Algebraic proofs of cut elimination*

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April 5, 2001

Abstract

Algebraic proofs of the cut-elimination theorems for classical and intuitionistic logic are presented, and are used to show how one can sometimes extract a constructive proof and an algorithm from a proof that is nonconstructive. A variation of the double-negation translation is also discussed: if φ is provable classically, then $\neg(\neg\varphi)^{nf}$ is provable in minimal logic, where θ^{nf} denotes the negation-normal form of θ. The translation is used to show that cut-elimination theorems for classical logic can be viewed as special cases of the cut-elimination theorems for intuitionistic logic.

1 Introduction

The cut-elimination theorems for classical and intuitionistic logic are a mainstay of proof theory, and with good reason. Even when it comes to pure first-order logic, cut-elimination is a remarkably powerful tool, allowing one to extract additional information from derivations in a wide range of axiomatic theories.

For the classical case, there is a simple, nonconstructive route to proving the cut-elimination theorem: first, one shows that the proof system *with* cut is sound with respect to standard first-order semantics, and then one shows that the fragment of the system *without* cut is complete. Together these facts imply that any sequent provable in the system with cut is valid, and hence has a cut-free proof. Of course, a priori this argument provides no information as to how to *translate* a proof with cuts to one that is cut free.

After setting forth the relevant preliminaries in Sections 2 and Section 3, in Section 4 I present a natural "constructivization" of the argument just described. The proof is akin to algebraic proofs of cut-elimination for higher-order intuitionistic logic found in [5, 7], and is also similar in spirit to algebraic model-theoretic constructions described in [2, 3, 6].

In Section 5, I discuss the algorithm that is implicit in the constructive proof. The developments in Sections 4 and 5 may therefore be interesting for two reasons: first, they provide an algorithm for cut-elimination which can be verified naturally in a suitable type-theoretic framework; and second, they

*Final version in the *Journal of Logic and Algebraic Programming* 49:15–30, 2001.

illustrate a way in which algebraic methods can be used to extract computational information from nonconstructive classical proofs.

When working with classical logic, it is often convenient to use a one-sided sequent calculus, in which formulae are assumed to be in negation-normal form. This is not really a restriction, since traditional two-sided calculi are easily interpreted in the one-sided version, a useful fact that suggests that in some contexts the negation-normal form representation is, in a sense, the "right" way to think about classical logic. In Section 6, I present a variation of the double-negation translation that serves to embed classical logic in the minimal fragment of intuitionistic logic. Using θ^{nf} to denote the negation-normal form of θ, the translation can be described simply as follows: if a formula φ is provable classically, then $\neg(\neg\varphi)^{nf}$ is provable in minimal logic. More generally, if a sequent $\{\varphi_1, \ldots, \varphi_k\}$ is provable in a one-sided classical sequent calculus, then the sequent $\{(\neg\varphi_1)^{nf}, \ldots, (\neg\varphi_k)^{nf}\} \Rightarrow \bot$ is provable in minimal logic.

In Section 7, I present an algebraic proof of the cut-elimination theorem for intuitionistic logic, essentially just the specialization of Buchholz [5] to the first-order setting. It will then be clear that the proof for classical logic given in Section 4 is just a special case of the intuitionistic version, via the translation above. What is new here, then, is the algebraic proof of the cut-elimination theorem for the classical sequent calculus; the new version of the double-negation translation; and the observation that the translation can be used to interpret classical logic as a fragment of intuitionistic logic, in a useful way. Since the translation works equally well for higher-order logic, Buchholz' proof can be seen to yield a cut-elimination theorem for classical higher-order logic as well.

In contrast, one can also consider normalization proofs in the style of Tait, Troelstra, and Girard, from an algebraic point of view; see, for example, Scedrov [15] and Altenkirch et. al [1]. Berger [4] extracts an algorithm from the Tait-Troelstra proof of strong normalization, in much the same way as an algorithm is extracted, in Section 5 below, from the proof of cut-elimination presented here. It would be interesting to have a better understanding of the relationship between cut-elimination and normalization, as well as the associated algorithms; Zucker [20] should be helpful in this respect. Section 8 poses some related questions.

2 Sequent calculi

Though they are not the most natural systems to work with when it comes to proving logical validities, proof theorists tend to be fond of sequent calculi. On the one hand, using the sequent calculus with cut, it is easy to simulate natural deduction or standard axiomatic systems. On the other hand, if one avoids the cut rule, sequents are proved only "from the bottom up," making it easy to extract additional information from proofs. In this section I will describe the sequent calculi that we will be concerned with below.

To have a uniform basis for the comparison of classical, intuitionistic, and minimal logic, I will take the first-order logical symbols to be $\forall$, $\exists$, $\wedge$, $\vee$, $\rightarrow$, and

$\bot$, with $\neg\varphi$ defined to be $\varphi \to \bot$. As is common, I will identify formulae that differ only in the names of the bound variables. If φ is a formula and t is a term, $\varphi[t/x]$ denotes the result of substitution t for x in φ, renaming bound variables if necessary; and once a formula has been introduced as $\varphi(x)$, $\varphi(t)$ denotes $\varphi[t/x]$. For simplicity, I will work with first-order logic without equality, though the modifications needed to accomodate equality are routine.

A formula is said to be in *negation-normal form* if it is built up from atomic and negated atomic formulae using $\wedge$, $\vee$, $\forall$, and $\exists$. Classically, every formula is equivalent to one in negation-normal form; if φ is any formula, I will use φ^{nf} to denote its canonical negation-normal-form representation. The negation operator, $\sim\varphi$, for negation-normal-form formulae is defined by $\sim\varphi \equiv (\neg\varphi)^{nf}$. More explicitly, $\sim\varphi$ is what you get if, in φ, you exchange $\wedge$ with $\vee$, $\forall$ with $\exists$, and atomic formulae with their negations. Note that $\sim\sim\varphi$ is just φ.

For classical logic, we will use a calculus in which one derives sets of formulae in negation-normal form, read disjunctively. If Γ and Δ are such sets and φ is a formula in negation-normal form, then Γ, φ abbreviates $\Gamma \cup \{\varphi\}$ and Γ, Δ abbreviates $\Gamma \cup \Delta$. The rules of the calculus are as follows:

$$\Gamma, A, \neg A$$

$$\frac{\Gamma, \varphi \qquad \Gamma, \psi}{\Gamma, \varphi \wedge \psi} \qquad\qquad \frac{\Gamma, \varphi_i}{\Gamma, \varphi_0 \vee \varphi_1}$$

$$\frac{\Gamma, \varphi}{\Gamma, \forall x\ \varphi} \qquad\qquad \frac{\Gamma, \varphi[t/x]}{\Gamma, \exists x\ \varphi}$$

$$\frac{\Gamma, \varphi \qquad \Gamma, \neg\varphi}{\Gamma}$$

In the first rule, expressing the law of the excluded middle, A denotes any atomic formula. In the rule for the universal quantifier, one has the usual restriction that x is not free in any formula of Γ. The last rule is the notorious *cut rule*, and proofs that do not use it are said to be *cut free*. An easy induction on proofs shows that "weakening" is a derived rule, which is to say, if Γ is provable and $\Gamma' \supseteq \Gamma$ then Γ' is provable as well. Below, we will find it convenient to add the weakening rule explicitly, allowing one to derive Γ' from Γ.

For intuitionistic or minimal logic, a simple one-sided calculus is not sufficient; one needs to use two-sided sequents of the form $\Gamma \Rightarrow \varphi$, where Γ is a set of formulae and φ is a formula. Such a sequent is interpreted as the assertion that the conjunction of the formulae in Γ entail φ. The rules of the intuitionistic calculus are as follows:

$$\Gamma, A \Rightarrow A \qquad\qquad \Gamma, \bot \Rightarrow A$$

$$\frac{\Gamma, \varphi_i \Rightarrow \psi}{\Gamma, \varphi_0 \wedge \varphi_1 \Rightarrow \psi} \qquad\qquad \frac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi}$$

$$\frac{\Gamma, \varphi \Rightarrow \psi \qquad \Gamma, \theta \Rightarrow \psi}{\Gamma, \varphi \vee \theta \Rightarrow \psi} \qquad\qquad \frac{\Gamma \Rightarrow \varphi_i}{\Gamma \Rightarrow \varphi_0 \vee \varphi_1}$$

$$\frac{\Gamma, \Rightarrow \varphi \qquad \Gamma, \theta \Rightarrow \psi}{\Gamma, \varphi \to \theta \Rightarrow \psi} \qquad\qquad \frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \varphi \to \psi}$$

$$\frac{\Gamma, \varphi[t/x] \Rightarrow \psi}{\Gamma, \forall x\ \varphi \Rightarrow \psi} \qquad\qquad \frac{\Gamma, \Rightarrow \psi}{\Gamma \Rightarrow \forall x\ \psi}$$

$$\frac{\Gamma, \varphi \Rightarrow \psi}{\Gamma, \exists x\ \varphi \Rightarrow \psi} \qquad\qquad \frac{\Gamma, \Rightarrow \psi[t/x]}{\Gamma \Rightarrow \exists x\ \psi}$$

$$\frac{\Gamma \Rightarrow \varphi \qquad \Gamma, \varphi \Rightarrow \psi}{\Gamma \Rightarrow \psi}$$

Once again, in the first line A is intended to denote any atomic formula, and the last rule is the two-sided version of the cut rule. If one omits the second axiom, one has minimal logic. Notice that the rest of the rules come in left/right pairs, one pair for each connective. The usual eigenvariable restrictions apply to the $\exists$ rule on the left and the $\forall$ rule on the right.

The two calculi I have presented are most similar to the ones denoted *G3i* and *GS* in [18]. Of course, there are also two-sided versions of the classical calculus: one obtains such a system by modifying the intuitionistic calculus above to allow sequents of the form $\Gamma \Rightarrow \Delta$, interpreted as the assertion that the conjunction of the formulae in Γ implies the disjunction of the formulae in Δ. If Γ is a set of formulae, let Γ^{nf} denote the set $\{\varphi^{nf} \mid \varphi \in \Gamma\}$, and if Π is a set of formula in negation-normal form, let $\sim\Pi$ denote $\{\sim\varphi \mid \varphi \in \Gamma\}$. So, writing $\sim\Gamma^{nf}$ instead of $\sim(\Gamma^{nf})$, we have

$$\sim\Gamma^{nf} = \{\sim(\varphi^{nf}) \mid \varphi \in \Gamma\} = \{(\neg\varphi)^{nf} \mid \varphi \in \Gamma\}.$$

The following theorem shows that the classical two-sided sequent calculus is inter-interpretable with the one-sided one.

Theorem 2.1 *A sequent $\Gamma \Rightarrow \Delta$ is provable in the two-sided classical sequent calculus if and only if $\sim\Gamma^{nf}, \Delta^{nf}$ is provable in the one-sided calculus. Moreover, there are efficient translations of derivations between the two systems, with the property that cut-free proofs are translated to cut-free proofs.*

Here and in Theorem 6.2 the word "efficient" means that the number of symbols in the translation is bounded by a polynomial in the number of symbols in the original. The proof of Theorem 2.1 is entirely routine: each rule in the two-sided calculus corresponds, under the translation, to a rule in the one-sided calculus.

3 The double-negation translation

A formula is said to be *negative* if it does not involve $\exists$ or $\vee$, i.e. it is built up using the logical symbols $\forall$, $\wedge$, $\rightarrow$, and $\perp$. The following version of the Gödel-Genzten double-negation translation maps an arbitrary first-order formula φ to a negative formula, φ^N:

- $\perp^N = \perp$
- $A^N = \neg\neg A$, for A atomic
- $(\varphi \wedge \psi)^N = \varphi^N \wedge \psi^N$
- $(\varphi \vee \psi)^N = \neg(\neg\varphi^N \wedge \neg\psi^N)$
- $(\varphi \rightarrow \psi)^N = \varphi^N \rightarrow \psi^N$
- $(\forall x\ \varphi)^N = \forall x\ \varphi^N$
- $(\exists x\ \varphi)^N = \neg\forall x\ \neg\varphi^N$

The following lemma and theorem are well known. The first is proved using induction on formulae, and the second is proved using induction on derivations.

Lemma 3.1 *If φ is any formula, then φ^N is equivalent to $\neg\neg\varphi^N$ in minimal logic.*

Theorem 3.2 *If φ is provable from Γ classically, then φ^N is provable from Γ^N in minimal logic.*

If one is interested specifically in translating proofs from the one-sided sequent calculus to minimal logic, one can prove the following lemma and theorem more directly.

Lemma 3.3 *If φ be any formula, then $({\sim}\varphi)^N$ is equivalent to $\neg\varphi^N$ in minimal logic.*

Theorem 3.4 *If $\{\varphi_1, \ldots, \varphi_k\}$ is provable in the classical sequent calculus, then $\neg(({\sim}\varphi_1)^N \wedge \ldots \wedge ({\sim}\varphi_k)^N)$ is provable in minimal logic.*

4 Cut elimination for classical logic

The cut-elimination theorem for classical logic states the following:

Theorem 4.1 *Any sequent provable in the classical one-sided sequent calculus has a cut-free proof.*

If Γ is a sequent, say that Γ is *valid* if the universal closure of $\bigvee \Gamma$ is true in every model. The next two lemmata provide a nonconstructive proof of the cut-elimination theorem.

Lemma 4.2 *The one-side sequent calculus with cut is sound for standard classical first-order semantics: if a sequent is provable, then it is valid.*

Proof. Use induction on the length of proofs. □

Lemma 4.3 *The one-sided sequent calculus without cut is complete: if a sequent is valid, then it has a cut-free proof.*

Proof (sketch). Here I will just outline the standard "tableau" construction; for details see, for example, [9, 16].

Let $p, q, r, \ldots$ stand for finite sets of formulae in negation-normal form, and read these conjunctively. We need to show that if Γ is any sequent and p is the set of negations of formulae in Γ, then either Γ has a cut-free proof or p has a model.

The idea is to construct, systematically, a tree of "attempts" at building a term model of p. Label the bottom node of the tree with p, and proceed upwards as follows. To build a model of $q, \varphi \wedge \psi$, build a model of q, φ, ψ. To build a model of $q, \varphi \vee \psi$, branch, and in parallel try to build a model of either q, φ or q, ψ. To build a model of $q, \forall x\ \varphi(x)$, build a model of $q, \forall x\ \varphi(x), \varphi(t)$, where t is the next term in some predetermined list. To build a model of $q, \exists x\ \varphi(x)$, pick a new constant symbol, say "y," and build a model of $q, \varphi(y)$. If one is ever called on to build a model of $q, A, \neg A$, where A is an atomic formula, one abandons the attempt that this branch represents, and hopes that another proves more fruitful.

Assuming one is systematic enough in choosing, at each node, which formula to deal with next, one of two things can happen. The first is that the process comes to an end at some finite stage, because each terminal node is of the form $q, A, \neg A$. In that case, replacing each node r in the tree by $\sim r$ yields, essentially, a cut-free proof of Γ. Otherwise, the tree is infinite, and by König's lemma has an infinite branch. The systematic construction of the tree should guarantee that the union of the sets appearing along this branch, H, is a *Hintikka* set: no atomic formula and its negation occurs in H; if $\varphi \wedge \psi$ is in H, then φ and ψ are both in H; if $\varphi \vee \psi$ is in H, then either φ or ψ is in H; if $\forall x\ \varphi(x)$ is in H and t is any term, then $\varphi(t)$ is in H; and if $\exists x\ \varphi(x)$ is in H, then for some "constant" y, $\varphi(y)$ is in H. Let $\mathcal{M}$ be the model whose universe is the set of terms in the language, and in which a relation symbol R is true of $t_1, \ldots, t_k$ if and only if $R(t_1, \ldots, t_k)$ is in H. Reading the variables as constants, it is then easy to verify that all the sentences in H, including those in p, are true in $\mathcal{M}$. □

This proof is curiously nonconstructive: it gives us no information on how to translate a proof of Γ with cuts to one that is cut-free. Of course, if Γ is provable with cut, then a blind search for the cut-free proof is guaranteed to succeed; but a priori the nonconstructive proof gives us no sense of how long the resulting derivation may be, and it gives us no way of using the original derivation to guide the search. The constructive proof presented below is similar to the nonconstructive one, but instead of worrying about the models obtained from infinite branches through a tree whose root is labelled with a set q, we reason about which formulae are *forced* to be true in any model obtained in this way. The argument will then read roughly as follows: if Γ is provable, then it is forced to be true in every model; and if it is forced to be true in every model, it has a cut-free proof.

Call the sets $p, q, r, \ldots$ *conditions*, and if p and q are conditions, write $q \preceq p$ ("q is stronger than or equivalent to p") if there is a proof of $\sim q$ from $\sim p$ in the one-sided sequent calculus, using only the rules for $\vee$, $\exists$, and weakening. Put differently,"stronger than or equivalent to" is the smallest transitive and reflexive relation satisfying the following clauses:

1. If $q \supseteq p$ then $q \preceq p$
2. If $\varphi \wedge \psi$ is in p, then $p \preceq p \cup \{\varphi\}$ and $p \preceq p \cup \{\psi\}$.
3. If $\forall x\ \varphi(x)$ is in p and t is any term, then $p \preceq p \cup \{\varphi(t)\}$.

The intuition is that if $p \preceq q$, then p implies q, or, better, every infinite branch through the tree containing p will also contain q. So if $p \preceq q$, any model of p (e.g., constructed from such a branch) is also a model of q. For a concrete example, the condition $\{\forall x\ \varphi(x), \forall y\ \psi(y), \theta, \eta, \sigma \wedge \tau\}$ is stronger than the condition $\{\varphi(t_1), \varphi(t_2), \varphi(t_3), \forall y\ \psi(y), \psi(s), \eta, \sigma\}$. It is not difficult to verify that the relation $\preceq$ is transitive and reflexive.

The following clauses define, inductively, a relation between sets of formulae p in negation-normal form, and negative formulae φ. Intuitively, $p \Vdash \varphi$ means that φ is true in any model obtained from a branch through a tree rooted at p, constructed according to the recipe above.

1. $p \Vdash \perp$ if and only if there is a cut-free proof of $\sim p$ in the one-sided sequent calculus.
2. If A is atomic, $p \Vdash A$ if and only if there is a cut-free proof of $\sim p, A$.
3. $p \Vdash \theta \wedge \eta$ if and only if $p \Vdash \theta$ and $p \Vdash \eta$.
4. $p \Vdash \theta \rightarrow \eta$ if and only if for every $q \preceq p$, if $q \Vdash \theta$ then $q \Vdash \eta$.
5. $p \Vdash \forall x\ \theta(x)$ if and only if for every term t, $p \Vdash \theta(t)$.

A formula φ is said to be *forced*, written $\Vdash \varphi$, if every condition forces φ.

Lemma 4.4 *The forcing relation is monotone: if $p \Vdash \varphi$ and $q \preceq p$ then $q \Vdash \varphi$. Also, for any formula φ, if $p \Vdash \perp$, then $p \Vdash \varphi$.*

Proof. Both claims are clearly true when φ is atomic or $\perp$. An easy induction shows that it holds for arbitrary negative formulae. $\square$

Since any two conditions p and q have a least upper bound, $p \cup q$, it is not difficult to show that $p \Vdash \theta \rightarrow \eta$ is equivalent to the assertion that for any condition q, if $q \Vdash \theta$, then $p \cup q \Vdash \eta$.

Lemma 4.5 *If φ is any negative formula provable in intuitionistic logic, then $\Vdash \varphi$.*

Proof. Take minimal logic to be given by a system of natural deduction (see, e.g. [19]), and show the following by induction on proofs: if φ is a negative formula provable from Γ, the free variables of Γ and φ are among $\vec{x}$, and $\vec{t}$ is any sequence of terms of the same length, then whenever $p \Vdash \theta[\vec{t}/\vec{x}]$ for each θ in Γ, then $p \Vdash \varphi[\vec{t}/\vec{x}]$. The second assertion of Lemma 4.4 is used to handle the rule *ex falso sequitur quodlibet*, "from $\perp$ conclude ψ" for arbitrary ψ. $\square$

In fact, Lemmata 4.4 and 4.5 hold for any forcing relation defined by clauses 2–5 above, as long as the conditions of Lemma 4.4 holds for atomic formulae and $\perp$.

Lemma 4.6 *If φ is any formula in negation-normal form, then $\{\varphi\} \Vdash \varphi^N$.*

Proof. The proof involves a routine induction on φ. I will carry out three illustrative cases.

In the case where φ is the atomic formula A, we need to show that if p is any condition and $p \Vdash \neg A$, then $p, A \Vdash \perp$. But this follows from the fact that $\{A\} \Vdash A$, and the observation after Lemma 4.4.

In the case where φ is of the form $\theta \wedge \eta$, from the induction hypothesis we have $\{\theta\} \Vdash \theta^N$ and $\{\eta\} \Vdash \eta^N$. But since $\{\theta \wedge \eta\}$ is stronger than both $\{\theta\}$ and $\{\eta\}$, we have that $\{\theta \wedge \eta\}$ forces both θ^N and η^N, and hence $(\theta \wedge \eta)^N$.

In the case where φ is of the form $\theta \vee \eta$, we need to show that whenever $p \Vdash \neg\theta^N \wedge \neg\eta^N$, then $p, \theta \vee \eta \Vdash \perp$. So suppose $p \Vdash \neg\theta^N \wedge \neg\eta^N$. Then, in particular, we have $p \Vdash \neg\theta^N$ and $p \Vdash \neg\eta^N$. Using the induction hypothesis, we have $\{\theta\} \Vdash \theta^N$ and $\{\eta\} \Vdash \eta^N$, and so $p, \theta \Vdash \perp$ and $p, \eta \Vdash \perp$. This means that there are cut-free proofs of $\sim p, \sim\theta$ and $\sim p, \sim\eta$, and hence a cut-free proof of $\sim p, \sim\theta \wedge \sim\eta$. But this is equivalent to saying $p, \theta \vee \eta \Vdash \perp$.

The cases for $\forall$, $\exists$, and negation atomic formulae are similar. $\square$

Lemmata 4.4–4.6 yield a short proof of the cut-elimination theorem.

Proof (of Theorem 4.1). Suppose $\{\varphi_1, \ldots, \varphi_k\}$ is provable in the sequent calculus with cut. By Theorem 3.4, $(\sim\varphi_1)^N \wedge \ldots \wedge (\sim\varphi_k)^N \to \perp$ is provable in minimal logic, and so is forced. By Lemma 4.6, for each i we have $\{\sim\varphi_i\} \Vdash (\sim\varphi_i)^N$. By monotonicity, we have $\{\sim\varphi_1, \ldots, \sim\varphi_k\} \Vdash (\sim\varphi_1)^N \wedge \ldots \wedge (\sim\varphi_k)^N$, and hence $\{\sim\varphi_1, \ldots, \sim\varphi_k\} \Vdash \perp$. By definition, this means that there is a cut-free proof of $\{\varphi_1, \ldots, \varphi_k\}$. $\square$

5 Extracting an algorithm

In what sense is the proof in the previous section algebraic? By assigning to each formula φ the set $[\![\varphi]\!] = \{p \mid p \Vdash \varphi^N\}$, one can view the forcing relation as providing a nonstandard semantics for classical first-order logic, mapping formulae to values in a boolean algebra of "regular" sets of conditions. See [5, 6, 7] for presentations more along these lines.

On the other hand, expressing the proof in terms of the forcing relation makes its constructive content more transparent. Assuming one has specified a language in which one can represent basic syntactic notions and operations, for each formula $\varphi(x_1, \ldots, x_1)$ with the free variables shown, the relationship $p \Vdash \varphi(t_1, \ldots, t_n)$ can be expressed with a first-order formula in $p, t_1, \ldots, t_n$. Note that the logical complexity of $p \Vdash \varphi(t_1, \ldots, t_n)$ increases with that of φ, so there is no single first-order formula that captures the notion uniformly. But for each fixed derivation in the classical sequent calculus, the proof in the last section can be carried out in intuitionistic first-order logic, using only quantifier-free axioms governing the syntactic notions. In other words, the argument above yields a method of assigning to any classical derivation d of a sequent Γ, an intuitionistic proof P_d that there exists a cut-free derivation of Γ. Normalizing P_d yields an explicit witness d', and a proof that d' is a cut-free derivation of Γ. If all one really cares about is the cut-free derivation (and not the associated proof that it is such), one can instead use modified realizability to extract a simply typed lambda term T_d realizing the existential conclusion of P_d (see [17]).

The argument in Section 4 used induction on d and the formulae in Γ, so the implicit assignment $d \mapsto P_d$ or $d \mapsto T_d$ can be obtained using the corresponding structural recursions. All told, then, the full proof of the cut-elimination theorem can be carried out in a theory that allows a modicum of recursion on syntax, together with a reflection principle for first-order intuitionistic logic. Alternatively, it can be carried out in an appropriate Martin-Löf-style type theory, allowing polymorphic recursion over a universe of basic types. Once again, if one is interested primarily in a procedure for eliminating cuts (and not a proof that the procedure is correct), one has the following algorithm: given a derivation d, find the term T_d, and normalize it.

The goal of this section is to describe the assignment $d \mapsto T_d$ in more detail. Suppose we would like to eliminate cuts from the sequent calculus for a fixed first-order language, L. Start with a basic type TERM for terms in this language, a type COND for conditions (or their negations, sequents), and a type DER for cut-free derivations, which we will take simply to be finite trees labelled with sequents.

Our strategy will be to assign to each negative formula φ a type Type_φ, and specify what it means for an object of this type to realize the statement "$p \Vdash \varphi$." Then, given a derivation d in the classical sequent calculus of a sequent $\{\varphi_1, \ldots, \varphi_k\}$, we will show how to obtain a term T_d denoting a realizer for $\{\sim\varphi_1, \ldots, \sim\varphi_k\} \Vdash \bot$. This realizer will be the cut-free derivation we are looking for.

For the moment, read "type" as "set," and if σ and τ are types, read $\sigma \times \tau$ as the ordinary set-theoretic cartesian product of σ and τ, and $\sigma \to \tau$ as the set of functions from σ to τ. Take the notation $\rho, \sigma \to \tau$ to abbreviate $\rho \to (\sigma \to \tau)$, and if f is of this type, take $f(a, b)$ to abbreviate $(f(a))(b)$. The assignment

$\varphi \mapsto \mathsf{Type}_\varphi$ is defined inductively, as follows:

$$\begin{aligned}
\mathsf{Type}_\varphi &= \mathsf{DER} \quad \text{if } \varphi \text{ is atomic or } \bot \\
\mathsf{Type}_{\varphi\wedge\psi} &= \mathsf{Type}_\varphi \times \mathsf{Type}_\psi \\
\mathsf{Type}_{\varphi\to\psi} &= \mathsf{COND}, \mathsf{DER}, \mathsf{Type}_\varphi \to \mathsf{Type}_\psi \\
\mathsf{Type}_{\forall \mathsf{x}\,\varphi} &= \mathsf{TERM} \to \mathsf{Type}_\varphi
\end{aligned}$$

Recall that if p and q are conditions, $p \preceq q$ means that there is a derivation of $\sim p$ from $\sim q$ using only the rules for $\vee$, $\exists$, and weakening. If $f \in \mathsf{DER}$, say "f realizes $p \preceq q$" if f is such a derivation. By induction on φ, given a condition p and an element f of Type_φ, we can now say what it means for f to realize $p \Vdash \varphi$:

- If φ is $\bot$, f realizes $p \Vdash \varphi$ if and only if f is a derivation of $\sim p$.
- If φ is atomic, f realizes $p \Vdash \varphi$ if and only if f is a derivation of $\sim p, \varphi$.
- f realizes $p \Vdash \varphi \wedge \psi$ if and only if $(f)_0$ realizes $p \Vdash \varphi$ and $(f)_1$ realizes $p \Vdash \psi$.
- f realizes $p \Vdash \varphi \to \psi$ if and only if the following holds: whenever d realizes $q \preceq p$ and g realizes $q \Vdash \varphi$, $f(q, d, g)$ realizes $q \Vdash \psi$.
- f realizes $p \Vdash \forall x\ \varphi(x)$ if and only if for every term t, $f(t)$ realizes $p \Vdash \varphi(t)$.

We will now work with a simply typed lambda calculus over the base types $\mathsf{TERM}, \mathsf{COND}, \mathsf{DER}$, with operations for pairing, projections, lambda abstraction, application, and a certain list of constant symbols, as described below. If f is a closed lambda term, we will say that f realizes $p \Vdash \varphi$ if the object denoted by f realizes $p \Vdash \varphi$. References to the full set-theoretic interpretation in the definition above is really just an expository convenience. The proofs which follow depend only on properties of β-conversion and syntactic properties of the objects denoted by the constant symbols, and so, as noted above, these proofs can be formalized in a weak theory.

The following is a list of constant symbols that we assume to be included in our calculus, together with their intended interpretations:

1. For each term $t(x_1, \ldots, x_k)$ of the language, L, with the free variables shown, a constant $\hat{t} \in \mathsf{TERM}^k \to \mathsf{TERM}$. In the intended interpretation, $\hat{t}$ is simply a function that maps terms $s_1, \ldots, s_k$ to the term $t(s_1, \ldots, s_k)$.
2. For each condition p with free variables $x_1, \ldots, x_k$, a constant $\hat{p} \in \mathsf{TERM}^k \to \mathsf{COND}$. In the intended interpretation, $\hat{p}(s_1, \ldots, s_k)$ is simply the condition obtained by subsituting $s_1, \ldots, s_k$ for $x_1, \ldots, x_k$.
3. For each k, a constant $\mathit{Tree}_k \in \mathsf{COND}, \mathsf{DER}^k \to \mathsf{DER}$. If p is a condition and $d_1, \ldots, d_k$ are derivations (labelled trees), then $\mathit{Tree}_k(p, d_1, \ldots, d_k)$ is the derivation with root labelled $\sim p$, and with subtrees $d_1, \ldots, d_k$.

4. A constant $Concat \in \mathsf{DER}, \mathsf{DER} \to \mathsf{DER}$. If d is a derivation of Γ from Δ using only the rules $\vee$, $\exists$, and weakening, and e is a similar derivation of Π from Γ, $Concat(d, e)$ represents the concatenation of these two derivations. In particular, if d realizes $q \preceq p$ and e realizes $r \preceq q$, then $Concat(d, e)$ realizes $r \preceq p$.

5. For each formula $\varphi(x_1, \ldots, x_k)$ with the free variables shown, a constant $Weaken_\varphi \in \mathsf{TERM}^k, \mathsf{DER} \to \mathsf{DER}$. If d is a proof of Γ from Δ using only $\vee$, $\exists$, and weakening, and $t_1, \ldots, t_k$ are terms, then $Weaken_\varphi(t_1, \ldots, t_k)$ denotes the derivation of $\Gamma, \varphi(t_1, \ldots, t_k)$ from $\Delta, \varphi(t_1, \ldots, t_k)$ obtained by adding $\varphi(t_1, \ldots, t_k)$ to each sequent.

6. For each k, a constant $NewVar \in \mathsf{TERM}^k \to \mathsf{TERM}$. If $t_1, \ldots, t_k$ are terms, $NewVar(t_1, \ldots, t_k)$ returns a variable that does not occur in any of the t_i.

The next three lemmata are the "concrete" versions of Lemmata 4.4, 4.5, and 4.6. In the statement of these lemmata, quantification over terms, conditions, and derivations amounts to quantification over the types TERM, COND, and DER.

Lemma 5.1 *For every formula $\varphi(x_1, \ldots, x_k)$ with the free variables shown, there is a term Mon_φ of type $\mathsf{DER}, \mathsf{Type}_\varphi \to \mathsf{Type}_\varphi$, with free variables $x_1, \ldots, x_k$ of type TERM, with the following property: whenever p is a condition, $t_1, \ldots, t_k$ are terms, f realizes $p \Vdash \varphi(t_1, \ldots, t_k)$, and d realizes $q \preceq p$, then the term $(Mon_\varphi[t_1/x_1, \ldots, t_k/x_k])(d, f)$ realizes $q \Vdash \varphi(t_1, \ldots, t_k)$.*

Also, with φ and $t_1, \ldots, t_k$ as above, there is a term $ExFalso_\varphi(p, f)$ such that whenever f realizes $p \Vdash \bot$, $(ExFalso_\varphi[t_1/x_1, \ldots, t_k/x_k])(p, f)$ realizes $p \Vdash \varphi(t_1, \ldots, t_k)$.

Proof. Both Mon_φ and $ExFalso_\varphi$ are obtained by structural recursion on φ, and induction on φ is used to show that these terms satisfy the statement of the lemma. I will focus on Mon_φ, and omit references to the variables $x_1, \ldots, x_k$ for simplicity.

Consider the case where φ is $\bot$. If f realizes $p \Vdash \bot$, then f is a proof of $\sim p$. Then, if d realizes $q \preceq p$, then d is a proof of $\sim q$ from $\sim p$, in which case $Concat(f, d)$ is a proof of $\sim q$. So we can set $Mon_\bot$ to be $\lambda d, f\ Concat(f, d)$.

Similarly, if φ is a atomic, we can take

$$Mon_\varphi \equiv \lambda d, f\ Concat(f, Weaken_\varphi(x_1, \ldots, x_k, d)).$$

Suppose φ is of the form $\theta \wedge \eta$. If f realizes $p \Vdash \theta \wedge \eta$ then $(f)_0$ realizes $p \Vdash \theta$ and $(f)_1$ realizes $p \Vdash \eta$. If d realizes $q \preceq p$, then by the inductive hypothesis, $Mon_\theta(d, (f)_0)$ realizes $q \Vdash \theta$ and $Mon_\eta(d, (f)_1)$ realizes $q \Vdash \eta$. So we can take

$$Mon_{\theta \wedge \eta} \equiv \lambda d, f\ \langle Mon_\theta(d, (f)_0), Mon_\eta(d, (f)_1) \rangle.$$

The reader can verify that we can take

$$Mon_{\theta \to \eta} \equiv \lambda d, f\ (\lambda r, e, g\ f(r, Concat(d, e), g))$$

and

$$Mon_{\forall x\ \theta} \equiv \lambda d, f\ \lambda y\ ((\lambda x\ Mon_\theta)(y, d, f(y)))$$

where y is a new variable of type TERM. $\square$

Lemma 5.2 *Suppose d is a proof in natural deduction of a negative formula $\varphi(x_1, \ldots, x_k)$ from negative formulae $\psi_1(x_1, \ldots, x_k), \ldots, \psi_l(x_1, \ldots, x_k)$. Then there is a term*

$$L_d[u, x_1, \ldots, x_k, z_1, \ldots, z_l] \in \mathsf{Type}_\varphi$$

with free variables $u \in \mathsf{COND}$, $x_1, \ldots, x_k \in \mathsf{TERM}$, $z_1 \in \mathsf{Type}_{\psi_1}, \ldots, z_l \in \mathsf{Type}_{\psi_l}$, such that the following holds: whenever p is a condition, $t_1, \ldots, t_k$ are terms, f_1 realizes $p \Vdash \psi_1(t_1, \ldots, t_k)$, …, and f_l realizes $p \Vdash \psi_l(t_1, \ldots, t_l)$, then

$$L_d[p/u, t_1/x_1, \ldots, t_k/x_k, f_1/z_1, \ldots, f_l/z_l]$$

realizes $p \Vdash \varphi(t_1, \ldots, t_k)$.

Proof. A straightforward recursion/induction on d. For an assumption, just use the variable for the corresponding realizer. For introduction and elimination for $\wedge$, use pairing and projection; for introduction and elimination for $\to$, use monotonicity (Lemma 5.1), lambda abstraction, and application; for introduction and elimination for $\forall$, use lambda abstraction and application; and for the rule *ex falso sequitur quodlibet*, use the terms $ExFalso_\varphi$ from Lemma 5.1. $\square$

Lemma 5.3 *Let $\varphi(x_1, \ldots, x_k)$ be a formula in negation-normal form, with the free variables shown. Then there is a term $K_\varphi[x_1, \ldots, x_k]$ with free variables $x_1, \ldots, x_k$ of type TERM, such that the following holds: whenever $t_1, \ldots, t_k$ are terms, $K_\varphi[t_1/x_1, \ldots, t_1/x_k]$ realizes $\{\varphi(t_1, \ldots, t_k)\} \Vdash \varphi^N(t_1, \ldots, t_l)$.*

Proof. Use recursion/induction on φ. I will do the same three illustrative cases as in the proof of Lemma 4.6, and again omit references to $x_1, \ldots, x_k$. Note, incidentally, that the function *NewVar* is needed to handle the rule for $\exists$.

In the base case, suppose φ is an atomic formula A. We want a term taking a condition p, a realizer d for $p \preceq \{A\}$, and a realizer f for $p \Vdash \neg A$ to a realizer for $p \Vdash \bot$. If d realizes $p \preceq \{A\}$ then $Weaken_A(x_1, \ldots, x_k, d)$ realizes $p \Vdash A$. Since $Tree_0(p)$ realizes $p \preceq p$, $f(p, Tree_0(p), Weaken_\varphi(x_1, \ldots, x_k, d))$ realizes $p \Vdash \bot$. So set

$$K_A \equiv \lambda p, d, f\ f(p, Tree_0(p), Weaken_A(x_1, \ldots, x_k, d)).$$

Suppose φ is of the form $\theta \wedge \eta$. By the induction hypothesis, K_θ realizes $\{\theta\} \Vdash \theta^N$ and K_η realizes $\{\eta\} \Vdash \eta^N$. Let e_0 be the derivation $Tree_1(\{\theta \wedge \eta\}, Tree_0(\{\theta\}))$ and let e_1 be the derivation $Tree_1(\{\theta \wedge \eta\}, Tree_0(\{\theta\}))$, so e_0 realizes $\{\theta \wedge \eta\} \preceq \{\theta\}$ and e_1 realizes $\{\theta \wedge \eta\} \preceq \{\eta\}$. Then $Mon_{\theta^N}(\{\theta \wedge \eta\}, e_0, K_\theta)$ realizes $\{\theta \wedge \eta\} \Vdash \theta^N$ and $Mon_{\eta^N}(\{\theta \wedge \eta\}, e_1, K_\eta)$ realizes $\{\theta \wedge \eta\} \Vdash \eta$. So we can take

$$K_{\theta \wedge \eta} \equiv \langle Mon_{\theta^N}(\{\theta \wedge \eta\}, e_0, K_\theta\}, Mon_{\eta^N}(\{\theta \wedge \eta\}, e_1, K_\eta\}\rangle.$$

Finally, suppose φ is of the form $\theta \vee \eta$. We need a term taking a condition p, a realizer d for $p \preceq \{\theta \vee \eta\}$, and a realizer f for $p \Vdash \neg\theta^N \wedge \neg\eta^N$ to a realizer for $p \Vdash \bot$. So, suppose p, d, and f are as above. Then $(f)_0$ realizes $p \Vdash \neg\theta^N$ and $(f)_1$ realizes $p \Vdash \neg\eta^N$. Let e_0 and e_1 realize $p \cup \{\theta\} \preceq \{\theta\}$ and $p \cup \{\eta\} \preceq \{\eta\}$, respectively. Then $Mon_{\theta^N}(p \cup \{\theta\}, e_0, K_\theta)$ realizes $p \cup \{\theta\} \Vdash \theta^N$ and $Mon_{\eta^N}(p \cup \{\eta\}, e_1, K_\eta)$ realizes $p \cup \{\eta\} \Vdash \eta^N$. Call these terms g_0 and g_1. Let h_0 and h_1 realize $p \cup \{\theta\} \preceq p$ and $p \cup \{\eta\} \preceq p$, respectively. Then $(f)_0(p \cup \{\theta\}, h_0, g_0)$ realizes $p \cup \{\theta\} \Vdash \bot$, i.e. is a derivation of $\sim p \cup \{\sim\theta\}$; and similarly, $(f)_1(p \cup \{\theta\}, h_1, g_1)$ is a derivation of $\sim p \cup \{\sim\eta\}$. Then

$$Tree_2(p \cup \{\theta \vee \eta\}, (f)_0(p \cup \{\theta\}, h_0, g_0), (f)_1(p \cup \{\theta\}, h_1, g_1))$$

is a derivation of $\sim p \cup \{\sim\theta \wedge \sim\eta\}$, and so realizes $p \cup \{\theta \vee \eta\} \Vdash \bot$. But since d realizes $p \preceq \{\theta \vee \eta\}$, $Weaken_{\theta\vee\eta}(x_1, \ldots, x_k, d)$ realizes $p \preceq p \cup \{\theta \vee \eta\}$; combining the two using *Concat* yields a realizer for $p \Vdash \bot$. □

Putting these together yields a "concrete" version of the cut-elimination theorem.

Theorem 5.4 *Let d be a proof of a sequent Γ in the classical sequent calculus. Then there is a typed lambda term T_d denoting a cut-free proof of Γ.*

Proof. For simplicity, assume that Γ has no free variables; otherwise, we can replace free variables x_i with constants $\hat{x}_i$ in the appropriate terms below.

Suppose Γ is the sequent $\{\varphi_1, \ldots, \varphi_k\}$ and d is a cut-free proof of Γ. Use the double-negation translation to find a derivation d' of $\bot$ from $(\sim\varphi_1)^N, \ldots, (\sim\varphi_k)^N$ in intuitionistic logic, and let $L_{d'}[u, z_1, \ldots, z_k]$ be the term given by Lemma 5.2.

For each i, let e_i realize $\{\sim\varphi_1, \ldots, \sim\varphi_k\} \preceq \{\sim\varphi_i\}$. Then for each i, the term $Mon_{\sim\varphi_i^N}(e_i, K_{\sim\varphi_i})$ realizes $\{\sim\varphi_1, \ldots, \sim\varphi_i\} \Vdash (\sim\varphi_i)^N$. Then the term

$$L_{d'}[\{\sim\varphi_1, \ldots, \sim\varphi_i\}/u, Mon_{\sim\varphi_1}(e_1, K_{\sim\varphi_1^N})/z_1, \ldots, Mon_{\sim\varphi_k}(e_k, K_{\sim\varphi_k^N})/z_k]$$

realizes $\{\sim\varphi_1, \ldots, \sim\varphi_i\} \Vdash \bot$, which is to say, it denotes a cut-free derivation of $\{\varphi_1, \ldots, \varphi_k\}$. □

6 Another double-negation translation

Lurking beneath the constructive proof in Section 4 there is an implicit translation from classical logic to minimal logic. Here I will make this translation explicit, and in the next section I will show how one can extend the proof of the cut-elimination theorem to intuitionistic logic, in such a way that proof for classical logic can be viewed as a special case.

Lemma 6.1 *Let φ be any formula in negation-normal form. Then $\varphi \to \varphi^N$ is provable in minimal logic.*

Proof. An easy induction on φ. For general formulae, dealing with implication in the induction step would be problematic; but the restriction to negation-normal form means that we only have to consider implication in the context of a negated atoms. $\square$

Theorem 6.2 *If a formula φ is provable classically, then $\neg(\neg\varphi)^{nf}$ is provable in minimal logic.*

Moreover: classical proofs of φ in a sequent calculus with cut, natural deduction, or an axiomatic proof system, can be translated efficiently to proofs of $\neg(\neg\varphi)^{nf}$ in minimal logic, and vice-versa. Similarly, cut-free classical proofs of φ in a two-sided sequent calculus, and cut-free classical proofs of φ^{nf} in a one-sided sequent calculus, can be translated efficiently to cut-free proofs of $\neg(\neg\varphi)^{nf}$ in the sequent calculus for minimal logic, and vice-versa.

Proof. If φ is provable classically, then φ^N and hence $\neg\neg\varphi^N$ are provable in minimal logic. By the previous lemma, $(\neg\varphi)^{nf} \to \neg\varphi^N$ is provable in minimal logic, and hence $\neg\neg\varphi^N \to \neg(\neg\varphi)^{nf}$ is also provable in minimal logic. So $\neg(\neg\varphi)^{nf}$ is provable in minimal logic as well.

We have shown that if φ is provable classically, then $\neg(\neg\varphi)^{nf}$ is provable in minimal logic. In the other direction, of course, $\neg(\neg\varphi)^{nf}$ is classically equivalent to φ. The reader can check that the proofs of the relevant implications and equivalences in classical and minimal logic, using any of the standard proof systems named in the theorem, are polynomial in the length of φ.

As far as cut-free proofs are concerned, one can show more generally that if $\{\varphi_1, \ldots, \varphi_k\}$ is provable in the classical one-sided sequent calculus without cut, then the sequent $\{\sim\varphi_1, \ldots, \sim\varphi_k\} \Rightarrow \bot$ has a cut-free proof in the minimal sequent calculus. The proof is easy: under the translation, the rules of the classical one-sided calculus correspond exactly to the left-rules of the minimal two-sided calculus. Then a classical proof of φ in a two-sided calculus corresponds to a classical proof of φ^{nf} in a one-side calculus, which in turn corresponds to a proof of $\{(\neg\varphi)^{nf}\} \Rightarrow \bot$ in minimal logic. $\square$

The moral behind Theorem 6.2 is that if we restrict our attention to negation-normal form, interpreting classical negation as $\sim\varphi$ and interpreting classical implication as $\sim\varphi \vee \psi$, then we can view classical logic as taking place on the left side of a sequent in minimal logic, with $\bot$ sitting on the right.

It is worth noting that with intuitionistic logic in place of minimal logic, these facts follow from a characterization of Glivenko formulae due to Orevkov; see the discussion in [12, Section 3.2.5].

7 Cut elimination for intuitionistic logic

Let us now extend the proof of the cut-elimination theorem to the intuitionistic calculus. Take conditions $p, q, r, \ldots$ to be finite sets of formulae, not necessarily in negation-normal form, and say p is *stronger than* q, written $p \preceq q$, if $p \supseteq q$.

The clauses below provide an inductive definition of a relation "S covers p," between conditions p and finite sets S of conditions stronger than p:

1. $\{p\}$ covers p.
2. If $\{q_1, \ldots, q_k\}$ covers p, and for each i, S_i covers q_i, then $\bigcup_{i=1}^{k} S_i$ covers p.
3. If $\varphi \vee \psi$ is in p, then $\{p \cup \{\varphi\}, p \cup \{\psi\}\}$ covers p.
4. If $\varphi \wedge \psi$ is in p, then $\{p \cup \{\varphi\}\}$ covers p, and $\{p \cup \{\psi\}\}$ covers p.
5. If $\varphi \rightarrow \psi$ is in p, and there is a cut-free proof of $p \Rightarrow \varphi$, then $\{p \cup \{\psi\}\}$ covers p.
6. If $\forall x\ \varphi$ is in p and t is any term, then $\{p \cup \{\varphi[t/x]\}\}$ covers p.
7. If $\exists x\ \varphi$ is in p and x is not free in p, then $\{p \cup \{\varphi\}\}$ covers p.

Below I will drop the extra set brackets, and say "$q_1, \ldots, q_k$ cover p" instead of "$\{q_1, \ldots, q_k\}$ covers p."

Lemma 7.1 *Suppose $q_1, \ldots, q_k$ cover p, and suppose there are cut-free proofs of $q_1 \Rightarrow \varphi, \ldots, q_k \Rightarrow \varphi$, for some formula φ. Then there is a cut-free proof of $p \Rightarrow \varphi$.*

Proof. Use induction on the covering relation. In fact, the hypotheses imply that there is a cut-free proof of $p \Rightarrow \varphi$ starting from the sequents $q_1 \Rightarrow \varphi, \ldots, q_k \Rightarrow \varphi$. $\square$

Now define the notion of $p \Vdash \varphi$ inductively, as follows:

1. $p \Vdash \bot$ if and only if there is a cut-free proof of $p \Rightarrow \bot$.
2. If A is atomic, $p \Vdash A$ if and only if there is a cut-free proof of $p \Rightarrow A$.
3. $p \Vdash \theta \wedge \eta$ if and only if $p \Vdash \theta$ and $p \Vdash \eta$.
4. $p \Vdash \theta \vee \eta$ if and only if there is a covering $q_1, \ldots, q_k$ of p such that for each i, $q_i \Vdash \theta$ or $q_i \Vdash \eta$.
5. $p \Vdash \theta \rightarrow \eta$ if and only if for every $q \preceq p$, if $q \Vdash \theta$ then $q \Vdash \eta$.
6. $p \Vdash \forall x\ \theta(x)$ if and only if for every term t, $p \Vdash \theta(t)$.
7. $p \Vdash \exists x\ \theta(x)$ if and only if there is a covering $q_1, \ldots, q_k$ of p and a sequence of terms $t_1, \ldots, t_k$ such that for each i, $q_i \Vdash \theta(t_i)$.

In the next two lemmata, a *renaming* of variables is just an injective map from the set of variables to the set of variables. If σ is a renaming, then φ^σ denotes the result of replacing each free variable x of φ by $\sigma(x)$, changing the names of the bound variables if necessary, to prevent collisions. Similarly, if p is a condition and σ is a renaming, then p^σ denotes $\{\varphi^\sigma \mid \varphi \in p\}$. Lemma 7.3

implies that our notion of covering induces a Grothendieck topology on the category with conditions as objects and an arrow $p \xrightarrow{\sigma} q$ for each renaming σ such that $p \preceq q^{\sigma}$. (A similar category is used in [3]. See also the definition of a base for a Grothendieck topology in [11, exercise III.3].) Lemma 7.4 then follows from the general results of [13], though it can just as well be verified directly.

Lemma 7.2 *Suppose p is any condition, $q_1, \ldots, q_k$ are conditions covering p, and r is a condition stronger than p. Then there is a renaming σ such that $q_1^{\sigma} \cup r, \ldots, q_k^{\sigma} \cup r$ covers r.*

Proof. A straightforward induction on the covering relation. The renamings are needed to handle the variable restriction in clause 7. □

Lemma 7.3 *The forcing relation defined above satisfies the following:*

1. *Stability under renaming: if σ is any renaming of variables and $p \Vdash \varphi$, then $p^{\sigma} \Vdash \varphi^{\sigma}$.*

2. *Monotonicity: if $p \Vdash \varphi$ and $q \preceq p$ then $q \Vdash \varphi$.*

3. *The covering property: if $q_1, \ldots, q_k$ cover p and for each i, $q_i \Vdash \varphi$, then $p \Vdash \varphi$.*

Proof. Each clause can be proved using a straightforward induction on the formula φ. Stability under renaming and Lemma 7.2 are needed to handle the clauses for $\vee$ and $\exists$ when proving monotonicity. □

An easy induction derivations shows that if there is a cut-free proof of $p \Rightarrow \bot$ and φ is any formula, then there is a cut-free proof of $p \Rightarrow \varphi$. This takes care of $\bot$ rule in the proof of the following lemma.

Lemma 7.4 *If φ is provable intuitionistically, then φ is forced.*

Proof. A straightforward induction on derivations, as in the proof of Lemma 4.5. □

Lemma 7.5 *Let φ be any formula. Then*

1. *$\{\varphi\} \Vdash \varphi$.*

2. *If $p \Vdash \varphi$, then there is a cut-free proof of $p \Rightarrow \varphi$.*

Proof. We can prove both claims simultaneously by induction on φ. I will focus on two illustrative cases.

For the first sample case, suppose φ is a formula of the form $\theta \vee \eta$. Using the induction hypothesis, we have $\{\theta \vee \eta, \theta\} \Vdash \theta$ and $\{\theta \vee \eta, \eta\} \Vdash \eta$. Hence, both these conditions force $\theta \vee \eta$; and since they cover $\{\theta \vee \eta\}$, we have $\{\theta \vee \eta\} \Vdash \theta \vee \eta$.

For the second claim, suppose $p \Vdash \theta \vee \eta$. By the definition of forcing and the induction hypothesis, there are conditions $q_1, \ldots, q_k$ covering p such that for each i, there is a cut-free proof of $q_i \Rightarrow \theta$ or $q_i \Rightarrow \eta$. In particular, for each i, there is a cut-free proof of $q_i \Rightarrow \theta \vee \eta$. By Lemma 7.1, there is a cut-free proof of $p \Rightarrow \theta \vee \eta$.

As a second example, suppose φ is of the form $\theta \to \eta$. For the first claim, we need to show that if p is any condition and $p \Vdash \theta$, then $p \cup \{\theta \to \eta\} \Vdash \eta$. So suppose $p \Vdash \theta$. By the induction hypothesis, there is a cut-free proof of $p \Rightarrow \theta$. By the definition of covering, $p \cup \{\eta\}$ covers $p \cup \{\theta \to \eta\}$. Again by the induction hypothesis, we have that $p \cup \{\eta\} \Vdash \eta$. By Lemma 7.1, we have that $p \cup \{\theta \to \eta\} \Vdash \eta$, as required.

For the second claim, suppose $p \Vdash \theta \to \eta$. By the induction hypothesis, we have that $\{\theta\} \Vdash \theta$, and so $p \cup \{\theta\} \Vdash \eta$. Again by the induction hypothesis, there is a cut-free proof of $p \cup \{\theta\} \Rightarrow \eta$. This yields a cut-free proof of $p \Rightarrow \theta \to \eta$. $\square$

Note that by Lemmata 3.3 and 4.6, with the forcing relation for classical logic, we have $\{\sim\theta\} \Vdash \neg\theta^N$ for any negative formula θ. Therefore, if $p \Vdash \varphi$, then $p \cup \{\sim\varphi^{nf}\} \Vdash \bot$, and so there is a cut-free proof of $\sim p, \varphi^{nf}$ in the one-sided sequent calculus. This is, in a sense, the classical analogue of clause 2 of Lemma 7.5.

Theorem 7.6 *Any sequent provable in the intuitionistic sequent calculus has a cut-free proof.*

Proof. Suppose there is a proof of $\{\varphi_1, \ldots, \varphi_k\} \Rightarrow \psi$ in the sequent calculus with cut. Then $\varphi_1 \wedge \ldots \wedge \varphi_k \to \psi$ is provable intuitionistically, and so it is forced. By Lemma 7.5, for each i we have $\{\varphi_i\} \Vdash \varphi_i$. By monotonicity, we have $\{\varphi_1, \ldots, \varphi_k\} \Vdash \varphi_1 \wedge \ldots \wedge \varphi_k$, and hence $\{\varphi_1, \ldots, \varphi_k\} \Vdash \psi$. By Lemma 7.5 again, there is a cut-free proof of $\{\varphi_1, \ldots, \varphi_k\} \Rightarrow \psi$. $\square$

The proof of the cut-elimination theorem for minimal logic instead of intuitionistic logic requires almost no changes; one simply replaces intuitionistic provability by minimal provability throughout, and ignores the comment before Lemma 7.4. Theorem 6.2 implies that the cut-elimination theorem for classical logic follows the cut-elimination theorem for minimal logic; in fact, the proof in Section 4 is just what one gets if one restricts the proof above to the negative fragment of minimal logic, replaces "stronger than" by "stronger than some covering of," and incorporates the translation in Section 6. In much the same way, the cut-elimination theorem for intuitionistic higher-order logic described in [5] can now be seen to provide a cut-elimination theorem for classical logic as well.

8 Questions

What is the relationship between the cut-elimination algorithm described in Section 5 and procedures that arise from syntactic proofs of the cut-elimination theorem (like the ones in [18])?

It is relatively easy to translate proofs in a sequent calculus with cut to proofs in natural deduction, and vice-versa; and it is well known that, at least as far as intuitionistic logic is concerned, cut-free proofs in the sequent calculus correspond to normal ones in the natural deduction setting. As a result, normalization proofs for natural deduction calculi can also be seen as establishing cut elimination. But methods like those used in [10, 14] yield additional information, in the form of *strong normalization*; which is to say, they guarantee termination for any procedure that follows a specified set of reductions. Can one say more about the relationship between the algebraic methods above and the methods of [10, 14]? Can the forcing arguments be modified to yield strong normalization?

On the other hand, more explicit cut-elimination procedures yield bounds on the increase in the length of proofs. In the current setting, one has to extract these bounds from the normalization procedure used in Section 5. Is there a more direct way to read off bounds from the algebraic proofs?

In sheaf-theoretic terms, the construction of Section 7 is not quite a sheaf model, but rather a "modified sheaf model" in the terminology of [3], or a "presheaf model" in the terminology of [13]. Similarly, the models constructed in [5] are not quite sheaves. Is there some way that these arguments can be recast as sheaf constructions?

The forcing relations described above can be viewed as providing a metalogical framework for reasoning about search space in a tableau search, in the sense that the forcing relation makes higher-order assertions about cut-free provability. Can this observation be put to use for the purposes of automated deuction?

References

[1] Thorsten Altenkirch, Martin Hofmann, and Thomas Streicher. Categorical reconstruction of a *reduction free* normalization proof. Logic in Computer Science, 1996.

[2] Jeremy Avigad. Saturated models of universal theories. Submitted.

[3] Jeremy Avigad and Jeffrey Helzner. Transfer principles for intuitionistic nonstandard arithmetic. To appear in the *Archive for Mathematical Logic*.

[4] Ulrich Berger. Program extraction from normalization proofs. In M. Bezem and J. F. Groote, eds., *Typed Lambda Calculi and Applications '93*, Lecture Notes in Computer Science 664, pages 91–106, 1993.

[5] Wilfried Buchholz. Ein ausgezeichnetes Modell für die intuitionistische Typenlogik. *Archive for Mathematical Logic*, 17:55–60, 1975.

[6] Thierry Coquand. Two applications of boolean models. *Archive for Mathematical Logic*, 37:143–147, 1997.

[7] Albert Dragalin. *Mathematical Intuitionism: Introduction to Proof Theory.* Translations of mathematical monographs. American Mathematical Society, 1988.

[8] J. E. Fenstad, editor. *Proceedings of the Second Scandinavian Logic Symposium.* North-Holland, 1971.

[9] Melvin Fitting. *Intuitionistic Logic, Model Theory, and Forcing.* North-Holland, 1969.

[10] Jean-Yves Girard. Une extension de l'interpretation de Gödel à l'analyse, et son application à l'élimination des coupures dans l'analyse et dans la théorie des types. In Fenstad [8], pages 63–92.

[11] Saunders Mac Lane and Ieke Moerdijk. *Sheaves in Geometry and Logic.* Springer, 1992.

[12] Grigori Mints. Proof theory in the USSR 1925–1969. *Journal of Symbolic Logic*, 56:385–424, 1991.

[13] Erik Palmgren. Constructive sheaf semantics. *Mathematical Logic Quarterly*, 43:321–327, 1997.

[14] Dag Prawitz. Ideas and results in proof theory. In Fenstad [8], pages 235–307.

[15] Andre Scedrov. Normalization revisited. J. Gray and A. Scedrov eds., *Categories in Computer Science and Logic*, AMS Contemporary Mathematics 92, pages 357–369.

[16] Raymond Smullyan. *First-Order Logic.* Dover, 1995.

[17] A. S. Troelstra. Realizability. In Samuel Buss ed., *The Handbook of Proof Theory.* North-Holland, 1998.

[18] A. S. Troelstra and Helmut Schwichtenberg. *Basic Proof Theory.* Cambridge University Press, 1996.

[19] A. S. Troelstra and Dirk van Dalen. *Constructivism in Mathematics: An Introduction*, volume 1. North-Holland, 1988.

[20] Jeffery Zucker. The correspondence between cut-elimination and normalization. *Annals of Mathematical Logic*, 7:1–112, 1974.