Distances, projections, and the mean ergodic theorem in subsystems of second-order arithmetic

Jeremy Avigad

Department of Philosophy

Carnegie Mellon University

avigad@cmu.edu

http://andrew.cmu.edu/∼avigad

Work in progress, with Ksenija Simic

Mathematics in restricted frameworks

Varying viewpoints, methods, and emphases:

- Constructive mathematics
- Recursive mathematics
- Subsystems of second-order arithmetic

The last tradition includes Weyl, Hilbert, Bernays, Kreisel, Feferman, Takeuti, Friedman, Simpson, . . .

- Emphasizes axiomatic derivability
- Excluded middle, nonrecursive constructions allowed
- Useful for "proof mining"

Subsystems of arithmetic

Language: $0, 1, +, \times, <, \in, x, y, z, \ldots X, Y, Z, \ldots$

Full second-order arithmetic has:

- Quantifier-free defining equations
- Induction
- Comprehension: $\exists Z \, \forall x \, (x \in Z \leftrightarrow \varphi(x))$

One can also consider various choice principles.

Restrict induction to Σ^0_1 formulas with parameters, and restrict set existence principles:

- RCA_0: recursive (Δ^0_1) comprehension
- WKL_0: paths through infinite binary trees
- ACA_0: arithmetic comprehension
- ATR_0: transfinitely iterated arithmetic comprehension
- Π^1_1-CA_0: Π^1_1 comprehension

Inequivalent definitions

Examples:

1. Reals are Cauchy sequences with fixed rates of convergence
2. Continuous functions may or may not be equipped with moduli of uniform continuity
3. Sequentally compact vs. compact

One of three things can happen:

1. Definitions may be equivalent.
2. One definition may prove to be more natural, or useful.
3. Different definitions may prove to be useful in different contexts.

All these phenomena are interesting.

Slogan: reverse mathematics as a study of mathematical *representations*.

Case studies

Contents of this talk:

1. Topological and metric notions
2. The theory of Hilbert spaces
3. The mean ergodic theorem

Complete metric spaces

Focus on separable metric spaces, presented as completions of a countable dense set.

Notions:

- Closed set
- Separably closed set
- Continuous function
- Distance from a point to a set
- Located set (= set with a distance function)

Complete metric spaces (cont'd)

In the base theory $\mathsf{RCA_0}$:

1. In compact spaces, "closed $\Rightarrow$ separably closed" is equivalent to ACA (Brown)
2. In general, "closed $\Rightarrow$ separably closed" is equivalent to Π^1_1-CA (Brown)
3. "separably closed $\Rightarrow$ closed" is equivalent to ACA (Brown)
4. In compact spaces, "closed implies located" is equivalent to ACA. (Giusto/Simpson)
5. In general, "closed implies located" is equivalent Π^1_1-CA. (Avigad/Simic)
6. "separably closed implies located" is equivalent to ACA. (Giusto/Simpson)

Complete metric spaces (cont'd)

Over $\mathsf{RCA_0}$, the following are equivalent:

1. In a compact space, if C is any closed set and x is any point, then $d(x, C)$ exists.
2. If C is any closed subset of $[0, 1]$, then $d(0, C)$ exists.
3. ACA

Over $\mathsf{RCA_0}$, the following are equivalent:

1. In an arbitrary space, if C is any closed set and x is any point, then $d(x, C)$ exists.
2. In a compact space, if S is any G_δ set and x is any point, then $d(x, S)$ exists.
3. If S is a G_δ subset of $[0, 1]$, then $d(0, S)$ exists.
4. Π^1_1-CA

On the other hand, in $\mathsf{RCA_0}$, the statement that "every F_σ set has a closure" is equivalent to ACA.

Hilbert spaces

A (code for) a Hilbert space H consists of a countable vector space A over $\mathbb{Q}$ together with a function $\langle \cdot, \cdot \rangle : A \times A \to \mathbb{R}$ satisfying

1. $\langle x, x \rangle \geq 0$
2. $\langle x, y \rangle = \langle y, x \rangle$
3. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$

Define $||x|| = \langle x, x \rangle^{\frac{1}{2}}$, $d(x, y) = ||x - y||$, and think of H as the completion of A.

RCA_0 proves:

- Every Hilbert space has an orthonormal basis.
- Every finite dimensional Hilbert space has a dimension.
- Two Hilbert spaces of the same dimension are isomorphic.
- $L_2([0, 1])$ is a Hilbert space.

Closed subspaces

Various notions:

1. *closed linear set*
2. *closed subspace* (i.e. a separably closed linear set)
3. *located closed linear set*
4. *located closed subspace*

2 is used in reverse mathematics, 3 in constructive mathematics.

- "2 implies 1" is equivalent to ACA
- "1 implies 2" is implied by Π^1_1-CA
- 3 and 4 are equivalent in RCA_0
- "2 implies 3/4" is equivalent to ACA
- "1 implies 3/4" is implied by Π^1_1-CA

Distances and projections

In RCA_0 the following are equivalent:

- The distance from x to M exists.
- The projection of x on M exists.

So are the following:

- M is located.
- The projection function, P_M, exists.

These are also equivalent:

- Every closed subspace is located.
- For every closed subspace M and every point x, $d(x, M)$ exists.
- For every closed subspace M, the projection on M exists.
- For every closed subspace M and every point x, the projection of x on M exists.
- ACA.

Norms and kernels

Let f be a bounded linear functional from H to $\mathbb{R}$.

RCA_0 proves that $ker f$ is a closed subspace and a closed linear subset.

In RCA_0, the following are equivalent:

- f has a norm.
- $ker f$ is located
- f is representable: for some y, $f(x) = \langle x, y \rangle$.

For arbitrary f, these are equivalent to ACA.

The mean ergodic theorem

Cast of characters:

- Let T be a contraction, $||Tx|| \leq ||x||$.
- Given x, let
  $x_n = (x + Tx + T^2x + \ldots + T^{n-1}x)/n$.
- Let $M = \{y \in H \mid Ty = y\}$
- Let N be the closure of $\{Ty - y \mid y \in H\}$

The mean ergodic theorem says:

- M is the orthogonal complement of N
- x_n converges in norm, to $P_M x$

Note: in RCA_0, M is closed and linear, and N is a closed subspace.

The mean ergodic theorem (cont'd)

In RCA_0, the mean ergodic theorem is equivalent to ACA.

Consider the three statements:

1. $\langle x_n \rangle$ converges to a point y
2. $P_N x$ exists
3. $P_M x$ exists

RCA_0 proves 1 and 2 equivalent, i.e. $y = x - P_N x$. Both imply 3, $P_M x = y$.

But: in general, showing 3 implies 1 or 2 requires ACA. Even the statement "if $P_M = 0$, then $\langle x_n \rangle$ converges" requires ACA.

Conclusions

Formalizations in subystems of analysis *are* sensitive to definitions. This makes it interesting.

Don't ask, "which is the right definition?"

Ask instead:

- What are the possible definitions?
- What are the relationships between them?
- In which contexts are they useful?