Hölzl, Keeley Hoek, Simon Hudon, Chris Hughes, Marc Huisinga, Kevin Kappelmann, Soonho Kong, Yury G. Kudryashov, Julian Küllshammer, Shing Tak Lam, Kenny Lau, Sean Leather, Robert Y. Lewis, Jannis Limperg, Amelia Livingston, Jean Lo, Patrick Lutz, Heather Macbeth, Paul-Nicolas Madelaine, Assia Mahboubi, Lewis Malone, Gihan Marasingha, Patrick Massot, Bhavik Mehta, Kyle Miller, Ramon Fernández Mir, Hunter Monroe, Scott Morrison, Joseph Myers, Wojciech Nawrocki, Oliver Nash, Paula Neeley, Filippo Nuccio, Grant Passmore, Yakov Pechersky, Stanislas Polu, Alexandre Rademaker, Jared Rosch, Cody Roux, Jason Rute, Peter Scholze, Calle Sönne, Justus Springer, Jalex Stark, Patrick Stevens, Neil Strickland, Abhimanyu Pallavi Sudhir, Kevin Sullivan, Adam Topaz, Devon Tuma, Sebastian Ullrich, Ruben Van de Velde, Koundinya Vajjha, Paul van Wamelen, Jens Wagemaker, David Wärn, Eric Wieser, Minchao Wu, Haitao Zhang, Zhouang Zhou, Sebastian Zimmer, Andrew Zipperer, . . .

Resources

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Borrowing from Andrew Carnegie:

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There are many questions to decide, involving investigation, careful study, and much labor.

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There are many questions to decide, involving investigation, careful study, and much labor.

But I am in a position to assure you that we are prepared to face the problem,

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Borrowing from Andrew Carnegie:

There are many questions to decide, involving investigation, careful study, and much labor.

But I am in a position to assure you that we are prepared to face the problem, and that our heart is in the work.