Efficiency and Adverse Selection: On The Role of Mutual Contracts

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Introduction

- Economies with adverse selection: classic examples of "inefficient" economies
    - Akerlof (1970): markets can fully shut down
    - Rothschild and Stiglitz (1976): pure strategy equilibria do not exist (with screening)
        - mixed strategy exists but is inefficient
    - Guerrieri, Shimer, and Wright (2011): existence but inefficiency (with capacity constraints)

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- Common result: equilibria do not exist or are often inefficient
- Common feature: contracts are not rich enough

This Paper

- Enrich contract space using insights from mechanism design
  - Facing many agents: contracts depend on composition of reports
- Main Results: once we allow for interdependence
  - Efficient equilibrium exists
  - Under some restriction all equilibria are constrained efficient
- Interdependence resembles mutual contracts/cooperatives
  - Interpretation: customers as shareholders

Customers as Shareholders

- Payoff of each customer depends on the aggregate loss experience of the firm
  - Insurance: mutual insurance is a prevalent form of insurance
- Life insurance in the U.S.
  - in 2014: 1/3 of all life insurance in force mutualized
- Health insurance in the U.S.
  - Aggregate loss experience leads to adjustment of future premia

Related Literature

- Blandin, Boyd, and Prescott (2016)
  - Use core as solution concept
- Wilson (1980)
  - Contracts depend on contracts offered by other firms
- Netzer and Scheuer (2014)
  - Give firms an option to exit
- Large literature on adverse selection and screening: often deliver inefficient market outcomes:
  - Dubey and Geanakoplos (2002), Guerrieri, Shimer and Wright (2010), Azevedo and Gottlieb (2017), among many many others.

Environment

Players

- Continuum of households of unit mass:
  - low risk (good) and high risk (bad): $j \in \{g, b\}$
  - endowment: $\omega \in \{\omega_2 < \omega_1\}$; 2 is loss state
    - risk: $\Pr(\omega_1|j) = \pi_j; \pi_g > \pi_b$
  - Population fractions: $\Pr(j) = \mu_j; \mu_g + \mu_b = 1$
  - Concave utility function $u(c)$
- 2 risk-neutral insurance companies (firms)

Allocations, Payoffs, ...

- Allocations: $\mathbf{c} = \{\mathbf{c}_g, \mathbf{c}_b\} = \{(c_{1j}, c_{2j})\}_{j \in \{g,b\}}$
- Payoffs:
  - Households:

$$U_j(\mathbf{c}_j) = \pi_j u(c_{1j}) + (1 - \pi_j) u(c_{2j})$$

  - Firms – from type j:

$$\Pi_j(\mathbf{c}_j) = \pi_j(\omega_1 - c_{1j}) + (1 - \pi_j)(\omega_2 - c_{2j})$$

  - Total firm profits:

$$\Pi(\mathbf{c}) = \sum_{j=b,g} \lambda_j \Pi_j(\mathbf{c}_j)$$

$\boldsymbol{\lambda} = (\lambda_b, \lambda_g)$ measure of types that a firm trades with

Incentive Compatibility

- Risk types: private information to household
- Focus on direct mechanisms: $(c_{1g}, c_{2g}, c_{1b}, c_{2b})$
- Incentive Compatibility:

$$\pi_b u(c_{1b}) + (1 - \pi_b) u(c_{2b}) \geq \pi_b u(c_{1g}) + (1 - \pi_b) u(c_{2g})$$
$$\pi_g u(c_{1g}) + (1 - \pi_g) u(c_{2g}) \geq \pi_g u(c_{1b}) + (1 - \pi_g) u(c_{2b})$$

- Relevant IC: b pretending to be g

Efficient Allocations

Efficiency

- Our Notion of Efficiency: constrained efficiency
- Defines an interim pareto frontier
- One example: low risk efficient allocation
  - Max welfare of g subject to
    - IC
    - resource constraint
    - participation by b: must be better off than autarkic full insurance
  - autarkic full insurance: full insurance with premium $(1 - \pi_b)(\omega_1 - \omega_2)$
  - One candidate for equilibrium

Interim Pareto Frontier

- Interim Pareto Frontier is characterized by

$$\max U_g(\mathbf{c}_g)$$

subject to

$$\text{IC}, \mu_g \Pi_g(\mathbf{c}_g) + \mu_b \Pi_b(\mathbf{c}_b) \geq 0$$
$$U_b(\mathbf{c}_b) \geq v_b$$

- Varying v_b traces out the frontier.
- Low-risk efficient: best from g's perspective

Low Risk Efficiency

For any composition of types (λ_b, λ_g)

$$V_g^{eff}(\lambda_b, \lambda_g) = \max_{c_{1j}, c_{2j}} \pi_g u(c_{1g}) + (1 - \pi_g) u(c_{2g})$$

subject to

$$\pi_b u(c_{1b}) + (1 - \pi_b) u(c_{2b}) \geq \pi_b u(c_{1g}) + (1 - \pi_b) u(c_{2g})$$
$$\sum_j \lambda_j \left[\pi_j(\omega_1 - c_{1j}) + (1 - \pi_j)(\omega_1 - c_{2j})\right] \geq 0$$
$$\pi_b u(c_{1b}) + (1 - \pi_b) u(c_{2b}) \geq V_b^f$$

- Equivalently defines $V_b^{eff}(\lambda_b, \lambda_g)$

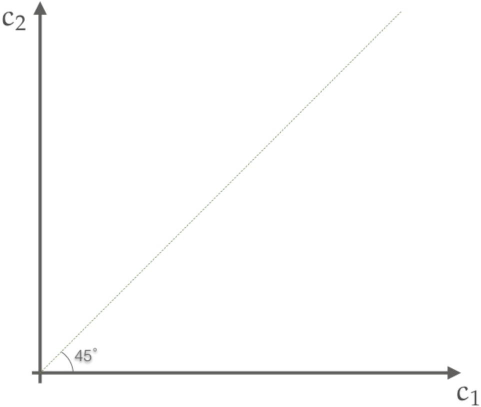
Low Risk Efficient Allocations

- Utilities are homogenous of degree 0 in (λ_b, λ_g)

Low Risk Efficient Allocations

- Utilities are homogenous of degree 0 in (λ_b, λ_g)
- If $\frac{\lambda_g}{\lambda_g + \lambda_b} \leq \lambda^*$ then
    - efficiency coincides with least-cost separating allocation
    - participation constraint binds
    - incentive constraint binds
    - no cross-subsidization; profits are zero on each type

Efficient Allocations



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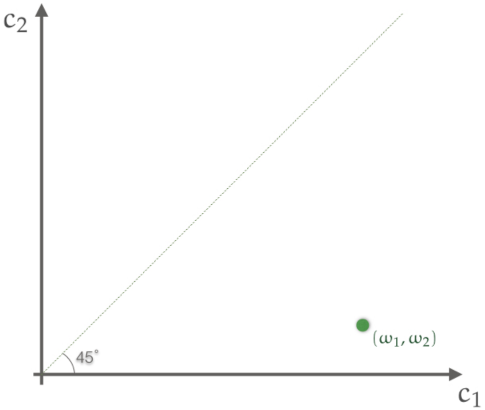

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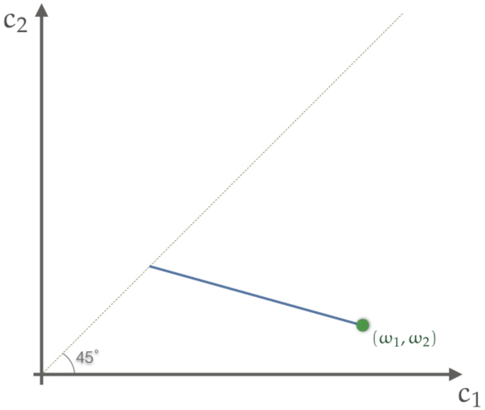

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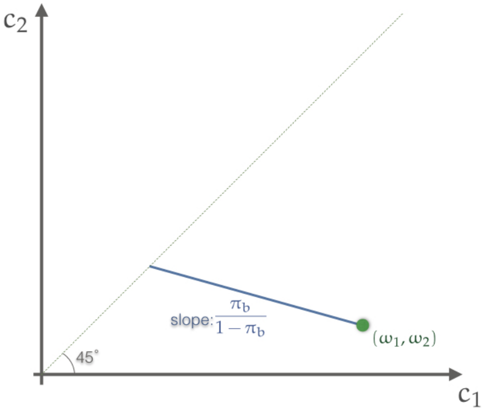

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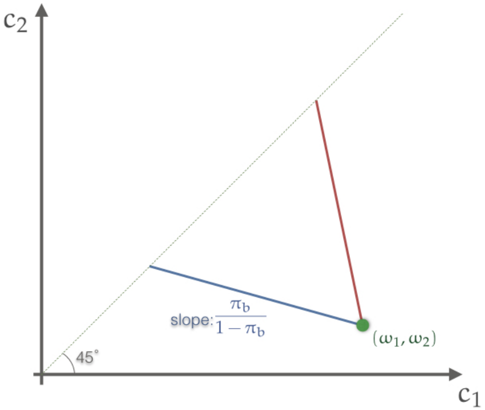

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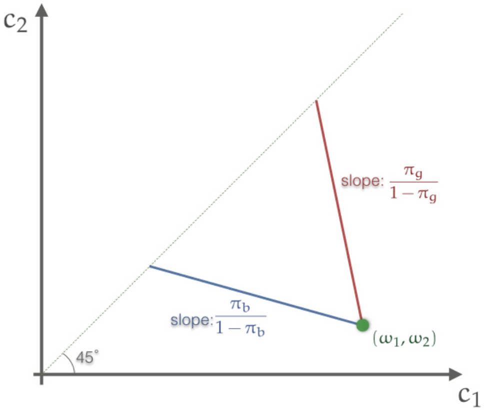

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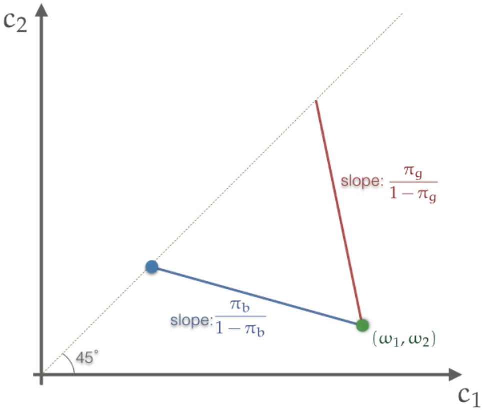

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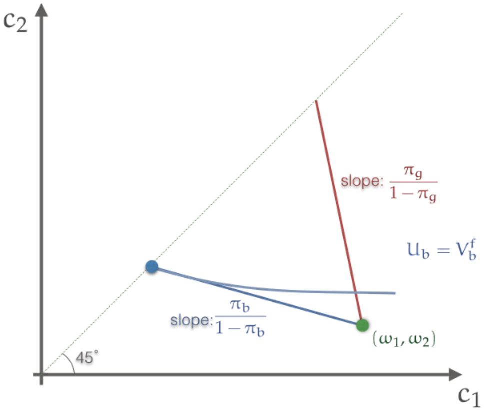

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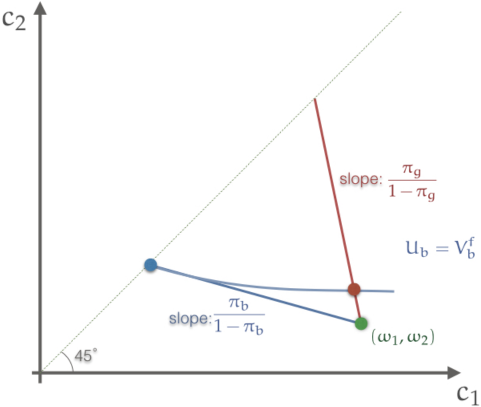

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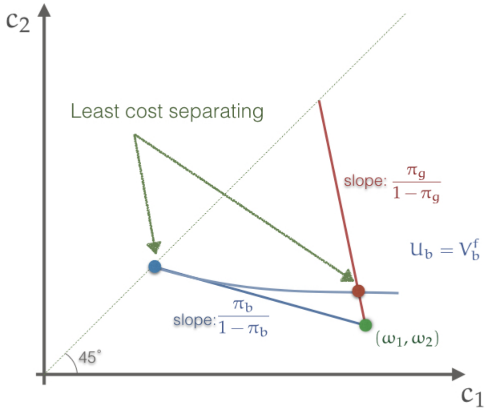

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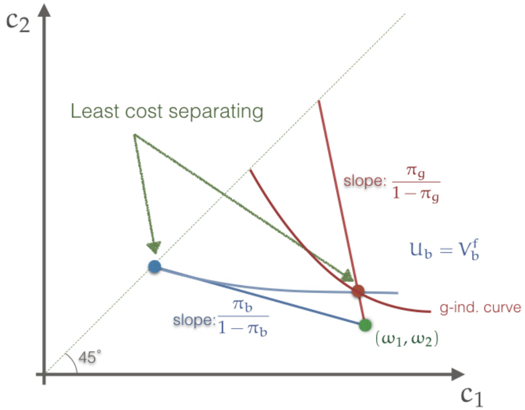

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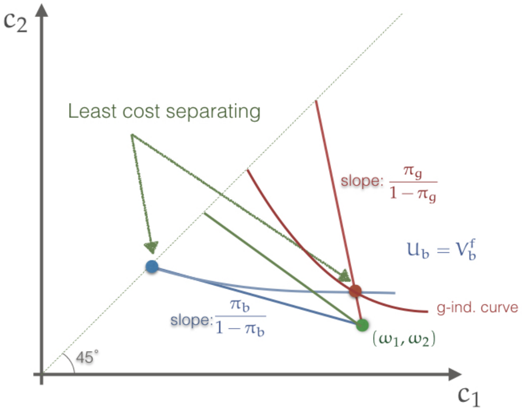

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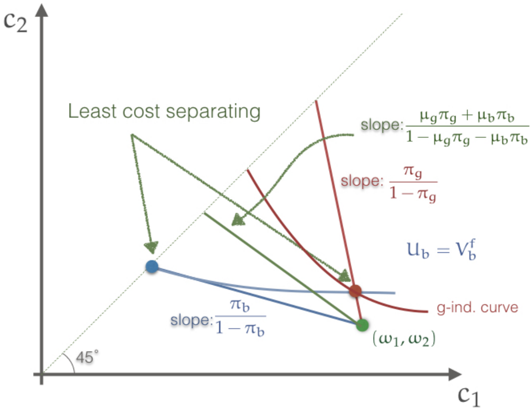

Low Risk Efficient Allocations

- If $\frac{\lambda_g}{\lambda_g+\lambda_b} > \lambda^*$ then
  - participation constraint slack
  - incentive constraint binds
  - cross-subsidization
    - positive profits on g
    - negative profits on b

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  - participation constraint slack
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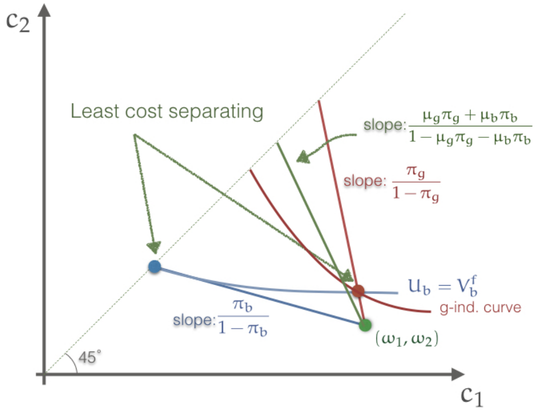
- Focus only on $\mu_g \geq \lambda^*$

Efficient Allocations



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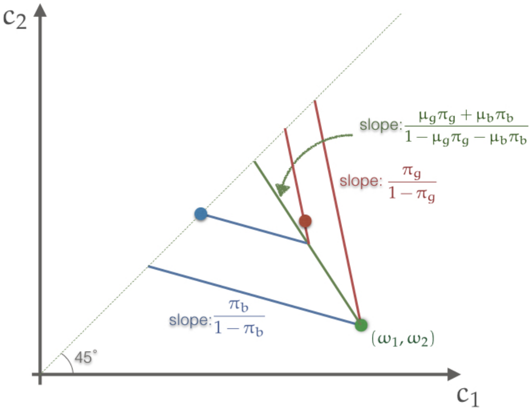

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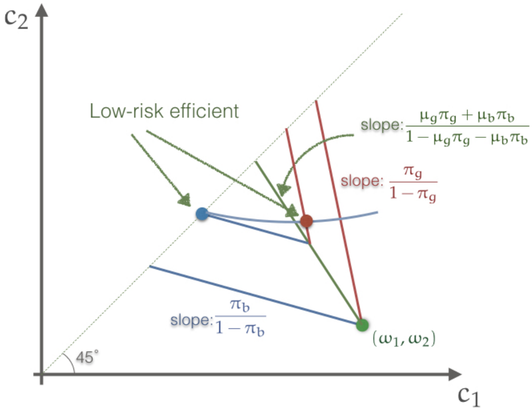

Low Risk Efficient Allocations

- The functions $V_j^{eff}(\lambda_g, \lambda_b)$:
  - increasing in $\frac{\lambda_g}{\lambda_g+\lambda_b}$ (constant below λ^*)
  - necessarily discontinuous at $(0,0)$
    - value at $(0,0)$ not defined
    - impossible to extend $V_j^{eff}\left(\lambda_g, \lambda_b\right)$ to $(0,0)$ in a continuous way

Our Extensive Form Game

Extensive Form Game

- Insurance companies move first:
  - Offer menus

$$i \in \{1, 2\} : \mathbf{c}^i(\boldsymbol{\lambda}^i) = \left(c^i_{1g}(\boldsymbol{\lambda}^i), c^i_{2g}(\boldsymbol{\lambda}^i), c^i_{1b}(\boldsymbol{\lambda}^i), c^i_{2b}(\boldsymbol{\lambda}^i)\right)$$

- Households choose between the two firms
  - $\sigma^i_j(\mathbf{c}^1, \mathbf{c}^2)$: probability of choosing firm i by type j
- $\boldsymbol{\lambda}^i = \left(\lambda^i_g, \lambda^i_b\right)$ measures of households at firm i;$\boldsymbol{\lambda} = \left(\boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2\right)$

Rothschild-Stiglitz as Restricted Version of Our Game

- Restriction: menus are independent of λ
- $\mu_g \leq \lambda^*$: Unique pure strategy equilibrium – least cost separating; interim efficient
- $\mu_g > \lambda^*$: no pure strategy equilibrium exists
  - Dasgupta and Maskin (1986):
    - mixed strategy equilibrium exists
    - equilibrium is interim inefficient

Standard Notion of Equilibrium

Definition. A Symmetric Equilibrium is defined by a pair of menus $\mathbf{c}^i(\boldsymbol{\lambda}) : [0, \mu_b] \times [0, \mu_g] \to \mathbb{R}^4$, $i = 1, 2$ together with households' strategies $\sigma^i_j : \left(\mathbf{c}^1, \mathbf{c}^2\right) \to \Delta\left(\{1, 2\}^2\right)$ such that:

- Households maximize: given any $\mathbf{c} = \left(\hat{\mathbf{c}}^1(\cdot), \hat{\mathbf{c}}^2(\cdot)\right)$

$$\sigma^i_j(\mathbf{c})\left[U_j(\sigma^i_g(\mathbf{c}), \sigma^i_b(\mathbf{c})) - U_j(\sigma^{-i}_g(\mathbf{c}), \sigma^{-i}_b(\mathbf{c}))\right] \geq 0$$

- Firms maximize

$$\mathbf{c}^i \in \arg\max_{\mathbf{c^i}} \Pi^i(\mathbf{c}(\sigma^i(\mathbf{c}^i, \mathbf{c}^{-i}))).$$

- Assumption: $\mathbf{c}^i(\boldsymbol{\lambda})$ is continuous everywhere but at $\boldsymbol{\lambda} = (0, 0)$

Main Theorems

Theorem 1. The game has a symmetric equilibrium whose outcome coincides with the low-risk efficient allocation.

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- Under appropriate restrictions/refinments

Theorem 2. The outcome of ***any*** symmetric restricted equilibrium is pareto optimal. Conversely, any pareto optimal allocation can be implemented as the outcomeof a symmetric restricted equilibrium.

Proof of Theorem in Steps

- Propose equilibrium strategies
- Show equilibrium in restricted strategy space
- Remove restrictions on strategies
  - subgame might not have an equilibrium for arbitrary pair of menus offered.

Equilibrium Strategies

- 1st step: construct “Mirror” Strategies
  - Construct strategy from the low-risk efficient allocation

$$V_j^*(\boldsymbol{\lambda}) = \max\left\{V_j^{\mathit{eff}}(\boldsymbol{\lambda}), V_j^{\mathit{eff}}(\boldsymbol{\lambda}^c)\right\}$$

    where

$$\boldsymbol{\lambda}^c = (\mu_b - \lambda_b, \mu_g - \lambda_g)$$

  - Associated menus are given by $\mathbf{c}^*(\boldsymbol{\lambda})$
- Note that both types rank low-risk efficient allocations the same way so this is well-defined

Proof in Restricted Strategy Set

- 2nd step: "Mirror" Strategies equilibrium in restricted strategy set

$$S = \{\mathbf{c}(\boldsymbol{\lambda}) : \text{ The subgame with } (\mathbf{c}(\boldsymbol{\lambda}), \mathbf{c}^*(\boldsymbol{\lambda})) \text{ has an equilibrium}\}$$

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Proposition 1. Consider the restricted game in which each firm offers menus in S. Then the low-risk efficient allocation is an equilibrium outcome of the game.

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- Why restriction: subgames are discontinuous non-atomic games:
  - Equilibrium does not necessarily exist!

Proof in Restricted Strategy Set

- Idea of proof:
  - Suppose that firm 2—incumbent—offers the mirror strategy $\mathbf{c}^*(\boldsymbol{\lambda})$
  - Firm 1—deviant—offers $\hat{\mathbf{c}}(\lambda) \in S$

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  - If deviant attracts type j:

$$U_j(\hat{\mathbf{c}}(\boldsymbol{\lambda}^1)) \geq V_j^*(\boldsymbol{\lambda}^{1c}) = \underbrace{\max\left\{V_j^{eff}(\boldsymbol{\lambda}^{1c}), V_j^{eff}(\boldsymbol{\lambda}^1)\right\}}_{\text{Mirror Strategy}} \geq V_j^{eff}(\boldsymbol{\lambda}^1)$$

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  - implies firm 1 cannot make positive profits

Removing Restriction on Strategies

- Every subgame is a discontinuous non-atomic game between a continuum of households
- Potentially does not have an equilibrium
- Our approach: discretize the game (finitely many households) and take limits (send number of households to infinity)
- Use Nash/Dasgupta-Maskin's existence result and convergence of binomial distributions
- We can show that Theorem 1 goes through under limit equilibria Discretization

Possible Problems with Mirror Strategies

- Main idea behind existence of equilibria with cross-subsidization:
  - Block deviations by committing to lose against cream-skimming
- Potentially too costly: why should firm commit to lose money in case somone tries to poach thei good customers?
- Similar logic can be used to show there are other equilibria
  - Similar to the literature on supply function equilibria: Klemperer and Meyer (1989)
- In what follows: restriction on strategies; use as refinement

Equilibrium Refinement

A restricted equilibrium is an equilibrium that satisfy the following properties:

R1. Off path non-negative profits:

$$\sum_{j=g,b} \lambda_j \Pi_j \left(\mathbf{c}_j\left(\boldsymbol{\lambda}\right)\right) \geq 0, \forall \boldsymbol{\lambda} \in \left[0, \mu_b\right] \times \left[0, \mu_g\right]$$

R2. Non-negative profits on each type at $(0,0)$:

$$\Pi_j \left(\mathbf{c}_j\left(0,0\right)\right) \geq 0, j = b, g$$

R3. For any pair of menus $\left(\mathbf{c}^1, \mathbf{c}^2\right)$, equilibria in the subgame should be pareto efficient.

R4. Equilibrium menus must be H.O.D. 0, i.e., $\mathbf{c}^i\left(\boldsymbol{\lambda}\right) = \mathbf{c}^i\left(\alpha\boldsymbol{\lambda}\right)$.

Second Theorem

Theorem 2. The outcome of ***any*** symmetric restricted equilibrium is pareto optimal. Conversely, any pareto optimal allocation can be implemented as the outcomeof a symmetric restricted equilibrium.

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- Idea of Proof:
  - For any pareto optimal allocation:
    - offer a menu that implements the allocation at population measure
    - Upon a deviation all household choose the incumbent

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    - Construct it so that all the other equilibria (in the subgame) are pareto dominated than everyone choosing deviant
    - by R3 the only equilibrium upon deviation would be everyone choosing the deviant
    - In the paper, we show such a construction is always possible

Conclusion

- A game theoretic construction of efficient market arrangements with adverse selection and screening

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- Ali and Ariel's conclusion:
  - Mutual contracts can achieve efficiency in markets with adverse selection
    - Perhaps policies which support mutualization more important than mandates
- Chari's conclusion:
  - Beware of theorists who say adverse selection leads to inefficiency!

Additional Slides

Discretization: A Clarifying Example

- Suppose two firms setting prices faces a continuum of consumers
- Suppose firms post $v^i(\alpha)$: the value of customer choosing firm i when fraction α also choose i

$$v^1(\alpha) = \begin{cases} 0 & \alpha \neq 0 \\ 2 & \alpha = 0 \end{cases}$$

$$v^2(\alpha) = 1$$

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- No symmetric equilibrium exists

Discretization

- Consider instead approximation with N customers
- Firm payoffs given by

$$v^1(\alpha) = \begin{cases} 0 & \alpha \neq \frac{1}{N} \\ 2 & \alpha = \frac{1}{N} \end{cases}$$

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- If p_N is probability of choosing firm 1, then

$$2(1 - p_N)^{N-1} = 1 \Rightarrow p_N = 1 - \left(\frac{1}{2}\right)^{\frac{1}{N-1}}$$

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- As $N \to \infty$, $p_N \to 0$

Discretization

- Discretization yields sensible equilibrium: everyone chooses firm 2
- Our equilibrium concept: discretize the game and take limits as number of households goes to infinity
- Next: apply discretization to our game

Discretized Subgame

- For any pair of contracts $\left(\mathbf{c}^1, \mathbf{c}^2\right)$, let $G(N_g, N_b)$ be the discretized subgame:
- N_j is number of households of type j
    - Payoffs:

$$U_j\left(\mathbf{c}^i\left(\mu_g \frac{n_g^i}{N_g}, \mu_b \frac{n_b^i}{N_b}\right)\right)$$

where n_j^i is number of households of type j at firm i

Discretized Subgame Equilibrium

- Symmetric mixed strategy $\mathbf{p} = \left\{ p_j^i \right\}_{j,i}$
- Payoffs using binomial expansion

$$U_j^i(\mathbf{p}) = \sum_{k_j=0}^{N_j-1} \sum_{k_{-j}=0}^{N_{-j}} \left(\begin{array}{c} N_j - 1 \\ k_j \end{array} \right) (p_j^i)^{k_j} (1 - p_j^i)^{N_j-1-k_j}$$
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Lemma (Nash (1950)). A symmetric Nash equilibrium exists in the discretized subgame.

Discretized Subgame Equilibrium

- Symmetric mixed strategy $\mathbf{p} = \left\{p_j^i\right\}_{j,i}$
- Payoffs using binomial expansion

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$$\times \begin{pmatrix} N_{-j} \\ k_{-j} \end{pmatrix} (p_{-j}^i)^{k_{-j}} (1 - p_{-j}^i)^{N_{-j} - k_{-j}} V_j^i \left(\mu_g \frac{k_g}{N_g}, \mu_b \frac{k_b}{N_b} \right)$$

- Nash Equilibrium: $p_j^i \left[U_j^i(\mathbf{p}) - U_j^{-i}(\mathbf{p}) \right] \geq 0, \forall j, i$

Lemma (Nash (1950)). A symmetric Nash equilibrium exists in the discretized subgame.

Subgame Limit Equilibrium

Definition (Limit Equilibrium). Given a subgame assiociated with menus $\mathbf{c} = \left(\mathbf{c}^1(\cdot), \mathbf{c}^2(\cdot)\right)$, an allocation $\left\{\boldsymbol{\lambda}^i\right\}_{i=1,2}$ in the subgame is a limit equilibrium if a sequence of discretized games $G^m = G\left(N_g^m, N_b^m\right)$ exists and their mixed strategy equilibria $\mathbf{p}^m$ satisfy

$$\lim_{m\to\infty} \frac{N_g^m}{N_b^m} = \frac{\mu_g}{\mu_b}$$

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Lemma. A limit equilibrium always exist.

Theorem 1– Restated

Theorem. If in any subgme the profits for the firms are given by a limit equilibrium, then the low-risk efficient allocation is an equilibrium outcome of the game.

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  - $X_j^{1,m}$ is binomially distributed

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  - Have already shown firm 1 cannot make positive profits in this case

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  - So firm 1 cannot make positive profits

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