Review of *Proofs and Confirmations: the story of the alternating sign matrix conjecture* by David Bressoud

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1 Overview

An *alternating sign matrix* is a square matrix whose entries are all either 0, 1, or -1, with the properties that each row and each column sums to 1, and that the non-zero entries in each row and column alternate in sign. For example,

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

is such a matrix. In the mid-1980's William Mills, David Robbins, and Howard Rumsey found themselves wondering how the number of $n \times n$ alternating sign matrices, A_n, grows as a function of n. On the basis of combinatorial fiddling and brute calculation, they came up with the following formula:

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

This equation came to be known as the *ASM conjecture*, and computational verification of the first 20 cases made it practically certain to be correct. But a proof remained elusive, despite the simplicity of the conjecture and a good deal of effort from the community of algebraic combinatoricists. Doron Zeilberger finally announced a proof in 1992, but it was not until 1995 that all the gaps were filled in and all the details were checked carefully; certifying the resulting 71 page manuscript employed the efforts of 88 referees and a computer. Soon after, Greg Kuperberg discovered another proof, exploiting connections with physicists' models of "square ice." Zeilberger was then able to use this connection to establish refinements of the alternating sign matrix conjecture as well.

Proofs and Confirmations fills out this story gracefully, guiding the reader through the long and elaborate proofs. In the introduction, David Bressoud writes:

> My intention in this book is not just to describe this discovery of new mathematics, but to guide you into this land and lead you up some of the recently scaled peaks. This is not an exhaustive account of the marvels that have been uncovered, but rather a selected tour that will, I hope, encourage you to return and pursue your own exploration.

This passage is representative of the tenor of Bressoud's exposition.

2 Contents and audience

In many ways, Combinatorics is a black sheep among the subjects of pure mathematics. It is often viewed as a study of isolated puzzles, lacking a deep conceptual framework and unifying ideas, and poorly integrated with the broader history and culture of mathematics.[1] *Proofs and Confirmations* may not do much to dispel the notion that combinatoricists like puzzles: alternating sign matrices arose incidentally in the study of an algorithm for evaluating determinants, and in Bressoud's account mathematical curiosity and the spirit of a challenge are the main motivating factors. But, in the end, the connection with square ice shows that the research has applications to statistical mechanics; and the book's final chapter indicates that there are additional ties to representation theory and the study of Lie algebras.

Indeed, Bressoud's book goes a long way towards meeting the objection that combinatorics is a subject without a broader context. In working through the book, one finds a wealth of generally useful concepts (matrices and determinants, binomial coefficients, generating functions, hypergeometric series, Pfaffians), and fundamental methods (including various forms of induction and recursive definition, and the strategy of exploiting symmetries that become apparent when one varies the representation of a problem). One also encounters connections to a number of branches of mathematics, including linear algebra, the study of differential equations, and the study of elliptic functions.

Finally, Bressoud shows us that algebraic combinatorics has a long and rich history. In the course of the narrative we encounter figures ranging from the fourteenth century Chinese mathematician Chu Shih-Chieh, to Viete, Newton, Gauss, Sylvester, Cauchy, Jacobi, Cayley, and Weyl, along with many others. These names are not dropped casually, but are, rather, introduced with brief contextual sketches that help the reader place them in the web of mathematical ideas. Even Charles Dodgson (better known as Lewis Carroll) makes an appearance, as the inventor of the algorithm for computing determinants that inspired the study of ASM's in the first place.

Proofs and Confirmations should appeal to a number of audiences. First, of course, are students and researchers in algebraic combinatorics and related

[1] For a discussion of these views and a spirited response, see W. T. Gowers' "The two cultures of mathematics" in V. Arnold et al., eds., *Mathematics: Frontiers and Perspectives*, American Mathematical Society, 2000, pages 65–78.

fields. The book is streamlined to bring the reader to the forefront of contemporary research as quickly and smoothly as possible. The nature of the subject makes it impossible to avoid pages of detailed calculations, but Bressoud has a gift for framing these calculations with the ideas and intuitions that guide them.

The fact that the book requires nothing more than familiarity with linear algebra, however, makes it accessible to a much wider audience. Its particular focus renders it unsuitable as a textbook for an introductory course in, say, Discrete Mathematics; but it could well be used in an undergraduate or graduate seminar, where the emphasis is on fostering the methods of mathematical exploration rather than conveying a standard body of content. Each chapter is followed by a long list of thoughtful and illuminating exercises, many of which encourage further experimentation with a symbolic computation package like Mathematica or Maple.

More generally, the book will appeal to anyone who likes algebra and combinatorics, and is curious as to what is currently going on at the intersection of these two disciplines. (I should make it clear that it is this group to which I myself belong.) *Proofs and Confirmations* is not light recreational reading, and following the proofs through to the end requires effort and endurance. But many of the topics developed along the way also stand alone, and can be enjoyed in their own right. For example, the reader can flip open to page 208 and learn that the determinant of a skew symmetric matrix is equal to the square of the associated Pfaffian. The book sketches the history behind this discovery, due to Cayley, and provides exercises that guide the reader through an elegant proof.

Finally, Bressoud has something to say about the development of mathematical ideas, based not only on the book's historical allusions, but also on comments from contemporary researchers. Bressoud provides us with snippets from verbal and written accounts from some of the key players in the proof of the ASM conjecture, and even a few photographs. The book's title was inspired by Imre Lakatos' *Proofs and Refutations*, which, in turn, borrows from Karl Popper's *Conjectures and Refutations*. In adapting Popper's analysis of the sciences to the history of mathematics, Lakatos' book created a stir when it was published in 1976; Lakatos aimed to show that formal models of mathematics based on the axiomatic method do not do justice to the "logic of discovery," i.e. the heuristic development of the subject.[2]

Although Bressoud tries to fill out the picture of mathematical discovery in ways that complement Lakatos', *Proofs and Confirmations* is a very different book. *Proofs and Refutations* was intended as a philosophical work that uses mathematical and historical examples to support its claims. In contrast, *Proofs and Confirmations* does not try to provide a sustained historical or philosoph-

[2]Lakatos' rhetoric is less damning when one recognizes that the axiomatic method was not designed to play this role. But Lakatos' writings do contain important insights, and are still widely discussed today. I am partial to a critique of Lakatos' work by Solomon Feferman, "The logic of mathematical Discovery versus the logical structure of mathematics," in P. D. Asquith and I. Hacking, eds., *PSA 1978: Proceedings of the 1978 Biennial Meeting of the Philosophy of Science Association, vol. 2*, 1981, pages 309–327, recently republished in *In the Light of Logic*, Oxford University, 1998, pages 77–93.

ical analysis; the primary emphasis is on the mathematics, with philosophical remarks confined mainly to the book's introduction and conclusion. The latter draw on the usual metaphors of mathematical inquiry—exploration, discovery, and creation—and supplement them with some new ones. For example:

> The doing of mathematics is more akin to being dropped on a distant and unknown mountain peak and then seeking to find one's way home.... Research in mathematics almost never begins with careful definitions and lemmas on which we build until something interesting is discovered. It starts with the discovery, and proof is the process of tying that discovery back to what is already known. (p. 258)

In the same discussion, mathematics is also compared to archeology, where objects are found in isolation and then related to their social, cultural, and historical context:

> This is the role of proof, to enrich the entire web of context that leads to understanding. The mathematician does not dig for lost artifacts of a vanished civilization but for the fundamental patterns that undergird our universe, and like the archeologist we usually only find small fragments. As archeology attempts to reconstruct the society in which this object was used, so mathematics is the reconstruction of these patterns into terms we can comprehend. (Ibid.)

Mathematics tends to resist simplifying characterizations, and one should be wary of drawing sweeping conclusions on the basis of one case study. Moreover, as a branch of mathematics, algebraic combinatorics has some atypical features. For example, the accessibility of its basic notions to a bright high school student, and the amenability of its conjectures to computational verification, give the subject a flavor that is distinct from, say, that of algebraic geometry or ergodic theory. Bressoud's remarks do, however, contain interesting insights into the attitudes and motives that drive mathematical inquiry, and they make it clear that there is much more to be said about the process of mathematical discovery than can be couched in terms of formal axiomatic proof.

3 Opinions

Proofs and Confirmations is a lovely book, a pleasure to read and to learn from. The exposition is clear and well organized, and anything but stodgy. As an example, consider the following colorful passage:

> Let us pause for the moment to appreciate the audacity of what we are proposing... Equation (5.4) is a system of $r+1$ equations in r unknowns. There is no guarantee that it has a solution. The only reason to proceed is the observation that we seem to have completed a missing symmetry. To a mathematician, there could be no better reason. (page 158)

This breathless exuberance may strike the reader as a little campy, but comments like these serve a useful purpose, providing structure to a long proof, and highlighting key ideas and intuitions.

It would be nice if mathematical narratives like Bressoud's were more common. This is not to deny the importance of traditional textbooks and references, where style and content is dictated by the need to lay out the fundamentals of a subject, rather than carry the reader in pursuit of a single thread. After all, setting forth the basics in an organized manner is certainly important. But so is the joy of buckling down and following a line of inquiry just for the fun of it, and few books manage to share that enjoyment as well as this one does.