# Proof Theory and Proof Mining IV: Ultraproducts and Nonstandard Analysis

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# Sequence of Topics

1. Computable Analysis
2. Formal Theories of Analysis
3. The Dialectica Interpretation and Applications
4. Ultraproducts and Nonstandard Analysis

# Nonstandard analysis

Seventeenth century:

- Newton initially develops calculus with notions of fluxions and fluents.
- Leibniz develops calculus with the notion of an infinitesimal.

Bishop Berkley (*The Analyst*, 1734) criticizes such talk.

Nineteenth century: the rigorization of calculus

Abraham Robinson (1966): *Nonstandard Analysis*.

# Nonstandard analysis

The idea:

- Build a model of mathematics, with lots of "stuff."
- From outside the model, distinguish the standard stuff from the nonstandard stuff.

The nonstandard stuff gives you infinitesimals, and more.

# Nonstandard analysis

Robinson's approach: let $\mathcal{M}$ be a structure. $\mathcal{M}$ is $\lambda$-saturated if whenever $A \subset |\mathcal{M}|$, $card(A) < \lambda$, then every type over $A$ is realized in $\mathcal{M}$.

Let $\mathcal{M}$ be any infinite model. Using model theory (just the compactness theorem), we can construct $\lambda$-saturated models $\mathcal{M}' \succeq \mathcal{M}$ for any $\lambda$.

# Nonstandard analysis

Let $\mathcal{N} = \langle \mathbb{N}, 0, 1, +, \times, <, \ldots \rangle$, and let $\mathcal{N}' \succeq N$ be $\aleph_0$ saturated.

There are elements $0, 1, 2, \ldots \in \mathcal{N}'$. These are the standard natural numbers.

There is also an element $\omega \in |\mathcal{N}'|$ such that

$$\omega > 0, \omega > 1, \omega > 2, \ldots,$$

a "nonstandard" natural number.

Also $\omega - 7$, $\omega + 1$, $\omega^2$, $\omega^\omega$. The model $\mathcal{N}'$ "thinks" these are all natural numbers.

# Nonstandard analysis

Consider the nonstandard rationals, $\mathbb{Q}^*$, in $\mathcal{N}'$.

Say $q \in \mathbb{Q}^*$ is *bounded* or *limited* if $|q| < n$ for some standard $n$.

Say $q > 0$ is a positive infinitesimal if $|q| < 1/n$ for every standard $n$.

Say $q_1 \sim q_2$ if $|q_1 - q_2|$ is zero or infinitesimal.

Observation: $\mathbb{R} \simeq \{q \in \mathbb{Q}^* \mid q \text{ bounded}\}/{\sim}$

# Nonstandard analysis

Why stop with numbers? Consider the structure

$$\mathcal{M} = (\mathcal{N}, \mathcal{P}(\mathcal{N}), \mathcal{P}(\mathcal{P}(\mathcal{N})), \ldots)$$

as a many-sorted structure.

Take a sufficiently saturated elementary extension.

Find $\mathbb{R}^*$ and $\mathbb{R}$, among other things.

# Nonstandard analysis

Robinson showed that this is a natural setting for analysis.

For example, a sequence $(s_n)_{n \in \mathbb{N}}$ of real numbers extends to a sequence $(s_n)_{n \in \mathbb{N}*}$ of nonstandard reals.

$(s_n)$ converges to $r$ iff $s_n \sim r$ for every unbounded $n$.

$f$ is continuous at $x$ iff $f(y) \sim f(x)$ whenever $y \sim x$.

# Ultraproducts

Here is another way to build a nonstandard model.

An *ultrafilter on* $\mathbb{N}$ is a collection $\mathcal{U} \subset \mathcal{P}(\mathbb{N})$ satisfying:

- $A \in \mathcal{U}, B \subseteq A \Rightarrow B \in \mathcal{U}$
- $A, B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$
- For every $A$, either $A \in \mathcal{U}$ or $\bar{A} \in \mathcal{U}$.

Think of $\mathcal{U}$ is a collection of "large" sets. Every set is either large or small; e.g. if $\{x \mid x \text{ is even}\}$ is large, then $\{x \mid x \text{ is odd}\}$ is small.

# Ultraproducts

A cheap way to get an ultrafilter: let $\mathcal{U} = \{A \mid n \in A\}$ for some fixed $n$.

Such an ultrafilter is *principal*.

Zorn's lemma can be used to how the existence of *nonprincipal ultrafilters on* $\mathbb{N}$.

(There is nothing special about $\mathbb{N}$ here.)

# Ultrapowers

Let $\mathcal{M}$ be a structure, and consider the product $\mathcal{M}^\omega$.

Say $f \sim g$ if $\{x \mid f(x) = g(x)\} \in U$, and similarly for other relations.

Form the structure $\mathcal{M}' = \mathcal{M}^\omega / \sim$.

**Theorem (Łoś).** This is an elementary extension.

**Theorem.** $\mathcal{M}'$ is $\aleph_1$ saturated.

# Nonstandard Analysis and Proof Theory

Nonstandard methods are abstract, and not obviously constructive.

- We can ask about their logical strength, and reductions to standard theories.
- We can ask about quantitative and computational aspects.

Nonstandard methods embody strong forms of compactness.

Understanding compactness has been a recurring theme in these lectures.

# Logical strength

Nonstandard methods seem strong.

Building saturated models requires choice.

In fact, the axiom of choice is equivelent to the statement that a product of a nonempty collection of nonempty sets is nonempty.

But we have seen that compactness alone is equivalent to Weak König's Lemma.

Proof theory tells us that some versions of nonstandardness come cheap.

# Logical strength

Nelson defined a nonstandard version of set theory, $IST$, conservative over $ZFC$.

Axioms:

- Idealization:
  $(\forall^{st} z\ (finite(z) \to \exists y\ \forall x \in z\ \varphi(x, y))) \leftrightarrow \exists y\ \forall^{st} x\ \varphi(x, y)$.
- Standardization: $\forall^{st} x\ \exists^{st} y\ \forall^{st} z\ (z \in y \leftrightarrow t \in x \wedge \varphi(t))$
- Transfer: $\forall^{st} x\ \varphi(x) \leftrightarrow \forall x\ \varphi(x)$ where $\varphi$ is an internal formula with standard parameters.

Idealization expresses that the model is saturated.

Transfer expresses that the standard part is an elementary substructure.

# Logical strength

Consider a nonstandard version *NPA* of *PA*. Add a predicate *st* for the "standard" numbers, and use the following axioms:

- The usual defining axioms for $0, S, +, \times$
- Induction for all formulas with respect to standard numbers.
- Induction for *st*-free formulas.
- The standard part contanis 0 and is closed under $S$.
- There exists a nonstandard integer.
- Transfer: an *st*-free formula with standard parameters holds on the standard part iff it holds on the full part.

**Theorem (Friedman).** *NPA* is a conservative extension of *PA*.

**Proof (idea).** Construct a nonstandard model in *PA*.

# Logical strength

Keisler, Ward, and Henson have shown that adding a saturation principle increases the strength to *second-order arithmetic*.

Takahashi, Keisler, and others have studied nonstandard methods from the point of view of reverse mathematics, including theories have the same strength as the “big 5”.

Suppes, Sommer, Avigad, Jerabek, Sanders, and others have studied weaker theories.

# Logical strength

For example, define $NPRA^\omega$ to be $PRA^\omega$ plus

- $\neg st(\omega)$
- $st(x) \wedge y < x \to st(y)$
- $st(x_1) \wedge \ldots \wedge st(x_k) \to st(f(x_1, \ldots, x_k))$, for each type 1 term $f$ with no free variables and no occurence of $\omega$

Add a schema of $\forall$-transfer without parameters:

- $\forall^{st} \vec{x}\, \psi(\vec{x}) \to \forall \vec{x}\, \psi(\vec{x})$

where $\psi$ is a q.f. and $st$-free and does not involve $\omega$.

**Theorem (Avigad).** Suppose $NPRA^\omega$ proves $\forall^{st} x\, \exists y\, \varphi(x, y)$, where $\varphi$ is quantifier-free in the language of $PRA$. Then $PRA^\omega$ proves $\forall x\, \exists y\, \varphi(x, y)$, and hence $PRA$ proves it as well.

# Logical strength

Townser has considered a system $ACA_0 + \mathcal{U}$.

$\mathcal{U}$ is a third-order constant, and there are axioms saying that it denotes a nonprincipal ultrafilter.

**Theorem (Towsner).** $ACA_0 + (\mathcal{U})$ is a conservative extension of $ACA_0$.

The proof uses a forcing interpretation. It works for other theories as well, like $ATR_0$ and $\Pi^1_1 - CA_0$.

# Logical strength

Independently, Kreuzer defined a higher-type theory, $ACA^\omega + (\mathcal{U})$.

**Theorem (Kreuzer).** $ACA_0 + (\mathcal{U})$ is $\Pi^1_2$ conservative over $ACA_0$.

His proof uses the Dialectica interpretation.

# Nonstandard methods in ergodic theory

Some results:

- Kamae gave a nonstandard proof of the ergodic theorem.
- Towsner gave a nonstandard proof of Tao's theorem on the norm convergence of diagonal averages.
- Tao gave a nonstandard proof of Walsh's theorem on the norm convergence of nilpotent ergodic averages.
- Tao recently gave a nonstandard "ergodic-theoretic" proof of Szemerédi's theorem.

Tao has blogged on the use of nonstandard methods in combinatorics and algebraic geometry as well.

Some of these raise a connection between nonstandard models and metastability.

# Metastability

Recall that proof mining shows that metastable convergence theorems often have very uniform bounds.

Such uniformities are can often obtained using nonstandard methods.

# Ultraproducts in analysis

Let $I$ be any infinite set, $D$ be a nonprincipal ultrafilter on $I$.

Suppose that for each $i$, $(X_i, d_i)$ is a metric space with a distinguished point $a_i$.

Let

$$X_\infty = \left\{ (x_i) \in \prod_{i \in I} X_i \mid \sup_i d(x_i, a_i) < \infty \right\} / \sim,$$

where $(x_i) \sim (y_i)$ if and only if $\lim_{i,D} d(x_i, y_i) = 0$.

Call this the "ultraproduct of the spaces $X_i$." This works for more general metric structures.

# Metastability and ultraproducts

**Theorem (Avigad and Iovino).** Let $C$ be a collection of pairs $((X, d), (a_n)_{n \in \mathbb{N}})$. Fix a nonprincipal ultrafilter. The following statements are equivalent:

1. There is a uniform bound on the rate of metastability for the sequences $(a_n)$.
2. For any sequence $((X_k, d_k), (a_n^k))_{k \in \mathbb{N}}$ of elements of $C$, the sequence $(\bar{a}_n)$ in the ultraproduct is Cauchy.

The first clause means: for every $F : \mathbb{N} \to \mathbb{N}$ and $\varepsilon > 0$, there is a $b$ with the following property: for every pair $((X, d), (a_n)_{n \in \mathbb{N}})$ in $C$, there is an $n \leq b$ such that $d(a_i, a_j) < \varepsilon$ for every $i, j \in [n, F(n)]$.

# Metastability and ultraproducts

What this means: if you have a convergence theorem, and

- the class of structures described by the theorem is closed under ultraproducts, and
- and the hypotheses are preserved by ultraproducts,

then there is a uniform bound on the rate of metastability.

There are sufficient syntactic conditions for these conditions to hold.

# Metastability and ultraproducts

A strong version of the mean ergodic theorem:

**Theorem (Lorch, Riesz, Yosida, Kakutani).** If $T$ is any power-bounded linear operator on a reflexive Banach space $\mathbb{B}$, and $x$ is any element of $\mathbb{B}$, then the sequence $(A_n x)_{x \in \mathbb{N}}$ converges.

Alas, the class of reflexive Banach spaces is not closed under ultraproducts.

But for fixed $p$, the $p$-uniformly convex spaces are, confirming the uniformity in that case.

# Metastability and ultraproducts

There are other collections of reflexive Banach spaces preserved under ultraproducts: uniformly nonsquare Banach spaces, $J$-$(n, \varepsilon)$ convex Banach spaces, etc.

The result also shows that mere convergence in the Tao / Walsh results implies uniformity.

It also provides short confirmations of other uniformities uncovered by Kohlenbach and students.

# Metastability and ultraproducts

If you prove a convergence theorem, you know it is true.

- Closure under ultraproducts then tells you that there are uniform bounds on the rate of metastability.
- Under general computability hypotheses, there is even a computable bound (Rute).

Alternatively, using Kohlenbach's methods:

- If the proof can be carried out in a certain (strong) theory, and the theorem has a certain logical form, you get uniformity and computability at once.
- Precise details of the theory give you more information about the computation.
- Analysis of the proof gives you an explicit bound.
- The methods can also handle non-continuous functions.