Indicator Choice in Pay-for-Performance

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Pay-for-Performance Contracts

- Performance pay is cornerstone of modern employment contracts
  - Executives, Athletes, Teachers, etc.
- Textbook moral hazard: given an indicator of output, how do we design contracts

Pay-for-Performance Contracts

- Performance pay is cornerstone of modern employment contracts
  - Executives, Athletes, Teachers, etc.
- Textbook moral hazard: given an indicator of output, how do we design contracts
- What if we can choose the indicator
  - Principal's choice: Informativeness Principle a la Holmstrom
  - Agent's choice: this paper

The Model

Textbook Moral Hazard Model

- A Principal (P or he) is employing an agent to perform a task.
- Agent (A or she) chooses effort $e \in E, |E| < \infty$ to perform the task.
- Effort is costly to the agent: $c : E \to \mathbb{R}_+$
  - $e_1 \in E$ represents the effort with the lowest cost: $c(e_1) = 0$

Performance Technology

- Effort e generates a performance measure $x \in X, |X| < \infty$
  - Distribution: $f(x|e) = \Pr(x|e) \in \Delta(X)$
- Output: $y = g(x)$
- Example: **Perfect performance technology**:

$$X = E, \Pr(x|e) = 1[x = e]$$

Model: Information Structure and Contracts

- P cannot observe A's effort
- A can influence the P's information about the output by choosing an information structure (S, π) where $\pi(\cdot|x) : X \to \Delta(S)$.
- P only observes the signal generated from this information structure and can thus only offer a contract contingent on this signal.
- P's choice: $w : S \to \mathbb{R}_+$
    - **Limited liability:** Cannot make the agent pay, $w(s) \geq 0$

Model: Payoffs

- P: $u_P = g(x) - w(s)$
- A: $u_A = w(s) - c(e)$
- P can always implement e_1 by offering $w(s) = 0, \ \forall s \in S$.
  - P's outside option: $\underline{U}_P = \sum_x g(x) f(x|e_1)$

Timing of the Game

1. A chooses an information structure π.
2. Observing the information structure π chosen by the A, P offers agent a contract $w : S \to \mathbb{R}_+$.
3. Given $w(\cdot)$, A chooses how much effort e to exert.
4. x is realized according to $f(x|e)$, signal $s \in S$ is realized according to $\pi(s|x)$, and payoffs are realized.

Timing of the Game

- First stage:

$$\pi^* \in \arg\max_{\pi} \mathbb{E}_{\pi}\left[w\left(s;\pi\right)|e^*\left(w^*\left(\cdot;\pi\right),\pi\right)\right] - c\left(e^*\left(w^*\left(\cdot;\pi\right),\pi\right)\right)$$

- Second stage:

$$w^*\left(s;\pi\right) \in \arg\max_{w(s)} \mathbb{E}\left[g\left(x\right)|e^*\left(w\left(\cdot\right),\pi\right)\right] - \mathbb{E}_{\pi}\left[w\left(s\right)|e^*\left(w\left(\cdot\right),\pi\right)\right]$$

- **Last stage:**

$$e^*\left(w\left(\cdot\right),\pi\right) \in \arg\max_{e} \mathbb{E}_{\pi}\left[w\left(s\right)|e\right] - c\left(e\right)$$

Timing of the Game

- First stage:

$$\pi^* \in \arg\max_{\pi} \mathbb{E}_\pi \left[w(s;\pi) | e^*\left(w^*(\cdot;\pi),\pi\right)\right] - c\left(e^*\left(w^*(\cdot;\pi),\pi\right)\right)$$

- **Second stage:**

$$w^*(s;\pi) \in \arg\max_{w(s)} \mathbb{E}\left[g(x)|e^*\right] - \mathbb{E}_\pi\left[w(s)|e^*\right]$$

- Third stage:

$$e^*(w(\cdot),\pi) \in \arg\max_{e} \mathbb{E}_\pi\left[w(s)|e\right] - c(e)$$

Timing of the Game

- **First stage:**

$$\pi^* \in \arg\max_{\pi} \mathbb{E}_\pi \left[w^* \left(s; \pi \right) | e^* \right] - c \left(e^* \right)$$

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- Third stage:

$$e^* \left(w \left(\cdot \right), \pi \right) \in \arg\max_{e} \mathbb{E}_\pi \left[w \left(s \right) | e \right] - c \left(e \right)$$

Interpretations

- Literal interpretation: information
  - The agent equivalent of the informativeness principle a la Holmstrom
  - Theoretical benchmark
- Contractibility interpretation:
  - A regulator or a union picks the performance measure:
  - It is binding for all parties
  - Examples: Union or regulators choose what is contractible
    - Teachers
    - Athletes
    - Hollywood Writers: the issue of residual payments

Perfect Performance

Perfect Performance

- Suppose that $X = E,\ f(x|e) = 1[x = e]$
- First best:

$$e_{FB} \in \arg\max_e g(e) - c(e)$$

- What if A chooses a deterministic signal?
  - $\pi(s|e) = 0$ or 1
  - no information and full information are special cases.
  - Payoff of the agent is 0!

Perfect Performance

Proposotion. Suppose that $g(e_{FB}) > g(e_1)$. Then, there exists a signal (π, S) such that:

1. FB effort, e^*, is implemented,
2. $u_A = g(e_{FB}) - c(e_{FB}) - g(e_1)$, i.e., A gets all the surplus.

Full Surplus Extraction

- Proof is by construction – for the sake of presentation $g(e_1) = 0$:

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$$S = \{L, H\}, \pi(H|e) = \begin{cases} 1 - \frac{c(e_{FB}) - c(e)}{g(e^*)} & c(e) < c(e_{FB}) \\ 1 & c(e) \geq c(e_{FB}) \end{cases}$$

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- If P wants to implement $\hat{e} : c(\hat{e}) \leq c(e_{FB})$, has to pay

$$w(H) = \max_{e' : c(e') \leq c(\hat{e})} \frac{c(\hat{e}) - c(e')}{\pi(H|\hat{e}) - \pi(H|e')} = \max_{e' : c(e') \leq c(\hat{e})} \frac{c(\hat{e}) - c(e')}{\frac{c(\hat{e}) - c(e')}{g(e_{FB})}}$$

$$= g(e_{FB})$$

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- P's payoff

$$g(\hat{e}) - \pi(H|\hat{e}) g(e_{FB}) =$$

$$g(\hat{e}) - c(\hat{e}) - [g(e_{FB}) - c(e_{FB})]$$

Full Surplus Extraction

- When A can choose any $\pi(s|e)$, she can guarantee a wage of $g(e^*)$ for all effort levels
- Cost to P (expected wage) is a shift of $c(e)$
- e^* is optimal for P
- Too much flexibility for choice of off-path information

General Performance

General Performance

- General performance: $x \in X, e \in E, f(x|e) \in \Delta(X)$
- A can only choose a garbling of x

$$p(s|e) = \sum_{x} \pi(s|x) f(x|e)$$

- We can think about choice of π as a sender-receiver game
- Complication:
    - P or receiver's choice of $w(\cdot)$ depends on the entire $\{p(s|e)\}$
- Next: Geometric method to deal with it.
- Key assumption:

Assumption. Performance has full support, i.e., $f(x|e) \neq 0, \forall x \in X, e \in E$

Likelihood Representation

- In what follows: what is the best way for the agent to implement e^*

Likelihood Representation

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- likelihood ratio; $E = \{e_1, \cdots, e_m\}$:

$$\ell_i^p(s) = 1 - \frac{p(s|e_i)}{p(s|e^*)}, \boldsymbol{l}^p(s) = \left(\ell_1^p(s), \ell_2^p(s), \cdots, \ell_m^p(s)\right)$$

can be embedded in $\mathbb{R}^{m-1}$; $\ell_i(s) = 0, e_i = e^*$.

Likelihood Representation

Claim.

1. Lowest cost of choosing $e_j \in E$ for the principal only depends on $\{\boldsymbol{l}(s)\}_{s \in S}$.
2. Choice of $w(s)$ associated with choosing a point in the convex hull of $\{\boldsymbol{l}(s)\}_{s \in S}$.

- Intuition:

$$\sum_{s} p\left(s|e_j\right) w(s) - c\left(e_j\right) \geq \sum_{s} p\left(s|e_i\right) w(s) - c\left(e_i\right)$$

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$$\sum_s p\left(s|e^*\right) w(s) \frac{p\left(s|e_j\right)}{p\left(s|e^*\right)} - c\left(e_j\right) \geq \sum_s p\left(s|e^*\right) w(s) \frac{p\left(s|e_i\right)}{p\left(s|e^*\right)} - c\left(e_i\right)$$

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$$\underbrace{\sum_s p\left(s|e^*\right) w\left(s\right)}_{\overline{w}} \frac{p\left(s|e_j\right)}{p\left(s|e^*\right)} - c\left(e_j\right) \geq \sum_s p\left(s|e^*\right) w\left(s\right) \frac{p\left(s|e_i\right)}{p\left(s|e^*\right)} - c\left(e_i\right)$$

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$$\overline{w} \sum_s \alpha_s \left[1 - \ell_j^p\left(s\right)\right] - c\left(e_j\right) \geq \overline{w} \sum_s \alpha_s \left[1 - \ell_i^p\left(s\right)\right] - c\left(e_i\right)$$

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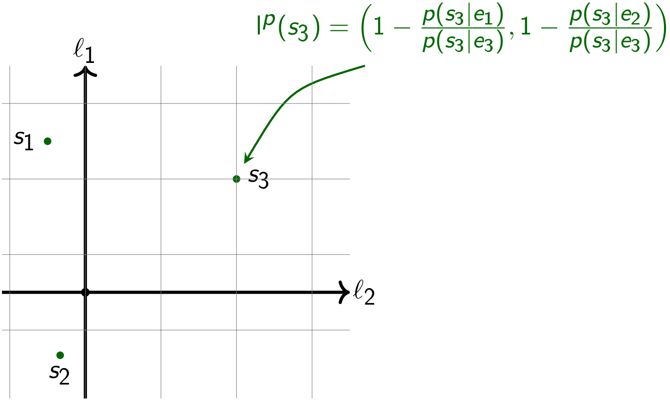
$$\overline{w}\left[1 - \ell_j\right] - c\left(e_j\right) \geq \overline{w}\left[1 - \ell_i\right] - c\left(e_i\right)$$

$$\mathrm{l} = (\ell_1, \cdots, \ell_m) \in \mathrm{co}(\mathrm{p})$$

Geometric Representation

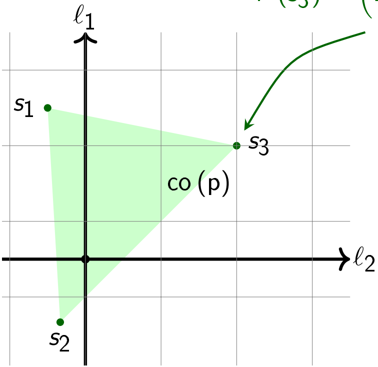
- **Example:** $E = \{e_1, e_2, e_3\},\ e^* = e_3.$



Geometric Representation

- **Example:** $E = \{e_1, e_2, e_3\},\ e^* = e_3.$

$$l^p(s_3) = \left(1 - \frac{p(s_3|e_1)}{p(s_3|e_3)}, 1 - \frac{p(s_3|e_2)}{p(s_3|e_3)}\right)$$



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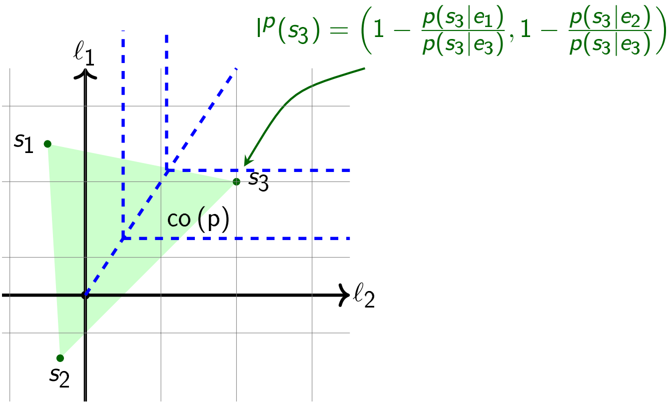


Geometry of Indicators

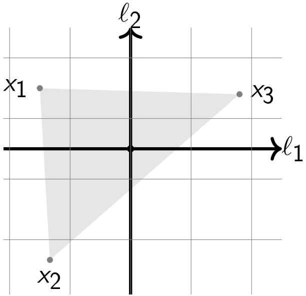
$$\mathrm{co}\left(\mathrm{f}\right) = \text{convex hull}\left(\left\{\left[1 - \frac{f\left(x|e_1\right)}{f\left(x|e^*\right)}, \cdots, 1 - \frac{f\left(x|e_m\right)}{f\left(x|e^*\right)}\right]\right\}_{x\in X}\right)$$

Proposition.

1. For any information structure (S, π) with $|S| < \infty$, its associated $\mathrm{co}\left(\mathrm{p}\right)$ is a subset of $\mathrm{co}\left(\mathrm{f}\right)$ that contains the origin $\mathbf{0} = (0, \cdots, 0)$.
2. For any convex subset C of $\mathrm{co}\left(\mathrm{f}\right)$ that contains the origin and has a finite set of extreme points, there exists an information structure (S, π) such that $\mathrm{co}\left(\mathrm{p}\right) = C$.

Geometric Game

1. A chooses a finite set of points L inside the convex set $\text{co}(\text{f})$ such that the convex hull of these points $\text{conv}(L)$ includes the origin.
2. Principal chooses an effort level $e_i \in E$ and a point $\text{l} \in \text{conv}(L) \cap \Omega_i$.
3. Agent chooses the effort level e_i.



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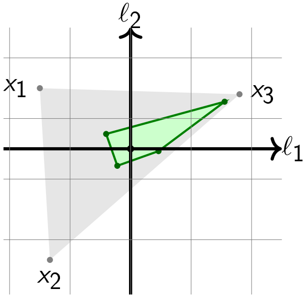
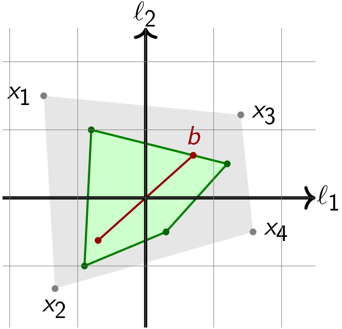


Binary Information Structures

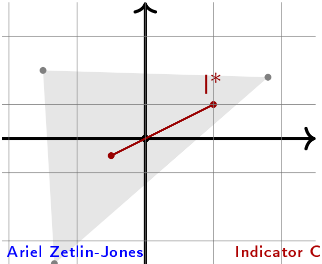
Proposition. If e^* is implementable by some information structure (S, π) and delivers expected wage $W(e^*, \pi)$ to the agent, then e^* is also implementable by a binary information structure $(\hat{S}, \hat{\pi})$, $|\hat{S}| = 2$ and $W(e^*, \hat{\pi}) = W(e^*, \pi)$.



Full Surplus Extraction

- First-best level of effort: $e^* \in \arg\max_{e \in E} \mathbb{E}[g(x)|e] - c(e)$
- Let $\ell_i^* = \frac{c(e^*) - c(e_i)}{\mathbb{E}[g(x)|e^*] - \mathbb{E}[g(x)|e_1]}$,

Proposition. Suppose that $\mathrm{l}^* \in \mathrm{co}(\mathrm{f})$. Then e^* is implementable and there exists an information structure for which the agent can capture the entire surplus.

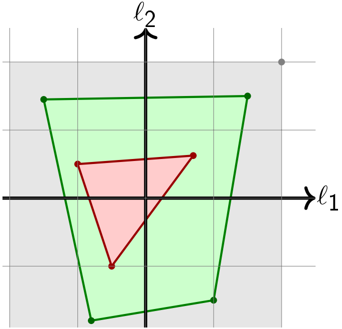


Lower Bound on Information

- Alternative constraint on (π, S):
  - P observes $x \sim f(x|e)$
  - A chooses what to show more

$$\pi \succcurlyeq_{\text{Blackwell}} f$$

- Similar geometric representation



Continuous Effort and Output

- $x \in X = [0,1],\ e \in E = [0,1],\ c'(e) \geq 0,\ c''(e) > 0,\ c(0) = 0$

Assumption.

1. Given any effort $e \in E$, the likelihood $\frac{f_e(x|e)}{f(x|e)}$ is strictly monotone in output x and its derivative $\frac{\partial}{\partial e}\frac{f_e(x|e)}{f(x|e)}$ is a convex function of the likelihood $\frac{f_e(x|e)}{f(x|e)}$.
2. First order approach is valid

Continuous Effort and Output

Proposition. The equilibrium information structure is characterized by at most two thresholds in the output space. If the equilibrium information structure has a single threshold, say x^*, then $\pi(H|x) = 1$ if and only if $x \geq x^*$. If the equilibrium information structure has two thresholds, say (x_1^*, x_2^*), then $\pi(H|x) = 1$ if and only if $x \in [x_1^*, x_2^*]$.

Concluding Remarks

- Developed the theoretical tool in the design of indicators for contracts in principal-agent settings.
- Geometric approach allows us to significantly simplify the problem and check optimality

Related Literature

Back

- **Moral Hazard**: Innes (1990), Poblete & Spulber (2012), Carrol (2015), Walton & Carroll (2022)
- **Information in Moral Hazard** HolmstrÃ¶m (1979), Chaigneau et al. (2019), Garrett et al. (2020), Georgiadis & Szentes (2020), Barron et al. (2020)
- **Incentives in Bayesian Persuasion**: Boleslavsky and Kim (2018), Rosar (2017), Perez-Richet and Skreta (2022), Ball (2019), Saeedi and Shourideh (2020), Zapechelnyuk (2020)