Forcing and Hilbert's program

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Proof theory in the Hilbert tradition

Proof theory: the general study of deductive systems

Structural proof theory: . . . with respect to structure, transformations between proofs, normal forms, etc.

Hilbert's program:

- Formalize abstract, infinitary, nonconstructive mathematics.
- Prove consistency using only finitary methods.

More general versions:

- Prove consistency relative to constructive theories.
- Understand mathematics in constructive terms.
- Study mathematical reasoning in "concrete" terms.

Proof theory in the Hilbert tradition

Proof-theoretic analysis can "reduce":

- infinitary to finitary reasoning
- nonconstructive to constructive reasoning
- impredicative to predicative reasoning
- nonstandard to standard reasoning
- arbitrary choices to choice-free reasoning

In a sense, this eliminates "ideal" elements.

A brief history of forcing

Cohen, '63: the independence of CH and AC from set theory.

Kripke, '59-'65: semantics for modal and intuitionistic logic.

Perspectives:

- Set theory: generic extensions, approximations
- Modal logic: possible worlds
- Recursion theory: diagonalization, conditions
- Model theory: existentially closed models
- Categorical logic: logic of sheaves
- Descriptive set theory: generic truth
- Effective descriptive set theory
- Complexity theory

Themes: diagonalization, local/global properties, construction via approximations

Overview

In this talk, I will explain how forcing is relevant (and ideally suited to) traditional, Hilbert-style proof theory.

1. The framework
   1.1 Minimal, intuitionistic, and classical logic
   1.2 The forcing relation
   1.3 Variations
2. Applications
   2.1 Subsystems of second-order arithmetic
   2.2 Intuitionistic theories
   2.3 "Point-free" model theory

From minimal to classical logic

Flavors of first-order logic:

- Minimal (M): nicest computational interpretation
- Intuitionistic (I): add "from $\bot$ conclude φ"
- Classical (C): add $\neg\neg\varphi \to \varphi$ or $\varphi \lor \neg\varphi$

Intuitionistic to minimal (F): replace atomic A by $A \lor \bot$ or $\neg\neg A$. Then

$$\vdash_M \bot \to \varphi^F$$

Classical to minimal (N): also replace $\varphi \lor \psi$ by $\neg(\neg\varphi \land \neg\psi)$ and $\exists x\, \varphi$ by $\neg\forall x\, \neg\varphi$. Then

- $\vdash_M \varphi^N \leftrightarrow \neg\neg\varphi^N$
- $\Gamma \vdash_C \varphi$ implies $\Gamma^N \vdash_M \varphi^N$

The Kuroda translation (K): instead, add $\neg\neg$ after each universal quantifier.

- $\vdash_M \neg\neg\varphi^K \leftrightarrow \varphi^N$
- $\vdash_C \varphi$ implies $\vdash_I \neg\neg\varphi^K$

Kripke semantics

Start with:

- a poset P (possible worlds)
- a domain $D(p)$ at each world
- for each $p \in P$ and atomic A, an interpretation of A at p

satisfying monotonicity: if $q \leq p$, then

- $D(q) \supseteq D(p)$
- If $p \Vdash A(a_0, \ldots, a_{k-1})$ then $q \Vdash A(a_0, \ldots, a_{k-1})$.

Extend the forcing relation to $L(D)$ inductively:

1. $p \Vdash \theta \wedge \eta$ iff $p \Vdash \theta$ and $p \Vdash \eta$
2. $p \Vdash \theta \vee \eta$ iff $p \Vdash \theta$ or $p \Vdash \eta$
3. $p \Vdash \theta \to \eta$ iff $\forall q \leq p \; (q \Vdash \theta \to q \Vdash \eta)$
4. $p \Vdash \forall x \; \varphi(x)$ iff $\forall q \leq p \; \forall a \in D(q) \; q \Vdash \varphi(a)$
5. $p \Vdash \exists x \; \varphi(x)$ iff $\exists a \in D(p) \; p \Vdash \varphi(a)$

Kripke semantics

Theorem.

- (monotonicity): $p \Vdash \varphi$ and $q \leq p$ imply $q \Vdash \varphi$
- $\vdash_M \varphi$ implies $\Vdash \varphi$

For intuitionistic logic, add

- $p \nVdash \bot$

Theorem.

- $p \Vdash \bot \to \varphi$
- $\vdash_I \varphi$ implies $\Vdash \varphi$.

Forcing for classical logic

Weak forcing: define $\Vdash_C \varphi$ by $\Vdash_M \varphi^N$.

For example:

- $p \Vdash_C \theta \vee \eta$ iff $\forall q \leq p \, \exists r \leq q \, ((r \Vdash_C \theta) \vee (r \Vdash_C \eta))$
- $p \Vdash_C \neg\neg\theta$ iff $\forall q \leq p \, \exists r \leq q \; r \Vdash_C \theta$

Theorem.

1. monotonicity: $p \Vdash_C \varphi$ and $q \leq p$ imply $q \Vdash_C \varphi$
2. genericity: $p \Vdash_C \varphi$ iff $\forall q \leq p \, \exists r \leq q \; r \Vdash_C \varphi$
3. soundness: $\vdash_C \varphi$ implies $\Vdash_C \varphi$

Strong forcing: define $\Vdash_{C'} \varphi$ by $\Vdash_M \varphi^K$.

Then

$$\Vdash_C \varphi \textit{ iff } \Vdash_{C'} \neg\neg\varphi$$

Notes and variations

1. $p \Vdash_C \varphi$ corresponds to "φ is true in every extension by a generic containing p"
2. Can replace $p \nVdash \bot$ by "if $p \Vdash \bot$ then $p \Vdash A(a_0, \ldots, a_{k-1})$."
3. Beth models:

   $$\begin{aligned} &p \Vdash \varphi \vee \psi \textit{ iff for some covering } C(p) \textit{ of } p, \\ &\forall q \in C(p)\ ((q \Vdash \varphi) \vee (q \Vdash \psi)) \end{aligned}$$

   and similarly for $\exists$.
4. Replace the poset by a category (presheaf models)
5. Replace Beth's coverings by a Grothendieck topology (sheaf models)
6. Extend to higher-order logic (and set theory)

“Internalized” constructions

Think syntactically:

- Work in a theory T.
- Use definable predicates, Cond, $\leq$, Name, $p \Vdash A(a_0, \ldots, a_{k-1})$.
- Assume T proves monotonicity, etc.

Then T can verify the soundness of forcing:

- Minimal logic verifies minimal forcing
- Intuitionistic logic verifies intuitionistic forcing
- Classical logic verifies classical forcing
- With modified falsity, minimal logic verifies intuitionstic forcing
- With additional negations, minimal logic verifies classical forcing
- One can also get genericity in minimal logic

Interlude

We've considered:

1. Minimal, intuitionistic, and classical logic
2. The forcing relation
3. Notes and variations

To interpret T_1 in T_2:

- Define a poset, basic forcing notions in T_2.
- Show axioms of T_1 are forced.
- Conclude: if T_1 proves φ, then T_2 proves "φ is forced."

For partial conservativity, show

- For $\varphi \in \Gamma$, if T_2 proves "φ is forced," then T_2 proves φ.

Applications

1. Subsystems of second-order arithmetic
   - Choice principles (Steele, Friedman)
   - Weak König's lemma
   - Baire Category Theorem
   - Ramsey's theorem
2. Intuitionistic theories
   - Goodman's theorem
   - Continuity, Bar recursion (Beeson, Grayson, Hayashi)
   - Interpreting classical theories in constructive ones
3. "Point-free" model theory
   - Nonstandard arithmetic and analysis
   - Eliminating Skolem functions
   - Algebraic proofs of cut elimination

I'll discuss some examples, favoring my own work.

Subsystems of arithmetic

Language: $0, 1, +, \times, <, \in, x, y, z, \ldots X, Y, Z, \ldots$

Full second-order arithmetic has:

- Quantifier-free defining equations
- Induction
- Comprehension: $\exists Z \, \forall x \, (x \in Z \leftrightarrow \varphi(x))$

One can also consider various choice principles.

Restrict induction to Σ^0_1 formulas with parameters, and restrict set existence principles:

- *RCA₀*: recursive (Δ^0_1) comprehension
- *WKL₀*: paths through infinite binary trees
- *ACA₀*: arithmetic comprehension
- *ATR₀*: transfinitely iterated arithmetic comprehension
- Π^1_1-CA_0: Π^1_1 comprehension

Weak König's lemma

König's lemma. Every infinite, finitely branching tree T has an infinite path

Kleene's basis theorem. The leftmost branch is computable in T'.

Weak König's lemma. Every infinite tree on $\{0, 1\}$ has an infinite path.

The Jockusch-Soare low basis theorem. Every such tree has a *low* path, i.e. satisfying $P' \leq_T T'$.

Iterative construction: at stage n, thin the tree to guarantee that $\varphi_n^P(0)$ will diverge, if possible; extend the path one step.

Weak König's lemma

Theorem (Friedman). *WKL_0* is conservative over primitive recursive arithmetic for Π^0_2 sentences.

Theorem (Harrington). *WKL_0* is, moreover, conservative over *RCA_0* for Π^1_1 sentences.

Proof.

- Start with a countable model of *RCA_0*.
- Pick an infinite binary tree.
- Add a generic branch (conditions: infinite subtrees).
- Show Σ^0_1 induction is preserved.
- Iterate.

Weak König's lemma

There are two ways of interpreting WKL_0 in RCA_0:

- Hájek: formalize a sharper version of the low basis theorem.
- Avigad: formalize the (iterated, proper-class) forcing argument. Conditions: sequences of names for infinite binary trees.

Variations:

- Brown and Simpson: use Cohen forcing to get a version of Baire Category theorem.
- Simpson and Smith: results for WKL and elementary arithmetic.
- Ferreira, Fernandes: results for WKL and feasible arithmetic.
- Simpson, Tanaka, Yamazaki: additional definability results.

Ramsey's theorem

Definition. $RT(k)$ is the statement that every for 2-coloring of k tuples of natural numbers there is an infinite homogeneous set.

Theorem (Jockusch). There is a recursive coloring of triples such that $0'$ is computable from any infinite homogenous set.

Theorem (Simpson). For each (standard) $k \geq 3$, $RT(k)$ is equivalent to arithmetic comprehension over RCA_0.

What about RT(2)?

Ramsey's theorem

Theorem (Jockusch). There is a recursive coloring such that no infinite homogeneous set is computable from $0'$.

Corollary. WKL_0 does not prove $RT(2)$.

Theorem (Seetapun). If A is not recursive, there is a recursive coloring such that A is not computable from any infinite homogeneous set.

Corollary. $RCA_0 + RT(2)$ does not prove ACA_0.

It is open as to whether WKL_0 proves $RT(2)$.

Ramsey's theorem

Theorem (Cholak, Jockusch, Slaman). Every 2-coloring C has an infinite homogeneous set H that is $low_2(C)$, i.e. $H'' = C''$.

Theorem (Cholak, Jockusch, Slaman). $RCA_0 + I\Sigma_2 + RT(2)$ is conservative over $RCA_0 + I\Sigma_2$ for Π^1_1 sentences.

> *first theorem : second theorem ::*
> *Jockusch-Soare : Harrington.*

Can the forcing argument be turned into a syntactic translation?

Goodman's theorem

Let HA^ω be a finite-type version of Heyting arithmetic (a conservative extension, without comprehension axioms).

The axiom of choice:

$$\forall x^\sigma \, \exists y^\tau \, \varphi(x,y) \rightarrow \exists f^{\sigma \rightarrow \tau} \, \forall x^\sigma \, \varphi(x, f(x)).$$

Classically, this implies comprehension. But intuitionistically:

Theorem (Goodman). $HA^\omega + AC$ is a conservative extension of HA^ω for arithmetic sentences.

Beeson's presentation:

- $HA^\omega + AC$ is realized in HA^ω, even with an extra function symbol.
- Force so that "φ is realized" implies "φ is true" for arithmetic sentences.

Interpreting classical theories constructively

The Gödel-Gentzen double-negation translation is a powerful tool:

- It reduces PA to HA, PA_2 to HA_2, ZF to IZF.
- The Friedman-Dragalin translation recovers Π^0_2 theorems.

But these methods do not work for S^1_2, $I\Sigma_1$, $\Sigma^1_1{-}AC$, KP.

What goes wrong? Some examples:

- The double-negation interpretation of Σ_1 induction involves induction on predicates of the form $\neg\neg\exists x\, A(x, y)$.
- The double negation translation of the Σ^1_1 axiom of choice is of the form

  $$\forall x\, \neg\neg\exists Y\, \varphi(x, Y) \rightarrow \neg\neg\exists Y\, \forall x\, \varphi(x, Y_x)$$

  where φ is arithmetic.

Interpeting classical theories

We can use the latitude in defining "$p \Vdash \bot$" to repair the double negation translation.

- Buchholz: theories of inductive definitions
- Coquand and Hofmann: Σ_1 induction, bounded arithmetic
- Avigad: bounded arithmetic, Σ^1_1-AC, admissible set theory

Interpreting classical theories

For arithmetic with Σ_1 induction, it suffices to obtain a forcing interpretation of Markov's principle:

$$\neg \forall x\, A(x) \rightarrow \exists x\, \neg A(x)$$

Take conditions p to be (codes for) finite sets of Π_1 sentences,

$$\{\forall x\, A_1(x), \forall x\, A_2(x), \ldots, \forall x\, A_k(x)\}.$$

Define $p \leq q$ to be $p \supseteq q$.

For θ atomic, define $p \Vdash \theta$ to be

$$\exists y\, (A_1(y) \wedge \ldots \wedge A_k(y) \rightarrow \theta).$$

In particular, $p \Vdash \perp$ is equivalent to

$$\exists y\, (\neg A_1(y) \vee \ldots \vee \neg A_k(y)).$$

Then it turns out that if $p \Vdash \neg \forall x\, A(x)$, then $p \Vdash \exists x\, \neg A(x)$. In other words, Markov's principle is forced.

Some details

Lemma. The following are provable in $I\Sigma_1^i$:

1. $\{\forall x\, A(x)\} \Vdash \forall x\, A(x)$
2. If $p \Vdash \neg\forall x\, A(x)$, then $p \Vdash \exists x\, \neg A(x)$.
3. $\Vdash \neg\forall x\, A(x) \to \exists x\, \neg A(x)$

Proof. For 1, we have

$$\begin{aligned} \forall x\, A(x) \Vdash \forall x\, A(x) &\equiv \forall z\, (\forall x\, A(x) \Vdash A(z)) \\ &\equiv \forall z\, \exists y\, (A(y) \to A(z)). \end{aligned}$$

For 2, let p be the set $\{\forall x\, B_1(x), \ldots, \forall x\, B_k(x)\}$, and suppose $p \Vdash \neg\forall x\, A(x)$. By 1, we have $p \cup \{\forall x\, A(x)\} \Vdash \bot$. In other words,

$$\exists y\, (B_1(y) \wedge \ldots \wedge B_k(y) \wedge A(y) \to \bot)$$

which implies

$$\exists x, y\, (B_1(y) \wedge \ldots \wedge B_k(y) \to \neg A(x)),$$

which is to say

$$\exists x\, (p \Vdash A(x)).$$

Point-free thinking

- Points in a topological space can be approximated by open neighborhoods.
- Real numbers can be approximated by rational intervals.
- A maximal ideal can be approximated by subideals.
- An ultrafilter can be approximated by filters.
- A maximally consistent sets can be approximated by finite consistent sets.

In constructive or restricted frameworks, it is often better to:

- Work with the approximations.
- Use generic objects.
- Reason about what is "forced" to be true.

Remember: genericity = Kripke models + double negation interpretation.

Weak theories of nonstandard arithmetic

Add to the language of *PRA*:

- a predicate, $st(x)$ ("x is standard")
- a constant, ω

Let *NPRA* consist of *PRA* plus the following axioms:

- $\neg st(\omega)$
- $st(x) \wedge y < x \rightarrow st(y)$
- $st(x_1) \wedge \ldots \wedge st(x_k) \rightarrow st(f(x_1, \ldots, x_k))$, for each function symbol f
- A very restricted transfer principle ($\forall$ sentences without parameters)

A short model-theoretic argument shows:

Theorem. Suppose *NPRA* proves $\forall^{st} x \, \exists y \, \varphi(x, y)$, with φ quantifier-free in the language of *PRA*. Then *PRA* proves $\forall x \, \exists y \, \varphi(x, y)$.

In particular, the conclusion holds if *NPRA* proves either

Weak theories of nonstandard arithmetic

Claims:

- The result extends to higher type theories.
- One can formalize arguments in analysis and measure theory.
- The conservation result can be obtained by an explicit forcing translation.

In the translation, for example:

- The standard natural numbers correspond to bounded sequences of natural numbers.
- Reals correspond to bounded sequences of rationals.
- Nonstandardly large intervals translate to sequences of arbitrarily large intervals.

Eliminating Skolem functions

A *Skolem axiom* has the form

$$\forall \vec{x}, y \; (\varphi(\vec{x}, y) \rightarrow \varphi(\vec{x}, f(\vec{x}))),$$

"if anything satisfies $\exists y \; \varphi(\vec{x}, y)$, $f(\vec{x})$ does."

These can be eliminated from first-order proofs.

- The model-theoretic argument is easy.
- Syntactic arguments are harder, and worse than exponential.

Pudlák: Is there an example of a single Skolem axiom that cannot be eliminated efficiently?

Eliminating Skolem functions

Theorem (Avigad). In any theory in which one can code finite partial functions, one can interpret Skolem axioms efficiently.

The idea: force with finite approximations to each Skolem function.

Conclusion

Metamathematical proof theory involves

- reflecting on the methods of mathematics, and
- representing them syntactically.

One hopes for

- mathematical,
- philosophical, and
- computational

insights.

Forcing can play a role, providing ways of

- interpreting “abstract” (or infinitary) principles, and
- reasoning with approximations.