Optimal Rating Design*

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Abstract

We study the design of optimal rating systems in the presence of moral hazard. First, an intermediary commits to a rating scheme. Then, a decision-maker chooses an action that generates value for buyers. The intermediary then observes a noisy signal of the decision-maker's choice and sends a signal to the buyer consistent with the rating scheme. . We provide a full characterization of the set of allocations that can arise in equilibrium under any arbitrary rating system. We use this characterization to study various design aspects of optimal rating systems. Specifically, we study the properties of optimal ratings when the DM's effort is productive and when they can manipulate the intermediary's signal with a noise. With manipulation, rating uncertainty is a fairly robust feature of optimal rating systems.

1 Introduction

Information disclosure policies are at the heart of designing markets with asymmetric information. For example, in the ever growing digital markets such as Airbnb, Amazon Marketplace and Uber, where transactions are not repeated, buyers rely on the market designer to convey information about the sellers to them. An important role of disclosure policies, henceforth ratings, is to address moral hazard. By providing better information, market designers can incentivize providers to offer better service. While better information alleviates moral hazard, it also affects the terms of trade among market participants. In this paper, we study the trade-offs involved in the design of rating systems under moral hazard. What properties do such ratings systems have?

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How informative are they? What are the pitfalls of using deterministic versus stochastic rating systems?

We study a setting in which an intermediary observes some information about the actions chosen by a decision maker (DM) and decides how to convey this information to a third party, i.e., market or a buyer – henceforth market. We assume that the actions taken by the DM are costly. Additionally, these actions are valued by the market. Finally, we assume that the market is willing to pay up to its expected value for the item or service based on the signal it received from the intermediary and its prior.

To answer the above questions in this setting, first one must describe the set of achievable outcomes. Perhaps surprisingly, relatively little is known about this question. The key difficulty is that DM's choice of action is endogenous and is affected by the disclosure policy. Our first result provides a parsimonious formulation of the set of achievable outcomes.

Our characterization of the set of achievable outcomes relies on the concept of interim prices. Interim prices characterize the DM's expectation of market's expectation of their valuation. In presence of random signals, the DM is uncertain about market's beliefs. We define interim prices as DM's expected price given the state and buyers' beleif. In other words, they are the second order expectation of the state. In general, these interim prices cannot be fully characterized in a simple fashion. However we have a sharp result when the interim prices and market valuation are comonoton, move in the same direction. We show that in these situations existence of a signal structure is equivalent to *second order stochastic dominance*. This characterization allows us to cast the information design problem as a mechanism design problem with transfers, i.e., interim prices, where transfers have to satisfy a certain feasibility constraint, i.e., second-order stochastic dominance relative to market valuation.

We use this mathematical characterization to study various applications. First, we study the problem of Pareto optimal rating design in standard moral hazard where the DM chooses a costly effort that changes the distribution of market payoffs. We find that under some general conditions, any Pareto-optimal rating system is deterministic and exhibits monotone partition. Additionally, we can identify the main forces that determine optimal ratings. There are two forces that shape the properties of the optimal rating system: redistributive forces and incentives forces. Pooling of realizations, allows for redistribution across different types of DMs with different costs which might be desirable. In contrast, since effort is productive, provision of incentives requires revelation of information. The relative tractability of our characterization allows us to characterize conditions under which optimal rating is pooling for middle values of realizations while it is fully revealing for extreme values where the incentive effects are the strongest.

Our second application is rating design under manipulation where the DM can ex-post manipulate the observed output by the intermediary. Optimal ratings in this case have to balance ex-ante incentive for productive effort with ex-post incentive for manipulation. Our main finding is that rating uncertainty is a robust feature of optimal rating design. We show this by showing that the interim price function associated with an optimal rating is continuous. Since manipulation implies that the rating system cannot be fully revealing, it must involve uncertainty. In other words, the partition ratings are never optimal because they lead to discontinuities in the interim price function.

Interestingly, rating uncertainty must be present when cost of manipulation is high, i.e., when manipulation is arduous. This is because when manipulation is arduous it is possible to use rating uncertainty – and thus partially pool ex-post realizations – to induce ex-ante effort without having to allow for manipulation. When manipulation is effortless, i.e., its marginal cost is low, optimal rating always involves manipulation. However, the feasibility restriction on interim prices imply that rating uncertainty should be mixed with deterministic disclosure. Some mid-value reports should be revealed fully to the market while extreme-values should face rating uncertainty. In other words, optimality of some level of manipulation together with full revelation of some outcomes implies that all the parties involved – the intermediary, the DM and the market is aware of manipulation and yet the rating system fully reveals the manipulated outcome.

Finally, we show that our characterization result for interim prices can also be used in more general environments. Namely, when the market has a dogmatic prior about actions of the DM and when the market uses the information disclosed by the intermediary to take an action which in turn affect interim prices. In each of these case, we can use the simplicity of our main characterization result to shed light on some properties of optimal rating systems.

Beyond its technical contributions, our paper has important implication for regulation and design of rating systems in practice. We should note that an important interpretation of rating uncertainty is that of opaqueness of the rating system. There are various examples in which rating opaqueness is used. For example, in the context of consumer credit, credit scores are notoriously opaque in that while it is possible to determine rough statistics that increase credit scores, the exact cutoffs and formulas are unclear. In the context of eBay, Nosko and Tadelis (2015) conduct an experiment where they use a particular measure of seller quality and use it in the search result ranking without announcing it to the sellers. They show that this change of policy improved the quality choice of the sellers. The results in our paper, particularly those on optimal ratings in the presence of manipulation, provide a justification for this experiment.

1.1 Related Literature

Our paper is related to a few strands of literature in information economics and mechanism design. Most closely, it is related to the Bayesian persuasion literature, as in Kamenica and

Gentzkow (2011), Rayo and Segal (2010), Alonso and Câmara (2016), and Dworczak and Martini (2019), among many others. However, in our setup, the states are endogenous and determined by the choice of information structure. A notable exception is the paper by Boleslavsky and Kim (2020) where they consider a model with moral hazard where an agent controls the distribution of state with her effort. They show that Kamenica and Gentzkow (2011)'s concavification method extends to their environment. In our setup, we are able to provide a sharp characterization of the set of implementable outcomes. Furthermore, we are able to solve the resulting mechanism design problem under fairly general assumptions on the cost function and distribution of types. Kolotilin et al. (2017) study a problem of information transmission where one of the parties is privately informed. However, in their setup, the informed party possesses information about her payoff which is independent of the state. In contrast, in our model sellers are informed about the state (their cost type), and the information disclosure affects their choice of quality.[1]

From a technical perspective, our paper is also related to a subset of the Bayesian persuasion literature that studies problems in which receivers' actions depend on their posterior mean. For example, Gentzkow and Kamenica (2016), Kolotilin (2018), Dworczak and Martini (2019), and Roesler and Szentes (2017) use Blackwell (1953)'s result that the existence of an information structure is equivalent to the distribution of the posterior mean second-order stochastically dominating (SOSD) the prior. However, in our study finding this posterior mean is not enough, since sellers' incentives depend on the expected prices, which are themselves determined by the expectation of the posterior mean *conditional on the state*. Our contribution to this literature is to show that any profile of second-order expectations that dominates full-information valuations in the sense of Second Order Stochastic Dominance can be derived from some information structure. Moreover, we use the majorization ranking in order to shed light on key properties of all the information structures that induce a certain distribution of second-order expectations.

In our formulation, we use the majorization ranking for the functions representing interim prices and action profiles by the DM. Thus our mechanism design problem is equivalent to a mechanism design problem with transfers in which the transfer function majorizes the market valuations function. Similar to this problem, Kleiner et al. (2020) solve a class of problems where majorization appears as a constraint. Their solution method uses the characterization of extreme points of the set of functions that majorizes a certain function. In contrast, our solution of the mechanism design problem involves calculus of variations due to the lack of linearity that is present in their model.[2]

[1]Few other papers have also focused on the joint problem of mechanism and information design; Guo and Shmaya (2019) and Doval and Skreta (2019) are notable examples.

[2]Gershkov et al. (2020) study optimal auction design with risk-averse bidders who have dual risk aversion a la Yaari (1987). In their problem, the feasibility of allocations implies a majorization constraint on quantities, i.e., probability of allocation of the object to each bidder. Similar to our paper, they use calculus of variations to solve

Our paper is also related to the extensive literature on contracting and mechanism design. Where as often the main assumption is that monetary transfers are available to provide incentives, in our setup incentives for quality provision are provided using the rating system. In fact, this is often the case in multi-sided platforms: seller badges in eBay and Airbnb as well as rider and driver ratings in Uber and Lyft are a few examples. A few notable exceptions are models that study the problem of certification and its interactions with moral hazard: Albano and Lizzeri (2001), Zubrickas (2015), and Zapechelnyuk (2020).[3] An important contribution is that of Albano and Lizzeri (2001) where a key assumption is that the intermediary can charge an arbitrary fee schedule. The presence of an unrestricted fee schedule potentially reduces the importance of the certification mechanism. This is in contrast with our model where monetary transfers are not flexible. More recently, Zubrickas (2015), Zapechelnyuk (2020) and **RayOhunchic also study variants of this problem. Their focus is, however, on deterministic ratings. As we show, rating uncertainty is an important feature of optimal ratings.

Finally, a recent series of papers have studied information design where "senders" (our DM) are strategic vis-a-vis the information structure. Notably **FrankelKartik, **Ball, and **Pereze-RichetSkreta all study similar problems. Compared to this strand of the literature, our mathematical result on second order expectations allows us to study a large class of problems without any restrictions on information structures (in contrast with **Ball and **FrankelKartik). While our focus is on moral hazard as opposed to adverse selection, we believe that our techniques can also be used to study models with adverse selection.

The rest of the paper is organized as follows: we start with an example in section 2; in section 3 we set up the model; in sections 4 and 5, we describe two applications of the model; in section 6 we consider some extensions of our model; finally, section 7 concludes. All the proofs are relegated to the appendix unless otherwise indicated.

2 An Example

Before diving into the model, we explore a simple example which illustrates the difficulties that can be arisen from endogenous state. Sellers choose to exert effort $a \in [0, 1]$ which determines

this problem. In contrast, our mechanism design problem is equivalent to a problem in which transfers must be majorized by qualities. This together with incentive compatibility puts more restriction on the set of implementable allocations.

[3]Evidently, our paper is also related to the extensive and growing literature that studies the problem of certification and information disclosure (e.g., Lizzeri (1999), Ostrovsky and Schwarz (2010), Boleslavsky and Cotton (2015), Harbaugh and Rasmusen (2018), and Hopenhayn and Saeedi (2020)).

the distribution of quality, $y \in [0,1]$, given by

$$G(\hat{y}|a) = \Pr(y \leq \hat{y}|a) = \frac{a^{\hat{y}} - 1}{a - 1}.$$

The seller are of two types, $\theta \in \{1, 4\}$, which determines the cost of exerting effort, $c(a, \theta) = \frac{a}{\theta}$. Suppose that $\Pr(\theta = 1) = 1/4$ and that θ is private information to the seller and not observed by other market participants.

An intermediary, such as a platform or a certifier, commits to an information structure $(S, \pi(\cdot|y))$ where $\pi(\cdot|y) \in \Delta(S)$.[4] The intermediary charges a tariff, t, to the seller in exchange for this information. The buyer payoff is y but she only observes the signal realization sent by the intermediary. She then uses her prior and the signal from the intermediary to update her beliefs, and pays her posterior mean, $p(s) = \mathbb{E}_{\pi}[y|s]$ to the seller.[5] The payoff of the seller from choosing effort a is thus given by

$$\int_0^1 \int_S p(s)\, \pi(ds|y)\, dG(y|a) - t - c(a, \theta).$$

Suppose that the outside option of the seller has a payoff of 0. The intermediary wishes to maximize its own revenue from the tariff t. What is the optimal signal structure that achieves this goal? Having a full disclosure policy leads to optimal choice of effort for the seller. On the other side hiding some information leads to redistribution of profits among sellers which can increase the revenue of the intermediary.

First, let's consider full disclosure policy, i.e., $S = [0,1]$, $\pi(\{y\}|y) = 1$. In this case, $p(y) = y$. Hence, conditional on participation each type of seller choose a to maximize

$$\int y dG(y|a) - \frac{a}{\theta} = \frac{a}{a-1} - \frac{1}{\log a} - \frac{a}{\theta}.$$

This objective is concave and single-peaked in a and is maximized at $a(\theta = 1) = 0.057$, $a(\theta = 4) = 0.312$. Given these choices, the before tariff payoff of each type of seller are $u(\theta = 1) = 0.2316$ and $u(\theta = 4) = 0.327$, respectively for the high- and the low-cost types. This implies that the intermediary can charge $u(1)$ and therefore keep both types of sellers, orshe can charge $u(4)$ in which only the low-cost sellers participate. Comparing the two cases, optimal tariff and the intermediary's expected revenue are given by $t = 0.327$, and Revenue $= 0.245$. Thus, here, full disclosure policy leads to exclusion of high-cost sellers.

[4]Throughout the paper, we will use $\Delta(S)$ to denote the set of all probability measures over the set S.

[5]Throughout the paper, we assume that buyers outside option is zero and they are on the long side of the market, so they are willing to pay their expected value of the item to the seller.

Next, we want to examine if the intermediary can increase its revenue by hiding some information. There are numerous possibilities that intermediary can choose from, various partitions, pooling various intervals of realizations while reporting the rest, or having a random signals. In this example, we consider a particular partial pooling that we later show can be optimal in this class of problems. Let's consider a partial pooling equilibrium where the intermediary pools the realizations of y above 0.16 but fully discloses realizations of y below 0.16. In this case, $S = [0, 0.16] \cup \{H\}$ and $\pi(\{y\}|y) = 1, y \leq 0.16, \pi(\{H\}|y) = 1, y > 0.16$. In response to this information structure, both seller types will shade their efforts which in turn changes the beleif of the market and thus value of $p(H)$, price paid following a realization of $s = H$. Specifically,

$$p(H) = \frac{1/4 \int_{0.16}^{1} y dG(y|a(1)) + 3/4 \int_{0.16}^{1} y dG(y|a(4))}{1 - 1/4 G(0.16|a(1)) - 3/4 G(0.16|a(4))}$$

In equilibrium, market's belief about effort by different seller types should be consistent with their choices. The unique pure strategy equilibrium of this game is given by $a(1) \approx 0.049$ and $a(4) \approx 0.2197$. In this case the optimal tariff includes both high and low types in the market and is given by $t = 0.254$ which is higher than the revenue under full disclosure policy.

Hiding information has two effects here. First, it reduces the incentives to exert effort by both types of sellers. Second, since the two types of sellers generate different distributions of qualities, by pooling some values, it redistributes profits from the low-cost seller to the high-cost one. If this redistribution is big enough, it leads to the intermediary charging a lower tariff to encourage participation by both types of sellers and higher revenue for the intermediary.

While the "upper-censorship" information structure considered here increases the revenue of the intermediary, it turns out not to be optimal. Indeed, as we show in section 4, the optimal information structure for this example is of "lower-censorship" form, i.e., one that pools an interval of qualities for low values of y and fully reveals the values of qualities outside of this interval. The threshold for censorship is given by $\hat{y} \approx 0.72$.

Several questions arise from this exercise. First, given that we have shown that it is optimal for the intermediary to hide some information, what is the optimal information structure? Second, would randomized signals ever be optimal? In what follows, we develop techniques to solve for a general solution of this problem. Specifically, in section 4, we show that in this example, lower-censorship is optimal.

3 General Model and Interim Prices

In this section, we describe our general model of rating design and provide a sharp characterization result for the set of feasible payoffs. In general, we are interested in settings in which an intermediary observes some information about the actions chosen by a decision maker (DM) and decides how to convey this information to a third party, i.e., market or a buyer, who then pays their posterior mean as a price to the DM.

More specifically, consider a DM who chooses an action $a \in A \subset \mathbb{R}^N$. This action creates a possibly random realization $y \in Y \subset \mathbb{R}^M$ with $\sigma(\cdot|a) \in \Delta(Y)$ describing the distribution of y given the action chosen by the DM. The action and the outcome generates value of $v(a, y)$ for a market or a buyer who then is willing to pay her expected payoff $\mathbb{E}[v(a, y)|s]$ conditional on the information available to her as well as her beliefs about equilibrium play.[6] The information structure is chosen by an intermediary who observes $y \in Y$ and chooses a signal structure $(S, \pi(\cdot|y))$ where S is a set of signal realizations and $\pi(\cdot|y) \in \Delta(S)$ for all $y \in Y$.[7] Figure 1 depicts the structure of the model and actions.

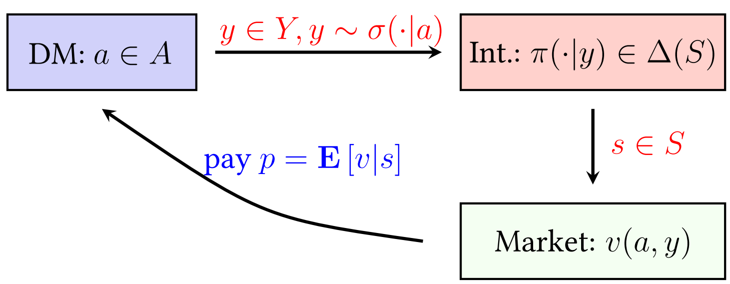

Figure 1: General Structure of the Model

The DM has a type $\theta \in \Theta$ with probability distribution given by $F \in \Delta(\Theta)$.[8] This type affects their cost of exerting effort, a. Hence, the payoff of the DM is given by

$$\int_Y \int_S \mathbb{E}[v|s]\, d\pi(s|y)\, d\sigma(y|a) - c(a, \theta) \tag{1}$$

[6]We maintain the assumption that the buyers are on the long-side of the market thus willing to pay their expected value. One can extend our analysis by allowing buyers to have positive outside options or positive bargaining power.

[7]More formally, an information structure is a family of probability spaces $\{(S, \mathscr{S}, \pi(\cdot|y))\}_{y \in Y}$ where S is the space of signal realizations and $\mathscr{S}$ is a σ-algebra. All throughout the paper, we work with S as a compact subset of some Euclidean space and $\mathscr{S}$ as the Borel σ-algebra associated with topology induced by the Euclidean norm and a compact space for S. Hence, we drop the σ-algebra in our analysis. Additionally, when describing subsets, we refer to Borel subsets.

[8]We will often assume that $\Theta \subset \mathbb{R}$ has a discrete distribution over a finite set of types or it has a continuous distribution with c.d.f. F. Using F as denoting the probability measure governing θ is a slight abuse of notation to avoid clutter.

where $c(\cdot,\cdot)$ is the cost of exerting effort. In a pure strategy equilibrium, DM chooses $a(\theta)$ to maximize (1).

In the above, the ex-post market price $\mathbb{E}[v|s]$ not only depends on the information structure, $\pi(\cdot|\cdot)$, but also it depends on the market's prior about the distribution of (a,y) which in turn depends on the strategy profile of the DM. In other words, it is an equilibrium object. More specifically, the market uses their prior about the distribution of θ – given by F – together with their beliefs about the equilibrium strategies of the DM types, $a(\theta)$, to form a prior $\mu \in \Delta(A \times Y)$ and uses Bayes' Rule to form the posterior expectation $\mathbb{E}[v|s]$ satisfying

$$\int_{A\times Y}\int_{S'} \mathbb{E}[v|s]\, d\pi(s|y)\, d\mu = \int_{A\times Y} v(a,y)\, \pi(S'|y)\, d\mu, \forall S' \subset S. \tag{2}$$

The above defines an equilibrium given the information structure. More specifically, given an information structure (S,π) an equilibrium is an action profile $a(\theta)$ by different types of the DM where: First, given market beliefs $\mu \in \Delta(A \times Y)$ and $\mathbb{E}[v|s]$, $a(\theta)$ maximizes (1); Second, given $a(\theta)$ market beliefs satisfy

$$\mu(A' \times Y') = \int_{A'}\int_{Y'} d\sigma(y|a(\theta))\, dF(\theta), \forall A' \subset A, Y' \subset Y$$

together with Bayesian updating as defined in (2).

3.1 Examples

To clarify the scope and applicability of our analysis, we describe a few examples of the above environment:

1. Reputation Mechanisms in Online Platforms: Consider the problem of an online platform such as Airbnb or eBay in designing its reputation system. It is a long-standing fact that online platforms suffer from adverse selection and moral hazard. The platform observes various performance parameters about a provider (a host on Airbnb and seller on eBay) that is not observed by the market.[9] These performance measures forms the basis of the platforms certification policy (such as Super Host in case of Airbnb or eBay Top rated Seller program in case of eBay) As noted by Hui et al. (2020), details of the certification policy affects the behavior of the providers. One can, thus, think about the certification policy of the platform as the information structure described in our model. Our model then addresses the issues and tradeoffs for the platform and

[9]As documented by Saeedi (2019), Hui et al. (2016), and Nosko and Tadelis (2015), there are many performances indicators available to eBay which are not conveyed to the market directly. Some examples include total quantities sold, previous claims and their outcomes, the standing of the seller with eBay, exact distribution of detailed sellers ratings, etc.

providers arising from the effect of certification policy on provider behavior.[10]

2. Ratings in asset markets: Similar to platforms, certification in financial markets is done based on data and forecasting models which are proprietary to the rating agencies. The major rating agencies in the United States, Moody's, Fitch, and S&P, operate under the so called "issuer-pay" model in which the issuer of a security pays for the rating and then the rating score for a security is freely available to the public. One can thus view our analysis as the effect of credit rating models on the behavior of issuers. Arguably, efforts such as investments that increase value for bond holders as opposed to equity holders or insiders are not fully observable by the credit rating agencies. However, their rating models affect the behavior of the issuers similar to our model. An important topic is the issue of regulation of the credit rating models, as discussed in Rivlin and Soroushian (2017). We show that a fully deterministic rating model, one that is set by a regulator and thus observable to the issuer, is desirable when manipulation of an indicator is not an issue, see section 4. In presence of manipulation, as we show in section 5, rating uncertainty is desirable and thus regulators should allow for some degree of uncertainty when designing the certification policies.

3. Manipulation of Ratings: Rating manipulations are very common.[11] This often occurs through misrepresentation of the data by the party being rated. In online platforms, data manipulation by providers has a constant presence.[12] For example, as discussed by He et al. (2022), third party sellers on Amazon.com, sometimes pay customers to leave positive reviews and inflate their ratings. One can view this in the context of our model. For example, the DM can have access to a costly action to increase the indicator y observed by the intermediary without affecting the market valuation. If this indicator is also related to another productive action – one that increases market valuation – this creates a trade-off for rating design. Information provision increases incentives for undertaking productive action while it increases incentives for data manipulation

[10] Given our assumption about inability of the intermediary to offer non-linear pricing schedules, this is most applicable to settings where pricing and transactions are not set and controlled by the platform.

[11] There has been several lawsuits involving manipulation of ratings in various industries. In the context of education, a notable example is the indictment of the Dean of Business School at Temple University who was convicted of U.S. News ranking manipulations using falsified data in 2022 and was sentenced to prison. See https://www.justice.gov/usao-edpa/pr/former-temple-business-school-dean-sentenced-over-one-year-prison-rankings-fraud-scheme, accessed August 16, 2022. Recently, Columbia University has been accused of manipulating its U.S. News ranking – see https://www.nytimes.com/2022/03/17/us/columbia-university-rank.html and http://www.math.columbia.edu/~thaddeus/ranking/investigation.html, accessed August 16, 2022. In the context of financial markets, the issue of "greenwashing" in ESG ratings (Environmental, Social and Governance) has gained significant interest by regulators, see https://www.bloomberg.com/news/articles/2022-05-31/deutsche-bank-s-dws-unit-raided-amid-allegations-of-greenwashing, accessed August 16, 2022.

[12] He et al. (2022) show that amazon sellers try to buy fake reviews to boost their ranking and overall rating on amazon marketplace. The issue of feedback manipulation has been a long debated issue in case of eBay, for example see Hui et al. (2017).

which makes the information less valuable. In section 5, we study this application.

3.2 Interim Prices: Definition and Characterization

In this section, we introduce a mathematical object, interim prices, that allows us to tractably analyze the problem of rating design in the environment described above. Our first major result is a novel characterization of these interim prices that allows us to solve the problem of rating design in various applications.

The notion of interim prices are simple. They are the mathematical object that indicates the DM's incentives. Specifically, we define *interim prices* as

$$p\left(y\right)=\int\mathbb{E}\left[v|s\right]d\pi\left(s|y\right) \tag{3}$$

Given that $\mathbb{E}\left[v|s\right]$ is an equilibrium object which depends on the beliefs of the market about the DM's action profile, so is $p\left(y\right)$. Nevertheless, it is a sufficient statistics for the information structure from the DM's perspective. Specifically, the payoff of the DM is given by

$$\int p\left(y\right)d\sigma\left(y|a\right)-c\left(a,\theta\right)$$

Interim prices are essentially the DM's beliefs about the beliefs of the market/buyer. More precisely, at the interim time of realization of y and before realization of the signal, the DM faces a distribution over the realization of signals – when random signals are used – and thus over the market's beliefs. One can thus interpret them as second order beliefs of the DM.

Example 1. Interim Prices To get a sense of interim prices and their relationship with an information structure, we consider two examples as depicted in Figure 2. Suppose that $A=Y=\left[0,1\right],v\left(a,y\right)=a,\sigma\left(Y'|a\right)=\mathbf{1}\left[a\in Y'\right]$. That is, market only values the DM's action and the intermediary observes it. Then an example of an information structure is one in which a deterministic signal is sent

$$\pi\left(s|a\right)=\begin{cases}1 & s=l,a<\underline{a}\\ 1 & s=a,a\in\left[\underline{a},\overline{a}\right]\\ 1 & s=h,a>\overline{a}\end{cases}$$

In words, the above information structure pools values of a below $\underline{a}$, fully separates values of

$a \in [\underline{a}, \overline{a}]$, and pools values of a above $\overline{a}$. Its associated interim prices are given by

$$p(a) = \begin{cases} \frac{\int_{a \le \underline{a}} a d\mu}{\mu((-\infty, \underline{a}))} & a < \underline{a} \\ a & a \in [\underline{a}, \overline{a}] \\ \frac{\int_{a > \overline{a}} a d\mu}{\mu((\overline{a}, \infty))} & a > \overline{a} \end{cases}$$

Another example of an information structure is a *partially mixing* one in which we have

$$\pi(s|a) = \begin{cases} \alpha(a) & s = a \\ 1 - \alpha(a) & s = N \end{cases}$$

In words, the above information structure reveals the state with probability $\alpha(a)$ and sends a generic signal with probability $1 - \alpha(s)$. Thus, the DM faces uncertainty regarding its ratings. In this case, interim prices are given by

$$p(a) = \alpha(a) a + (1 - \alpha(a)) \frac{\int [1 - \alpha(a)] a d\mu}{\int [1 - \alpha(a)] d\mu}$$

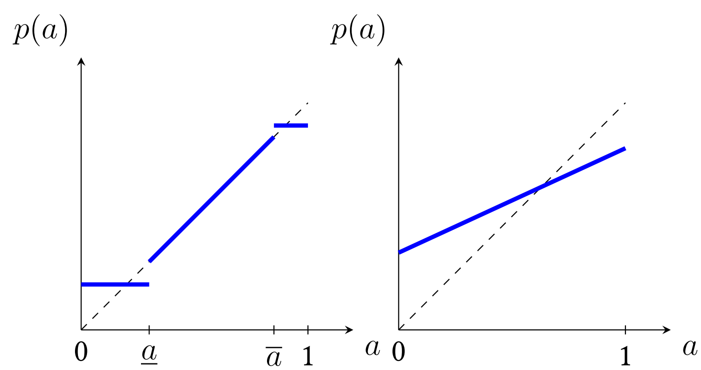

Figure 2: Examples of interim prices as in Example 1. Left panel is a deterministic signal structure; right panel is partial mixing signal structure with $\alpha(a)$ constant

Figure 2 depicts the interim prices associated with each signal structure and its relationship with market valuations.

Given our definition of interim prices, instead of viewing an equilibrium as an action profile $a(\theta)$ and the distribution of market prices, $\mathbb{E}[v|s]$, it induces, we can view it as an action profile $a(\theta)$ and an interim price function $p(y)$. Evidently, given $p(y)$, $a(\theta)$ must be incentive compatible, i.e.,

$$a(\theta) \in \arg\max_{a \in A} \int p(y) d\sigma(y|a) - c(a, \theta).$$

An important question that arises in characterization of interim prices is whether simple conditions exist to characterize the set of interim price profiles that result from a particular information structure and action profiles. As we discuss below through a series of examples, in general, the answer to this question is no. However, our main result is that under some restriction on information structures, a simple characterization exists.

First, let us examine interim prices. Suppose that the sets Y, A, and S are finite so we can easily write conditional expectations. We have that interim prices are given by

$$p(y) = \sum_{s\in S} \frac{\sum_{(a,\hat{y})\in A} v(a,\hat{y})\,\mu(a,\hat{y})\,\pi(s|\hat{y})}{\sum_{(a,\hat{y})\in A} \mu(a,\hat{y})\,\pi(s|\hat{y})}\pi(s|y).$$

Since $\sum_s \pi(s|y) = 1$, it can readily be seen that $p(y)$ is weighted average of values of $v(a,\hat{y})$ where the weights depend on y. In fact, we can go further and write the above as

$$\begin{aligned} p(y) &= \sum_{s\in S} \frac{\sum_{(a,\hat{y})\in A} v(a,\hat{y})\,\mu(a,\hat{y})\,\pi(s|\hat{y})}{\sum_{(a,y')\in A} \mu(a,y')\,\pi(s|y')}\pi(s|y) \\ &= \sum_{s\in S} \frac{\sum_{\hat{y}\in Y} \left(\sum_{a\in A} v(a,\hat{y})\,\mu(a,\hat{y})\right)\pi(s|\hat{y})}{\sum_{y'\in Y}\left(\sum_{a\in A}\mu(a,y')\right)\pi(s|y')}\pi(s|y) \\ &= \sum_{s\in S} \frac{\sum_{\hat{y}\in Y} \overline{v}(\hat{y})\,\mu_y(\hat{y})\,\pi(s|\hat{y})}{\sum_{y'\in Y}\mu_y(y')\,\pi(s|y')}\pi(s|y) \\ &= \sum_{\hat{y}\in Y} \overline{v}(\hat{y}) \sum_{s\in S} \frac{\pi(s|\hat{y})\,\pi(s|y)\,\mu_y(\hat{y})}{\sum_{y'\in Y}\mu_y(y')\,\pi(s|y')} \end{aligned} \tag{4}$$

where in the above $\overline{v}(\hat{y}) = \mathbb{E}[v(a,y)\,|\hat{y}]$ is the mean of $v(a,y)$ conditional on realization of y while $\mu_y(\hat{y}) = \sum_{a\in A}\mu(a,\hat{y})$ is the marginal distribution of μ along the y-direction. We make the following assumption about $\overline{v}$:

Assumption 1. *The range of $\overline{v}(\cdot)$, i.e., $\overline{v}(Y)$ is a finite collection of closed subintervals of $\mathbb{R}$.*

Assumption 1 is a technical assumption that allows us to prove our main result on characterization of interim prices, i.e., Theorem 1. It holds for example if Y is a finite collection of disjoint connected sets and $\overline{v}(\cdot)$ is continuous.

The expression (4) states that $p(y)$ is a weighted average of $\overline{v}(y)$. Hence, it is natural to think that $p(y)$ is a less dispersed version of $\overline{v}(y)$, i.e., a mean-preserving contraction. Indeed, we have the following lemma:

Lemma 1. *For any information structure (S,π) and $p(y)$ defined by (3), $p(\cdot)$ second order stochas-*

tically dominates $\overline{v}(\cdot)$, i.e., for all concave and increasing function $u : \mathbb{R} \to \mathbb{R}$,

$$\sum_{y\in} \mu_y(y)\, u(\overline{v}(y)) \leq \sum_{y\in} \mu_y(y)\, u(p(y))$$
$$\sum_{y\in} \mu_y(y)\, \overline{v}(y) = \sum_{y\in} \mu_y(y)\, p(y)$$

While the above result is a necessary requirement for interim prices, in general its reverse is not true. To see this, consider the following example:

Example 2.

Suppose that $A = Y = \{0, 1, 3\}$, $v(a, y) = \overline{v}(a) = a$, $\sigma(Y'|a) = \mathbf{1}[a \in Y']$, and $\mu(\{a\}) = 1/3$. In words, the consumers only care about the action of the seller and y coincides with it. Figure 3 depicts the values of $p(0)$ and $p(1)$, note that the sum of the three interim prices are always equal to 4 given the Bayes' Rule. Area A shows the set of vectors $\mathbf{x} = (x_1, x_2, x_3)$ that second order stochastically dominate $(0, 1, 3)$. Each random variable is represented by (x_1, x_2), the third element is the distance from the $x_1 + x_2 = 4$ line. The conditions are: $0 \leq x_i \leq 3, 1 \leq x_i + x_j \leq 4$, for all i, j and $x_1 + x_2 + x_3 = 4$. However, the set of interim prices does not coincide with the set A and is depicted by set B. To find the set of all interim prices, one can consider cases where the intermediary reveal one of the states fully while mixing the other two, in another two extremes the intermediary reveals all the information or non. Moreover, interim prices are not necessarily monotone. A signal that pulls $a = 0, 3$ and reveals $a = 1$ leads to an interim price of $3/2$ for $a = 0$ and 1 for $a = 1$ – depicted by point C in Figure 3.

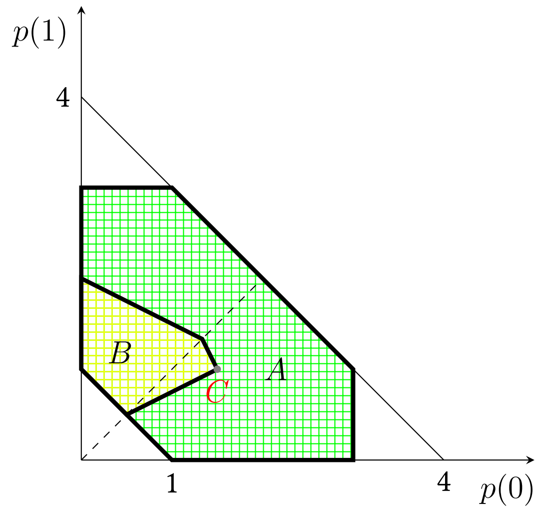

Figure 3: Depiction of the set of interim prices and mean-preserving contractions of market valuations for Example 2; The green area, A, represents the three state random variables that are mean-preserving contraction of a. The yellow area, B, is the set of interim prices arising from some information structure.

The above example illustrates the difficulties associated with identifying the set of all interim prices for all information structure. Nevertheless, we are able to show a somewhat general result when market valuations have the same ranking as interim prices, comonotonicity. Our main mathematical result is that when market valuations $\overline{v}\left(y\right)$ and $p\left(y\right)$ are comonotone then existence of a signal structure is equivalent to second order stochastic dominance. The following theorem states this result:

Theorem 1. *Consider an action profile $a\left(\theta\right)$ and its associated $\overline{v}\left(y\right)$ as defined in (4). Suppose that $p\left(y\right)$ is a function that maps Y into $\mathbb{R}$ such that:*

1. *$p\left(\cdot\right)$ is comonotone with $\overline{v}\left(\cdot\right)$, i.e., $p\left(y\right) > p\left(y'\right) \Rightarrow \overline{v}\left(y\right) > \overline{v}\left(y'\right)$,*
2. *$p\left(\cdot\right)$ second order stochastically dominates $\overline{v}\left(\cdot\right)$.*

Then, there exists an information structure $\left(S,\pi\right)$ such that $p\left(y\right) = \sum_{s\in S}\mathbb{E}\left[v|s\right]\pi\left(s|y\right)$.

The proof is relegated to the Appendix. Here are the intuitive steps. We prove this theorem in two steps: First, we show it when Y is finite. Second, we use an approximation argument for an arbitrary compact Y. For finite Y, our proof is based on the separating hyperplane theorem. In particular, we consider the set

$$\mathcal{S} = \left\{\hat{p}\left(\cdot\right) \middle| \exists\left(S,\pi\right) : \hat{p}\left(y\right) = \sum_{s\in S}\pi\left(s|y\right)\mathbb{E}\left[\overline{v}|s\right]\right\}.$$

In Example 2, the set $\mathcal{S}$ is the part of set B in Figure 3 above the 45-degree line. If $n = |Y|$, then $\mathcal{S} \subset \mathbb{R}^n$. It is fairly straightforward to show that $\mathcal{S}$ is a convex and compact subset of $\mathbb{R}^n$. Thus, by the separating hyperplane theorem, for a $p(y)$ not to belong to $\mathcal{S}$, there must exist $\lambda \in \mathbb{R}^n$, with $\lambda \neq 0$ such that $\sum_y \lambda(y) p(y) > \sum_y \lambda(y) \hat{p}(y), \forall \hat{p} \in \mathcal{S}$. Hence, if we show that for any $\lambda \neq 0$, there exists $\hat{p}_\lambda \in S$ such that $\sum_y \lambda(y) p(y) \leq \sum_y \lambda(y) \hat{p}_\lambda(y)$, then it must be that $p \in \mathcal{S}$. We show existence of $\hat{p}_\lambda$ by construction of signal structures which depend on the monotonicity properties of $\lambda(y)$. Specifically, we index members of Y in the increasing direction of $\overline{v}$, i.e., $\overline{v}(y_i) \geq \overline{v}(y_{i-1}), n \geq i \geq 2$. When $\lambda(y_i)/\mu(y_i) \geq \lambda(y_{i-1})/\mu(y_{i-1}), \forall i \geq 2$, then $\hat{p}_\lambda(y) = \overline{v}(y)$, i.e., full disclosure. When $\lambda(y_i)/\mu(y_i) < \lambda(y_{i-1})/\mu(y_{i-1})$, for some i, we pool states y_{i-1} and y_i and use an inductive argument to construct $\hat{p}_\lambda(y)$.

For arbitrary compact Y, we approximate the distribution of $\mu_y(\cdot)$ with a sequence of discrete distributions whose supports are ordered according to the subset order, i.e., they are a filtration. We can then apply the result from the finite case to construct an information structure associated with each of these discrete approximations. By compactness of the space of measures over the posterior mean and y, these information structures must have a convergent subsequence with a limiting information structure. It thus remains to be shown that the expectation of the posterior mean conditional on y under this limiting information structure coincides with $p(y)$. To show that, we resort to the martingale convergence theorem. This filtration and the realization of y and posterior mean form a bounded martingale. As a result, we can apply Doob's martingale convergence theorem to show that the limiting information structure generates interim price $p(y)$. We formalize this argument in the appendix.

The above Theorem implies that we can characterize "co-monotone" equilibria of the game for arbitrary information structure with an action profile $\{a(\theta)\}_{\theta \in \Theta}$ and interim prices $p(y)$ such that:

1. it is incentive compatible:

$$a(\theta) \in \arg\max \int p(y)\, d\sigma(y|a) - c(a,\theta), \forall \theta \in \Theta, \tag{5}$$

2. Interim prices $p(y)$ dominate $\overline{v}(y) = \mathbb{E}[v(a,y)|y]$ according to the second order stochastic order.

3. interim prices and market valuations are co-monotone.

We use this implication of Theorem 1 in the rest of the paper to characterize optimal rating systems in various settings.

Remark on Theorem 1 The result in Theorem 1 is reminiscent of the result of Blackwell (1953) and Rothschild and Stiglitz (1970) – the general version can be found in Strassen (1965). That result states that for any two random variables x and y, there exists a random variable s such that $\mathbb{E}[x|s]$ has the same distribution as y if and only if y second-order stochastically dominates x. While similar, this result is different in two ways. First, it is stated for the second order *conditional* expectation and thus Blackwell's result cannot be applied. The key intricacy is that the same signal structure that generates the random variable $\mathbb{E}[\overline{v}|s]$ must be used to generate $\mathbb{E}[\mathbb{E}[\overline{v}|s]|y]$. Second, as illustrated by Example 2, the equivalent of Blackwell's result does not hold in general and can only be shown when $\overline{v}$ and p are co-monotone.

Majorization In the rest of the paper, we will use Theorem 1 to characterize optimal rating systems in various applications. When $Y \subset \mathbb{R}$, the majorization formulation – see Hardy et al. (1934) – of second order stochastic dominance helps us use a Lagrangian method to solve for the optimal rating systems. When $Y = \mathbb{R}$, we can write

$$p \succcurlyeq_{SOSD} \overline{v} \iff \int_{-\infty}^{y} p(\hat{y})\, d\mu_y(\hat{y}) \geq \int_{-\infty}^{y} \overline{v}(\hat{y})\, d\mu_y(\hat{y}), \forall y \in \mathbb{R}.$$

4 Application 1: Rating Design with Moral Hazard

Our first application of rating design is a general version of the model in section 3. We will show that for a large class of objective functions, optimal ratings are deterministic. We then provide specific characterization of optimal ratings and characterize how they depend on the distribution of outcomes.

More specifically, suppose that $A = [0, \overline{a}]$ for some $\overline{a} > 0$, $Y = [0,1]$, $v(y,a) = y$ and $\Theta = \{\theta_1, \cdots, \theta_m\}$. We assume that the cost function $c(a,\theta)$ is decreasing in θ, increasing in a and submodular. This implies that higher θ's are more efficient in exerting effort. Additionally, we assume that y conditional on a is distributed according to a continuously differentiable c.d.f. $G(y|a)$. We make the following assumption:

Assumption 2. *The distribution of y conditional on a is decreasing with respect to a, i.e., $G_a(y|a) \leq 0$.*

This assumption implies that an increase in a shifts the distribution of y to the right, i.e., increases $G(\cdot|a)$ according to first order stochastic dominance.

We consider the class of pareto optimal objectives. That is, if payoff of the DM of type θ is given by $u(\theta)$, then the objective is given by $\int \lambda(\theta)\, u(\theta)\, dF$. For example, the monopoly problem of section 2 is a special case of this objective. There, the intermediary wishes to maximize his revenue from certification fees which translate into maximizing the utility of the lowest quality seller who is active in the market, if that type is $\underline{\theta}$ then the objective will be to maximize $u(\underline{\theta})$.

Under our characterization result, the problem of rating design is to choose $p(y)$ and $a(\theta)$ to maximize

$$\sum_{\theta \in \Theta} f(\theta) \lambda(\theta) \left[\int p(y) \, dG(y|a(\theta)) - c(a(\theta), \theta) \right]$$

subject to the incentive compatibility constraint (5), monotonicity of $p(y)$ and majorization constraint

$$\sum_{\theta \in \Theta} f(\theta) \int_0^y p(\hat{y}) \, dG(\hat{y}|a(\theta)) \geq \sum_{\theta \in \Theta} f(\theta) \int_0^y \hat{y} dG(\hat{y}|a(\theta)) \tag{6}$$

$$\sum_{\theta \in \Theta} f(\theta) \int_0^1 p(\hat{y}) \, dG(\hat{y}|a(\theta)) = \sum_{\theta \in \Theta} f(\theta) \int_0^1 \hat{y} dG(\hat{y}|a(\theta))$$

Our first result illustrates that Pareto optimal ratings are always deterministic:

Proposition 1. *Suppose that the first-order approach is valid, then Pareto-optimal rating systems are monotone partitions.*

It is worth mentioning that, first, we assume that the first-order approach is valid. This is because in our proof of this theorem, we use the existence of Lagrange multipliers on the incentive compatibility constraint. The first-order approach ensures that such Lagrange multipliers exist and our proof is valid. We suspect that the result holds even in the absence of the first-order approach yet our proof no longer works. Second, our proposition illustrates that not only optimal rating systems are deterministic but also monotone partitions. In other words, for any y, either y is revealed perfectly or there exists an interval around y where only it is revealed that y belongs to this interval.

The above proposition implies that moral hazard in the form described above does not lead to rating uncertainty. While this is informative about general properties of optimal ratings, the number of partitions and the precise design of optimal ratings depends on the distribution function $g(y|a)$. In what follows, we discuss two cases for the optimal disclosure policy of the monopolist intermediary.

Two-Type Case Suppose that $\Theta = \{\theta_1, \theta_2\}$ with $\theta_1 < \theta_2$, $f(\theta_j) = f_j$, and $\lambda(\theta_1) = 1, \lambda(\theta_2) = 0$. This is the problem of a monopolist that wishes to serve both types of sellers and thus wants to maximize the pre-tariff payoff of the high-cost seller.

Before stating our formal result, we provide a heuristic analysis of the main determinants of optimal rating. Suppose that the Lagrange multipliers on the incentive compatibility constraints of each type of seller are γ_j, $j = 1, 2$. Then, given effort level of a_1 and a_2, finding the optimal interim price is equivalent to

$$\max_{p(\cdot)} \int \Gamma(y)\, p(y)\, h(y)\, dy$$

subject to the majorization and monotonicity constraints. In this formulation $h(\cdot)$ is the unconditional density of y, i.e., $h(y) = f_1 g(y|a_1) + f_2 g(y|a_2)$. The function $\Gamma(y)$ is the *gain* function associated with optimal rating design and is given by:

$$\Gamma(y) = \frac{g(y|a_1)}{h(y)} \left(1 + \gamma_1 \frac{g_a(y|a_1)}{g(y|a_1)} + \gamma_2 \frac{g_a(y|a_2)}{g(y|a_2)} \frac{g(y|a_2)}{g(y|a_1)}\right). \tag{7}$$

Analyzing the terms in the gain function identifies two forces that shape the properties of the optimal rating system:

1. *Redistributive*: The first term in the gain function $g(y|a_1)/h(y)$ is a decreasing function of the likelihood ratio $g(y|a_2)/g(y|a_1)$. Under the monotone likelihood ratio property, this likelihood function is increasing in y and as a result the term $g(y|a_1)/h(y)$ is decreasing in y. Thus, when $\gamma_1 = \gamma_2 = 0$, i.e., when we do not have to worry about the effect of the rating system on incentives, then optimal rating system is simply one that provides no information. Intuitively, absent the incentive constraints, holding a_1 and a_2 fixed, the payoff of the high-cost seller is maximized when no information is provided.

2. *Incentive*: The second and third term in the gain function represents the importance of incentive provision for types 1 and 2. The function $g_a(y|a)/g(y|a)$ is an increasing function; negative for low values of y and positive for higher values of y. As a result, the second term creates a force for information revelation. In fact, when γ_1 and γ_2 are very large, the gain function $\Gamma(y)$ becomes increasing and as a result it is optimal to reveal all information.

At the optimum, the exact nature of the optimal rating system depends on how these forces interact. While the guiding principle for the design of rating systems is Proposition 2, in what follows, we provide conditions for which revelation must occur for high and low values. As we show later, various classes of distribution functions $g(y|a)$ satisfy this assumption:

Assumption 3. *The distribution function $g(y|a)$ satisfies:*

1. *Monotone likelihood ratio, i.e., $g_a(y|a)/g(y|a)$ is increasing in y for all a.*

2. *For arbitrary $a_2 > a_1$, define the function $\hat{y}(z)$ as the solution of $z = g(\hat{y}(z)|a_2)/g(\hat{y}(z)|a_1)$. The function $\hat{y}(z)$ must satisfy the following properties:*

   (a) The function $\phi(z) = g_a(\hat{y}(z)|a)/g(\hat{y}(z)|a)$ satisfies $\phi''(z) \le 0$,

(b) *The function* $\psi(z) = zg_a(\hat{y}(z)|a)/g(\hat{y}(z)|a)$ *satisfies* $\psi''(z) \geq 0$*,*

(c) *The function* $\phi''(z)/\psi''(z)$ *is increasing in* z*.*

Using the above assumption, we have the following proposition:

Proposition 2. *Suppose that Assumption 3 hold. If at the optimum* $a_2 \geq a_1$*, then there exists two thresholds* $y_1 < y_2$ *where optimal monopoly rating system is fully revealing for values of* y *below* y_1 *and above* y_2 *while it is pooling for values of* $y \in (y_1, y_2)$*.*

Under Assumption 3, the incentive effects are strongest for extreme values of y while the the redistributive force is strongest for mid values of y. As the Proposition illustrates optimal rating system pools intermediate values of y while fully reveals extreme values. Roughly speaking the full revelation of extreme values of y are associated with incentive provision for types 1 and 2. Under Assumption 3, the incentive effect for type 1 – the second term in the gain function (7) – is steepest for low values of y while the incentive effect for type 2 – the third term in the gain function (7) – is steepest for high values of y. Thus, mid-values of y are pooled since redistributive effect dominates, while at the extremes incentive effects dominate.

Some examples of distributions that satisfy Assumption 3 are:

1. C.d.f. is a power of y: $G(y|a) = y^{\alpha \cdot a}$ for $\alpha > 0$.

2. C.d.f. is a power of $1-y$: $G(y|a) = 1-(1-y)^{\frac{\alpha}{a}}$ for $\alpha > 0$.

3. C.d.f. is exponential of y : $G(y|a) = \left(e^{\psi(a)y} - 1\right)/\left(e^{\psi(a)} - 1\right)$ for some increasing function $\psi(a)$. A similar property holds for $G(y|a) = \frac{1-e^{-\psi(a)y}}{1-e^{-\psi(a)}}$.

Note that the example in section 2 is the last example above for $\psi(a) = \log a$. We can then simply use Proposition 2 and impose the "mid-pooling" information structure. It turns out that for that example, the information structure is lower-censorship, i.e., the lower bound of the mid-pooling region is $y_1 = 0$ and the upper bound is given by $y_2 \approx 0.72$.

5 Application 2: Rating Design under Manipulation

In this section, we consider another application of our result in section (3) to a setting where the DM can manipulate the statistic observed by the intermediary.

More specifically, consider a special case of the model in section (4) where there is only one type θ. Suppose that market valuation is $v(y, a) = y$ and that $Y = [0, 1]$. More importantly, suppose that the intermediary does not observe the true realization of y. It instead observes x which the DM can manipulate at a cost. In particular, the DM, after observing y can pay a cost and reveal x to the intermediary. The cost of manipulation is given by $c_m(x-y) = k\frac{(x-y)^2}{2} + \tau|x-y|$ where $k, \tau \geq 0$ and $\tau < 1$. We assume that the intermediary wishes to maximize the surplus generated by the DM which is given by $\int [y - c_m(\hat{x}(y) - y)]\, dG(y|a) - c(a)$ where $\hat{x}(y)$ is the manipulation strategy of the DM for each realization of $y \in [0,1]$.

We make the following assumption about the distribution:

Assumption 4. *The c.d.f of y, $G(\cdot|\cdot)$ satisfies the following properties:*

1. *$G(y|a)$ is a C^2 function of y and a.*

The above assumption helps us prove our main result about the shape of optimal ratings in Proposition 2.

In this setup, a rating system is a signal structure $(S, \pi(s|x))$, i.e., an information structure that maps manipulated values x into signals for the market/buyers. In equilibrium, there is common knowledge of strategies by the DM and thus the buyers' interpretation of the signals depends on manipulation strategy of the DM. This updating takes the form of

$$\mathbb{E}[y|s] = \mathbb{E}\left[\hat{x}^{-1}(x)\,|s\right]$$

where $\hat{x}^{-1}$ is the inverse correspondence of the DM's equilibrium manipulation strategy.[13] The DM's interim price from reporting x' to the intermediary is given by

$$\mathbb{E}\left[\mathbb{E}[y|s]\,|x'\right] = \mathbb{E}\left[\mathbb{E}\left[\hat{x}^{-1}(x)\,|s\right]|x'\right] = \hat{p}(x') \tag{8}$$

Since payoff function of the DM is supermodular between x and y, this implies that the equilibrium manipulation function $\hat{x}(y)$ is increasing in y. Moreover, equilibrium interim price $\hat{p}(\hat{x}(y))$ is also increasing. Therefore, the co-monotonicity assumption of Theorem 1 holds. Hence, existence of a signal structure that satisfies (8) is equivalent to $p(y) = \hat{p}(\hat{x}(y))$ dominating y according to second order stochastic dominance. Thus, we have the following corollary:

Corollary 1. *Consider any manipulation strategy $\hat{x}(y)$ together with an information structure (π, S). Then $\hat{x}(\cdot)$ is an equilibrium strategy if and only if there exists an increasing interim price function, $p(y)$, such that:*

1. *The function $p(y)$ second order stochastically dominates y (given the distribution of y, $G(y|a)$),*

[13]When there are multiple y's that report x to the intermediary, set $\hat{Y} \subset [0,1]$, the conditional expectation $\mathbb{E}\left[\hat{x}^{-1}(x)\,|s\right]$ pools them together and treats them as one observation with its value given by conditional expectation of $\mathbb{E}\left[y|y \in \hat{Y}\right]$.

2. *The pair of functions* $p(\cdot), \hat{x}(\cdot)$ *satisfies incentive compatibility*

$$p(y) - c_m(\hat{x}(y) - y) \geq p(y') - c_m(\hat{x}(y') - y), \forall y, y' \in [0,1] \tag{9}$$

Corollary 1 implies that the problem of optimal rating design in this application is given by

$$\max_{p(\cdot),\hat{x}(y),a} \int [p(y) - c_m(\hat{x}(y) - y)] g(y|a) \, dy - c(a) \tag{P1}$$

subject to:

1. ex-post incentive compatibility of the manipulation strategy:

$$p(y) - c_m(\hat{x}(y) - y) \geq p(y') - c_m(\hat{x}(y') - y), \forall y, y' \in [0,1] \tag{10}$$

2. ex-ante incentive compatibility of the effort:

$$a \in \arg\max_{a'} \int [p(y) - c_m(\hat{x}(y) - y)] g(y|a') \, dy - c(a') \tag{11}$$

3. Feasibility of the interim prices for some information structure

$$\int_0^{\hat{y}} p(y) g(y|a) \, dy \geq \int_0^{\hat{y}} y g(y|a) \, dy \tag{12}$$

$$\int_0^{1} p(y) g(y|a) \, dy = \int_0^{1} y g(y|a) \, dy \tag{13}$$

It is perhaps worth commenting that in this environment, there is no reason for the intermediary to rule out manipulation in equilibrium. When the cost of manipulation is high, e.g., τ is close to 1, we will show that it is optimal for the interim prices to have a slope of τ. This will then imply that marginal cost of manipulation at $\hat{x}(y) = y$ is equal to the increase in the interim prices and thus it rules out manipulation in equilibrium. However, when τ is low, e.g., $\tau = 0$, then manipulation always occur in equilibrium.

We proceed by stating our main result of this section:

Theorem 2. *Suppose that Assumption 4 holds. If* $p(\cdot)$ *is an interim price function that achieves the maximum in (P1), then* $p(\cdot)$ *is continuous.*

The result in Theorem 2 is in sharp contrast with that in Proposition 1. In Proposition 1, we established that optimal information structures are monotone partitions. This means that the DM does not face any uncertainty with respect to the determination of its rating. In contrast, when

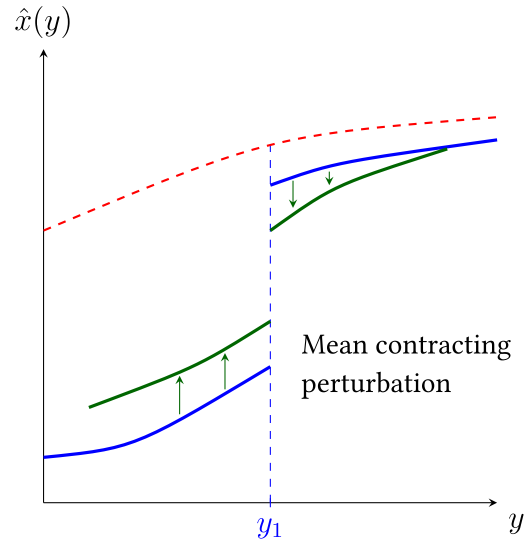

Figure 4: A Mean Contracting Perturbation of Interim Prices for Discontinuous Allocations

$p(\cdot)$ is continuous and when not all information is revealed, then the rating system must involve randomization or rating uncertainty.

The proof of Theorem 2 uses a perturbation argument as depicted in Figure 4. A useful notion is to consider the optimal manipulation when majorization is not taken into account, i.e., when there are no restrictions on prices. This is depicted by the dashed red line in Figure 4. If optimal manipulation – when majorization constraint (12) is imposed – has a discontinuity at y_1 (blue curve in Figure 23), then $\hat{x}(y_1+)$ is either higher than the unconstrained optimal manipulation or lower. In Figure 23, it is lower. The green line shows a perturbed manipulation strategy wherein manipulation is reduced above y_1 and increased below. It is possible to do this perturbation in such a way as to not violate the majorization constraint. Intuitively, this decreases dispersion of prices and thus preserves majorization. Since cost of manipulation is convex, the benefit of the increase for values of y below y_1 is higher than the cost of decrease above y_1. Hence, the manipulation function in Figure 23 cannot be optimal. If $\hat{x}(y_1+)$ is higher than the unconstrained manipulation, then a simpler perturbation of lowering manipulation above y_1 and increasing prices for all realizations of y is mean-contracting and thus improves upon the original allocation.

A key insight of Theorem 22 is that we can divide the domain $[0, 1]$ into a collection of subintervals – possibly points – where the optimal interim price function alternates between the identity function – for which majorization constraint is binding – and one in which majorization is slack and thus involves rating uncertainty. In other words, partitions often used by various disclosure mechanisms where various states are pooled are not optimal. Additionally, the theorem illustrates an algorithm in order to find optimal ratings via optimization over alternating intervals.

We can provide two further characterization of optimal rating systems under manipulation. To do so, we make the following assumption:

Assumption 5. *In addition to Assumption 4, the c.d.f of y, $G(\cdot|\cdot)$ satisfies the following properties:*

1. *First order stochastic dominance: $\frac{\partial G(y|a)}{\partial a} \leq 0$ for all $y \in [0,1], \forall a \in \mathbb{R}_+$.*
2. *The function $\frac{\partial G(y|a)/\partial a}{g(y|a)}$ is bounded below.*

The first assumption simply ranks the distribution of y as a function of a according to first order stochastic dominance. The second assumption is less standard. The second assumption effectively states that the density of y is non-zero and bounded below over $[0,1]$.

Our first main result is on optimal rating when manipulation is arduous, i.e., its marginal cost at 0 is high.

Proposition 3. *Suppose that Assumption 5 holds. Then, there exists $\overline{\tau} \in (0,1)$ such that for all $\tau \geq \overline{\tau}$, there is no manipulation under optimal rating, i.e., $\hat{x}(y) = y$, and the optimal rating system satisfies*

$$S = \{N\} \cup [0,1], \pi(\{s\}|y) = \begin{cases} \tau & s = y \\ 1-\tau & s = N \end{cases} \tag{14}$$

The above proposition states that when τ is high enough, then the intermediary should not allow any manipulation at the optimum. Note that $\tau = \partial c_m/\partial x\,(y+, y)$ is the marginal cost of manipulation at $\hat{x} = y$. Naturally, one way to implement no manipulation under ex-post incentive compatibility is to have a price function where the marginal benefit of manipulation, the slope of the price function, is equal to its marginal cost. This gives rise to a pricing function whose slope at y is equal to the marginal cost of manipulation τ. Similar to Example 1, one way to achieve this is using a partial mixing information structure as described in (14). Note that ex-ante incentive compatibility implies that we would want to reward productive effort as much as possible and hence the slope of price function should be τ; the steepest price function that implements no manipulation.

The following proposition provides another insight into optimal rating systems when manipulation is effortless, i.e., when $\tau = 0$:

Proposition 4. *Suppose that $\tau = 0$. Then there exists thresholds $0 < k_1 < k_2$ such that:*

1. *For all $k \geq k_2$, the majorization constraint is slack at the optimum and the optimal rating system involves mixing.*

The above result states that when manipulation is effortless rating uncertainty is a robust feature of optimal ratings.

The intuition behind this result can be understood via trade-offs in manipulation. In this setting, since marginal cost of manipulation at $x = y$ is zero, as long as interim prices are increasing, there will be manipulation. Hence, there is a trade-off between allowing for manipulation and providing ex-ante incentives for productive efforts.

To see this, note that the envelope condition associated with the ex-post incentive constraint states that

$$u'(y) = k(x(y) - y)$$

where $u(y) = p(y) - c_m(x(y), y)$ is the ex-post utility of the DM. As this shows, an increase in $x(y)$ leads to a steeper utility profile and thus increases the return to productive effort. Thus the optimal rating system balance the welfare cost of manipulation with its incentive benefits. When k is large, a small level of manipulation can make $u(y)$ steep and provide strong incentives for productive effort. For high enough values of k, the level of manipulation required is small and thus price does not need to be steep enough which then means that majorization is not going to be violated. However, as k decreases the required level of manipulation increases and prices become too steep and violate majorization. When this is the case, the majorization constraint binds for mid-values of y and full revelation must occur.

6 Extensions

In this section, we discuss two other extensions of our majorization result in order to illustrate the extent and depth of its applicability.

6.1 Different Priors

Consider a variation of the model in section 3, wherein the market has dogmatic prior beliefs about the distribution of (a, θ); this is in contrast with the market having rational expectations which coincide with the equilibrium behavior of DM. More specifically, let $\phi \in \Delta(A \times \Theta)$ be the prior of the market and suppose that the market uses this prior and the true signal distribution to do Bayesian updating. When A, Θ, and Y are finite and we can write $\phi(a, \theta)$ as the probability of (a, θ) under market prior, interim prices are given by

$$p(y) = \sum_{s \in S} \frac{\sum_{(a,\hat{y}) \in A} v(a, \hat{y}) \phi(a, \hat{y}) \pi(s|\hat{y})}{\sum_{(a,\hat{y}) \in A} \phi(a, \hat{y}) \pi(s|\hat{y})} \pi(s|y).$$

The above is identical to interim prices in section 3 except for the fact that ϕ is used instead of the true distribution of (a, y). One can then conclude that the following holds:

Lemma 2. *If market prior is given by ϕ and $\overline{v}_{\phi}(y) = \mathbb{E}_{\phi,\sigma}[v(a, y)|y]$, if $p(y)$ is co-monotone with $\overline{v}_{\phi}(y)$ and $p(y) \succcurlyeq_{S.O.S.D} \overline{v}_{\phi}(y)$, both distributed according to ϕ_y, then there exists an information structure that induces $p(y)$.*

To see the benefit of this result, consider a simple setting in which there is only one type of DM who has a cost $c(a)$ and market has a biased belief that the density of y is given by $h(y)$. Moreover, suppose that $\mathbb{E}_{\phi}[v(a, y)|y] = \overline{v}_{\phi}(y) = \alpha y + (1-\alpha)\overline{y}$. Then the problem of optimal rating design is to find $p(y)$ and a to solve the following:

$$\max_{p(\cdot),a} \int_0^1 p(y) g(y|a) dy - c(a)$$

subject to

$$\int_0^y p(y') h(y') dy' \geq \int_0^y p(y') h(y') dy'$$
$$\int_0^1 p(y) h(y) dy = \int_0^1 \overline{v}_{\phi}(y) h(y) dy$$

We can then use simple arguments from calculus of variations, similar to those in section 4, to show the following:

Proposition 5. *Optimal ratings are:*

1. *Upper-censorship if for all a, $h(y)/g(y|a)$ is hump-shaped, i.e., increasing-then-decreasing. Moreover, if a^* achieves the maximum of $\mathbb{E}[y|a] - c(a)$ and if $h(y)$ is strictly dominated by $g(y|a^*)$ according to first order stochastic dominance, then optimal rating is not full information.*

2. *Lower-censorship if for all a, $h(y)/g(y|a)$ is U-shaped, i.e., decreasing-then-increasing. Moreover, if a^* achieves the maximum of $\mathbb{E}[y|a] - c(a)$ and if $h(y)$ strictly dominates $g(y|a^*)$ according to first order stochastic dominance, then optimal rating is not full information.*

Proposition 5 again illustrates the power of our result on characterization of interim prices. It describe how the shape of the bias determines the structure of optimal ratings. More specifically, in the class of the distributions considered, more pessimism leads to upper-censorship while more optimism leads to lower-censorship.

6.2 Allowing for Market Action

In the above, one of our key assumptions is that information does not lead to an informed action by the market. This then implies that the interim prices, i.e., expectation of posterior mean conditional on the state, are linear in posterior mean. A question that arises is whether this can be extended to a situation in which information is valuable for the market and thus interim prices are not necessarily linear in posterior means.

Here, we show that it is possible to extend our result to settings where the information content of the intermediary's signal leads to a change of action. More specifically, suppose that market participants (or alternatively a buyer) have a payoff of $\alpha \cdot \left(y - \underline{y}\right)$ with $\alpha \in \{0, 1\}$ being chosen by the market. Suppose as before that the price paid to the DM is $\left(\mathbb{E}\left[y|s\right] - \underline{y}\right) \mathbf{1}\left(\mathbb{E}\left[y|s\right] \geq \underline{y}\right)$. One interpretation of this example is one in which $\underline{v}$ is the cost of shopping, such as search, cognitive, etc., and thus the market participants only make the purchase when their posterior mean value of the object is above $\underline{y}$.

Suppose for simplicity that Y and S are finite. For any arbitrary rating system (S, π), we can define the following objects:

$$q\left(y\right) = \Pr\left(\left\{s : \mathbb{E}\left[y|s\right] \geq \underline{y}\right\} |y\right)$$
$$\hat{p}\left(y\right) = \frac{1}{q\left(y\right)} \mathbb{E}\left[\left(\mathbb{E}\left[y|s\right] - \underline{y}\right) \mathbf{1}\left(\mathbb{E}\left[y|s\right] \geq \underline{y}\right) |y\right], \text{if } q\left(y\right) > 0$$

In words, $q\left(y\right)$ is the probability that market action is equal to 1 conditional on the state being equal to y. Furthermore, $p\left(y\right) = q\left(y\right) \hat{p}\left(y\right)$ is the interim price vector faced by the DM. Thus, $\hat{p}\left(y\right)$ is the interim price conditional on $\alpha = 1$.

Note that given this change of variable, we can write

$$\hat{p}\left(y\right) = \sum_{s \in \hat{S}} \hat{\pi}\left(s|y\right) \frac{\sum_{y' \in Y} \hat{\pi}\left(s|y'\right) \left(y' - \underline{y}\right) q\left(y'\right) \mu_y\left(y'\right)}{\sum_{y' \in Y} \hat{\pi}\left(s|y'\right) q\left(y'\right) \mu_y\left(y'\right)}$$

where $\hat{S} = \left\{\mathbb{E}\left[y|s\right] \geq \underline{y}\right\}$ and $\hat{\pi}\left(s|y\right) = \frac{\pi(s|y)}{q(y)}$. This implies that $\hat{\pi}\left(\cdot|y\right) \in \Delta\left(\hat{S}\right)$. Thus, $\hat{p}\left(y\right)$ becomes the interim price function associated with the signal structure $\left(S, \hat{\pi}\left(\cdot|y\right)\right)$ with the prior distribution of y given by $q\left(y\right) \mu_y\left(y\right)$.

The following proposition summarizes this logic:

Proposition 6. *Suppose that market payoff is given by $v\left(a, y\right) \cdot \alpha$ where $\alpha \in \{0, 1\}$ is the action optimally taken by the market and that marginal distribution of y is given by a probability measure μ_y. Then if a positive measure ρ and a pricing function $\hat{p}\left(y\right)$ exists such that*

1. *$\rho\left(\hat{Y}\right) \leq \mu_y\left(\hat{Y}\right)$ for all Borel subsets $\hat{Y} \subset Y$,*

2. *interim price function satisfies* $\hat{p}(y)$

$$\int \hat{p}(y)\, d\rho = \int \overline{v}(y)\, d\rho$$
$$\int u(\hat{p})\, d\rho \geq \int u(\overline{v}(y))\, d\rho, \forall u\text{: concave}$$

3. $\hat{p}(y)$ *is co-monotone with* $\overline{v}(y)$ *and* $\min_{y\in Y} \hat{p}(y) \geq 0$

Then, there exists (S, π) *such that*

$$\hat{p}(y) = \mathbb{E}\left[\mathbb{E}\left[\max_{\alpha\in\{0,1\}} v\cdot\alpha \middle| s\right]\middle| y\right]$$

It is worth mentioning that, first, the positive measure ρ can be thought of as the probability of state being y and posterior mean of $v(a, y)$ being positive. Since posterior mean of v can be sometimes negative – which leads to $\alpha = 0$, ρ is not necessarily a probability but a positive measure. Moreover, by the first condition, ρ is absolutely continuous with respect to μ_y and thus by Radon-Nykodim theorem there must exist $0 \leq q \leq 1$ such that $\rho = \int q(y)\, d\mu_y$. The function q must then be the probability of positive posterior mean conditional on $y \in \hat{Y}$. Second, the above is not an exact application of Theorem 1 since we are imposing an additional constraint that interim prices are positive. This is indeed because the market participants can guarantee a positive value for themselves which then guarantees a positive value for the DM.

7 Conclusion

We explored the design of optimal rating systems in the presence of moral hazard. We started by introducing the concept of interim prices and to show the conditions where we can substitute the information problem with majorization conditions. We show that when the value created for the consumers and the interim prices are comonotone our mechanism design and information design problem can be simplified by just considering majorization conditions for the interim prices and using them in the incentive conditions.

We then used our method to solve two applications. One for the optimal information mechanism in presence of moral hazard. The other application is exploring the optimal design in the presence of manipulation.

In this paper, we did not consider the case of matching where buyers/receivers are heterogeneous in their taste for quality and the presence of information can help with matching as well as giving incentives to sellers to exert effort. This will remain as a topic for future research.

References

ALBANO, G. L. AND A. LIZZERI (2001): "Strategic certification and provision of quality," *International economic review*, 42, 267–283. 5

ALIPRANTIS, C. D. AND K. BORDER (2013): *Infinite Dimensional Analysis: A Hitchhiker's Guide*, Springer-Verlag Berlin and Heidelberg GmbH & Company KG. 38

ALONSO, R. AND O. CÂMARA (2016): "Persuading voters," *American Economic Review*, 106, 3590–3605. 4

BLACKWELL, D. (1953): "Equivalent comparisons of experiments," *The annals of mathematical statistics*, 265–272. 4, 17

BOLESLAVSKY, R. AND C. COTTON (2015): "Grading standards and education quality," *American Economic Journal: Microeconomics*, 7, 248–79. 5

BOLESLAVSKY, R. AND K. KIM (2020): "Bayesian persuasion and moral hazard," *Working Paper, Emory University.* 4

DOOB, J. L. (1994): *Measure theory*, Springer Science & Business Media. 38

DOVAL, L. AND V. SKRETA (2019): "Mechanism design with limited commitment," *arXiv preprint arXiv:1811.03579.* 4

DWORCZAK, P. AND G. MARTINI (2019): "The simple economics of optimal persuasion," *Journal of Political Economy*, 127, 1993–2048. 4

GENTZKOW, M. AND E. KAMENICA (2016): "A Rothschild-Stiglitz Approach to Bayesian Persuasion," *American Economic Review*, 106. 4

GERSHKOV, A., B. MOLDOVANU, P. STRACK, AND M. ZHANG (2020): "Optimal Auctions for Dual Risk Averse Bidders: Myerson meets Yaari," *Available at SSRN.* 4

GUO, Y. AND E. SHMAYA (2019): "The interval structure of optimal disclosure," *Econometrica*, 87, 653–675. 4

HARBAUGH, R. AND E. RASMUSEN (2018): "Coarse grades: Informing the public by withholding information," *American Economic Journal: Microeconomics*, 10, 210–35. 5

HARDY, G., J. LITTLEWOOD, AND G. POLYA (1934): *Inequalities*, Cambridge Universtiy Press, Cambridge, UK. 17, 42

He, S., B. Hollenbeck, and D. Proserpio (2022): "The market for fake reviews," *Marketing Science.* 10

Hopenhayn, H. and M. Saeedi (2020): "Optimal Quality Ratings and Market Outcomes," *National Bureau of Economic Research Working Paper.* 5

Hui, X., M. Saeedi, Z. Shen, and N. Sundaresan (2016): "Reputation and regulations: evidence from eBay," *Management Science*, 62, 3604–3616. 9

Hui, X., M. Saeedi, G. Spagnolo, and S. Tadelis (2020): "Raising the Bar: Certification Thresholds and Market Outcomes,"," *Working Paper, Carnegie Mellon University.* 9

Hui, X., M. Saeedi, and N. Sundaresan (2017): "Adverse Selection or Moral Hazard: An Empirical Study," *Working paper.* 10

Kamenica, E. and M. Gentzkow (2011): "Bayesian persuasion," *American Economic Review*, 101, 2590–2615. 3, 4

Kleiner, A., B. Moldovanu, and P. Strack (2020): "Extreme Points and Majorization: Economic Applications," *mimeo.* 4

Kolotilin, A. (2018): "Optimal information disclosure: A linear programming approach," *Theoretical Economics*, 13, 607–635. 4

Kolotilin, A., T. Mylovanov, A. Zapechelnyuk, and M. Li (2017): "Persuasion of a privately informed receiver," *Econometrica*, 85, 1949–1964. 4

Lizzeri, A. (1999): "Information revelation and certification intermediaries," *The RAND Journal of Economics*, 214–231. 5

Luenberger, D. G. (1997): *Optimization by Vector Space Methods*, John Wiley & Sons. 39

Nosko, C. and S. Tadelis (2015): "The limits of reputation in platform markets: An empirical analysis and field experiment," Tech. rep., National Bureau of Economic Research. 3, 9

Ostrovsky, M. and M. Schwarz (2010): "Information disclosure and unraveling in matching markets," *American Economic Journal: Microeconomics*, 2, 34–63. 5

Rayo, L. and I. Segal (2010): "Optimal information disclosure," *Journal of political Economy*, 118, 949–987. 4

Rivlin, A. M. and J. B. Soroushian (2017): "Credit rating agency reform is incomplete," *Brookings Institution, https://www. brookings. edu/research/credit-rating-agency-reform-is-incomplete.* 10

Roesler, A.-K. and B. Szentes (2017): "Buyer-optimal learning and monopoly pricing," *American Economic Review*, 107, 2072–80. 4

Rothschild, M. and J. E. Stiglitz (1970): "Increasing risk: I. A definition," *Journal of Economic theory*, 2, 225–243. 17

Royden, H. L. and P. Fitzpatrick (1988): *Real Analysis*, vol. 32, Macmillan New York. 38

Saeedi, M. (2019): "Reputation and adverse selection: theory and evidence from eBay," *The RAND Journal of Economics*, 50, 822–853. 9

Strassen, V. (1965): "The existence of probability measures with given marginals," *The Annals of Mathematical Statistics*, 36, 423–439. 17

Yaari, M. E. (1987): "The dual theory of choice under risk," *Econometrica*, 95–115. 4

Zapechelnyuk, A. (2020): "Optimal quality certification," *American Economic Review: Insights*, 2, 161–76. 5

Zubrickas, R. (2015): "Optimal grading," *International Economic Review*, 56, 751–776. 5

A Proofs

A.1 Proof of Theorem 1

Proof. We will first prove the theorem when Y is finite. We will then show that the theorem can be extended to cases when Y is a compact Euclidean space.

Before doing so, first we define the following set of interim price functions

$$\mathcal{S} = \left\{ \hat{p}\left(\cdot\right) \middle| \exists \left(S, \pi\right) : \hat{p}\left(y\right) = \int_S \mathbb{E}\left[\overline{v}|s\right] d\pi\left(s|y\right) \right\}$$

The following lemma is a key property of $\mathcal{S}$ that will be used in our proof.

Lemma 3. *The set $\mathcal{S}$ is convex.*

Proof. Let $\hat{p}_1\left(\cdot\right), \hat{p}_2\left(\cdot\right) \in \mathcal{S}$ and their associated information structures be given by $\left(S_1, \pi_1\right)$ and $\left(S_2, \pi_2\right)$, respectively. Let μ_1 and μ_2 be the probability measures representing the distributions of $\left(s_1, y\right)$ and $\left(s_2, y\right)$. Let $\mu_{1,s}$ and $\mu_{2,s}$ be the marginal probability measures associated with μ_1 and μ_2 along s_1 and s_2, respectively. Note that μ_y is the marginal probability measure of both μ_1 and μ_2 along y.

Additionally, we have the standard conditional probability measures $\mu_1\left(\cdot|s_1\right), \mu_2\left(\cdot|s_2\right) \in \Delta\left(Y\right)$. We can then define probability measures $\tau_1, \tau_2 \in \Delta\left(\Delta\left(Y\right)\right)$ given by

$$\tau_1\left(\Phi\right) = \int_{S_1} \mathbf{1}\left[\mu_1\left(\cdot|s_1\right) \in \Phi\right] d\mu_{1,s}, \forall \Phi \subset \Delta\left(Y\right)$$
$$\tau_2\left(\Phi\right) = \int_{S_2} \mathbf{1}\left[\mu_2\left(\cdot|s_2\right) \in \Phi\right] d\mu_{2,s}, \forall \Phi \subset \Delta\left(Y\right)$$

Given this construction, we have

$$\begin{aligned}
\int_Y \hat{p}_1\left(y\right) \mathbf{1}\left[y \in \hat{Y}\right] d\mu_y &= \int_{S_1 \times Y} \mathbb{E}\left[\overline{v}|s_1\right] \mathbf{1}\left[y \in \hat{Y}\right] d\mu_1\left(s_1|y\right) d\mu_y \\
&= \int_{S_1 \times Y} \mathbb{E}\left[\overline{v}|s_1\right] \mathbf{1}\left[y \in \hat{Y}\right] d\mu_1\left(y|s_1\right) d\mu_{1,s} \\
&= \int_{S_1} \mathbb{E}\left[\overline{v}|s_1\right] \mu_1\left(\hat{Y}|s_1\right) d\mu_{1,s} \\
&= \int_{\Delta(Y)} \mathbb{E}_\phi\left[\overline{v}\right] \phi\left(\hat{Y}\right) d\tau_1.
\end{aligned}$$

In other words, $\hat{p}_1$ is the Radon-Nikodym derivative of the measure $\int_{\Delta(Y)} \mathbb{E}_\phi\left[\overline{v}\right] \phi\left(\cdot\right) d\tau_1$ with respect to μ_y. By Radon-Nikodym theorem, this is unique up to a measure zero change. The same property holds for $\hat{p}_2$.

Now consider the measure $\lambda\tau_1+(1-\lambda)\tau_2 \in \Delta(\Delta(Y))$. Let $\mu_\lambda \in \Delta(\Delta(Y)\times Y)$ be defined as

$$\mu_\lambda (B_1 \times B_2) = \int_{\Delta(Y)} \phi(B_2)\, d(\lambda\tau_1 + (1-\lambda)\tau_2), \forall B_1 \subset \Delta(Y), B_2 \subset Y$$

Similar to construction of the product measure, the above can be extended to a Borel measure over all Borel subsets of $\Delta(Y)\times Y$. We can then compute the conditional probability measure $\mu_\lambda(\cdot|y)$, $\forall y \in Y$. This is a signal structure wherein $S = \text{Supp}(\lambda\tau_1 + (1-\lambda)\tau_2)$.

Given the above, we have

$$\int_Y (\lambda\hat{p}_1(y) + (1-\lambda)\hat{p}_2(y))\, \mathbf{1}\left[y \in \hat{Y}\right] d\mu_y =$$
$$\int_{\Delta(Y)} \mathbb{E}_\phi[\overline{v}]\, \phi\left(\hat{Y}\right) d(\lambda\tau_1 + (1-\lambda)\tau_2)$$

This implies that $\lambda\hat{p}_1 + (1-\lambda)\hat{p}_2 \in \mathcal{S}$ which concludes the proof. □

□

1. When Y is finite.

Let $Y = \{y_1, y_2, \cdots, y_n\}$ such that $\overline{v}(y_1) \le \overline{v}(y_2) \le \cdots \le \overline{v}(y_n)$. When $p(y)$ is co-monotone with $\overline{v}(y)$, we must have that $p(y_1) \le p(y_2) \le \cdots \le p(y_n)$. In this case, $\mathcal{S} \subset \mathbb{R}^n$. For simplicity, we also let $f_i = \mu_y(\{y_i\})$.

We show that if p satisfies $p(y_1) \le \cdots \le p(y_n)$ and $p \succcurlyeq_{\text{SOSD}} \overline{v}$, then $p \in \mathcal{S}$. To do so, we show that for any $\lambda \in \mathbb{R}^n$, $\lambda \neq 0$, then there exists $p' \in \mathcal{S}$ such that $\lambda \cdot p \le \lambda \cdot p'$. This result combined with separating hyperplane theorem implies that $p \in \mathcal{S}$.

(i) We only need to focus on $\lambda > 0$. This is because for all $p' \in \mathcal{S}$,

$$\sum_{i=1}^n f_i p(y_i) = \sum_{i=1}^n f_i p'(y_i) = \sum_{i=1}^n f_i \overline{v}(y_i)$$

Thus,

$$(\lambda + \alpha f)\cdot p \le (\lambda + \alpha f)\cdot p' \iff \lambda \cdot p \le \lambda \cdot p', \forall \alpha \in \mathbb{R}$$

where in the above $f = (f_1, \cdots, f_n)$. Thus choosing a large enough α implies that $\lambda + \alpha f > 0$.

(ii) we prove existence of p' *by induction on* n.

Step 1. claim holds when $n = 2$. When $n = 2$, then stochastic dominance is equivalent to

$$f_1 p(y_1) + f_2 p(y_2) = f_1\overline{v}(y_1) + f_2\overline{v}(y_2)$$
$$\overline{v}(y_1) \le p(y_1) \le p(y_2) \le \overline{v}(y_2)$$

Now, there are two possibilities: 1. $\lambda_1/f_1 \geq \lambda_2/f_2$. Then Chebyshev's sum inequality implies that

$$\begin{aligned}\lambda \cdot p &= \sum_i f_i \frac{\lambda_i}{f_i} p(y_i) \\ &\leq \sum_i f_i \frac{\lambda_i}{f_i} \sum_i f_i p(y_i) \\ &= (\lambda_1 + \lambda_2) \sum_i f_i \overline{v}(y_i) = \lambda \cdot p'\end{aligned}$$

In the above $p'(y_i) = \sum_i f_i \overline{v}(y_i)$ is the interim price vector associated with no information.
2. $\lambda_1/f_1 < \lambda_2/f_2$. Then

$$\begin{aligned}\lambda \cdot p &= \sum_i f_i \frac{\lambda_i}{f_i} p(y_i) \\ &= \frac{\lambda_1}{f_1} \sum_i f_i p(y_i) + f_2 \left(\frac{\lambda_2}{f_2} - \frac{\lambda_1}{f_1}\right) p(y_2) \\ &\leq \frac{\lambda_1}{f_1} \sum_i f_i \overline{v}(y_i) + f_2 \left(\frac{\lambda_2}{f_2} - \frac{\lambda_1}{f_1}\right) \overline{v}(y_2) \\ &= f_1 \frac{\lambda_1}{f_1} \overline{v}(y_1) + f_2 \frac{\lambda_2}{f_2} \overline{v}(y_2) \\ &= \lambda_1 \overline{v}(y_1) + \lambda_2 \overline{v}(y_2) = \lambda \cdot p'\end{aligned}$$

where in the above $p'(y_i) = \overline{v}(y_i)$ is the interim price associated with full information.
Step 2. If the claim holds when $|Y| = n-1$, then it must hold when $|Y| = n$.
To show this step, consider $\lambda \in \mathbb{R}^n$. There are two possibilities:
Case 1. The vector λ satisfies $\lambda_1/f_1 \leq \lambda_2/f_2 \leq \cdots \leq \lambda_n/f_n$. In this case, we can write

$$\begin{aligned}\lambda \cdot p &= \sum_i \lambda_i p(y_i) \\ &= \sum_i f_i \frac{\lambda_i}{f_i} p(y_i) \\ &= \frac{\lambda_1}{f_1} \sum_i f_i p(y_i) + \sum_{i \geq 2} f_i \left(\frac{\lambda_i}{f_i} - \frac{\lambda_1}{f_1}\right) p(y_i) \\ &= \frac{\lambda_1}{f_1} \sum_i f_i p(y_i) + \sum_{i \geq 2} \sum_{j=2}^{i} \left(\frac{\lambda_j}{f_j} - \frac{\lambda_{j-1}}{f_{j-1}}\right) f_i p(y_i) \\ &= \frac{\lambda_1}{f_1} \sum_i f_i p(y_i) + \sum_{i \geq 2} \left(\frac{\lambda_i}{f_i} - \frac{\lambda_{i-1}}{f_{i-1}}\right) \sum_{j=i}^{n} f_j p(y_j)\end{aligned}$$

Given that $p \succcurlyeq_{\text{SOSD}} \overline{v}$ and that p is increasing in i, second order stochastic dominance implies that

$$\sum_{j=1}^{i} f_j p\left(y_j\right) \geq \sum_{j=1}^{i} f_j \overline{v}\left(y_j\right) \rightarrow \sum_{j=i}^{n} f_j p\left(y_j\right) \leq \sum_{j=1}^{i} f_j \overline{v}\left(y_j\right)$$

Thus, we can write

$$\begin{aligned}
\lambda \cdot p =& \frac{\lambda_1}{f_1} \sum_i f_i p\left(y_i\right) + \sum_{i \geq 2} \left(\frac{\lambda_i}{f_i} - \frac{\lambda_{i-1}}{f_{i-1}}\right) \sum_{j=i}^{n} f_j p\left(y_j\right) \\
\leq& \frac{\lambda_1}{f_1} \sum_i f_i \overline{v}\left(y_i\right) + \sum_{i \geq 2} \left(\frac{\lambda_i}{f_i} - \frac{\lambda_{i-1}}{f_{i-1}}\right) \sum_{j=i}^{n} f_j \overline{v}\left(y_j\right) \\
=& \sum_i \lambda_i \overline{v}\left(y_i\right) = \lambda \cdot p'
\end{aligned}$$

where in the above p' is the interim price vector associated with full information.
Case 2. There exists $k \geq 2$ such that $\lambda_k / f_k < \lambda_{k-1} / f_{k-1}$. In this case, consider the following

$$\begin{aligned}
\tilde{Y} &= \left\{y_1, \cdots, y_{k-2}, y_k, y_{k+1}, \cdots, y_n\right\} \\
\tilde{v}\left(y_i\right) &= \begin{cases} \overline{v}\left(y_i\right) & i \neq k, k-1 \\ \frac{f_k \overline{v}\left(y_k\right) + f_{k-1} \overline{v}\left(y_{k-1}\right)}{f_k + f_{k-1}} & i = k \end{cases} \\
\tilde{\mu}\left(y_i\right) &= \begin{cases} f_i & i \neq k, k-1 \\ f_k + f_{k-1} & i = k \end{cases}
\end{aligned}$$

$$\begin{aligned}
\tilde{p}\left(y_i\right) &= \begin{cases} p\left(y_i\right) & i \neq k, k-1 \\ \frac{f_k p\left(y_k\right) + f_{k-1} p\left(y_{k-1}\right)}{f_k + f_{k-1}} & i = k \end{cases} \\
\tilde{\lambda}_i &= \begin{cases} \lambda_i & i \neq k, k-1 \\ \lambda_k + \lambda_{k-1} & i = k \end{cases}
\end{aligned}$$

This modification of p and $\overline{v}$ satisfies all the properties of p and $\overline{v}$. That is, $\tilde{p}$ is comonotone with $\tilde{v}$ and satisfies stochastically dominates $\tilde{v}$. Since $\left|\tilde{Y}\right| = n - 1$, the hypothesis of induction implies that there must exists an information structure $\left(\tilde{S}, \tilde{\pi}\right)$ and its associated interim price vector $\check{p}$ that satisfies

$$\tilde{\lambda} \cdot \tilde{p} \leq \tilde{\lambda} \cdot \check{p} \tag{15}$$

Now, using $\left(\tilde{S}, \tilde{\pi}\right)$, we construct a new information structure that satisfies the claim for p. We define

$$S=\tilde{S},\pi\left(\{s\}\,|y_i\right)=\begin{cases}\tilde{\pi}\left(\{s\}\,|y_i\right) & i\neq k,k-1\\ \tilde{\pi}\left(\{s\}\,|y_k\right) & i=k,k-1\end{cases}$$

In words, this information structure pools y_k and y_{k-1} (they always send the same signals) and otherwise replicates $\tilde{\pi}$. Since this is the case, the posteriors induced by this information structure is the same as those induced by $\tilde{\pi}$. Therefore

$$\sum_s \pi\left(\{s\}\,|y_i\right)\mathbb{E}\left[\overline{v}\left(y_i\right)|s\right]=\check{p}_i, i\neq k,k-1$$
$$\sum_s \pi\left(\{s\}\,|y_i\right)\mathbb{E}\left[\overline{v}\left(y_i\right)|s\right]=\check{p}_k, i=k,k-1$$

Hence, the interim prices associated with (S,π) is given by

$$p'=\left(\check{p}\left(y_1\right),\cdots,\check{p}\left(y_{k-2}\right),\check{p}\left(y_k\right),\check{p}\left(y_k\right),\check{p}\left(y_{k+1}\right),\cdots,\check{p}\left(y_n\right)\right)$$

We have

$$\begin{aligned}\lambda\cdot p&=\sum_{j\neq k-1,k}\lambda_j p\left(y_j\right)+f_{k-1}\frac{\lambda_{k-1}}{f_{k-1}}p\left(y_{k-1}\right)+f_k\frac{\lambda_k}{f_k}p\left(y_k\right)\\&\leq\sum_{j\neq k-1,k}\lambda_j p\left(y_j\right)+\left(\lambda_{k-1}+\lambda_k\right)\left(\frac{f_{k-1}p\left(y_{k-1}\right)+f_k p\left(y_k\right)}{f_{k-1}+f_k}\right)\\&=\tilde{\lambda}\cdot\tilde{p}\end{aligned}$$

where the inequality holds because $\lambda_{k-1}/f_{k-1}>\lambda_k/f_k$ and $p\left(y_{k-1}\right)\leq p\left(y_k\right)$. Therefore, (15) implies

$$\begin{aligned}\lambda\cdot p\leq\tilde{\lambda}\cdot\tilde{p}\leq&\tilde{\lambda}\cdot\check{p}\\&=\sum_{j\neq k,k-1}\lambda_j\check{p}\left(y_j\right)+\left(\lambda_k+\lambda_{k-1}\right)\check{p}\left(y_k\right)\\&=\sum_{j=1}^{n}\lambda_j p'\left(y_j\right)\end{aligned}$$

which proves the claim.

2. When Y is an arbitrary compact subset of a Euclidean Space.

Let $V = \overline{v}(Y)$ be the range of $\overline{v}$ and a subset of $\mathbb{R}$. Furthermore, let us define

$$\forall v \in V, \hat{p}(v) = p(y), \overline{v}(y) = v.$$

This function is well-defined since p is co-monotone with $\overline{v}$. That is if for two values y_1 and y_2, $\overline{v}(y_1) = \overline{v}(y_2)$, then we must have that $p(y_1) = p(y_2)$. We also let $\mu_v \in \Delta(V)$ be the probability measure induced on V using μ_y and $\overline{v}(\cdot)$. Clearly, we also must have that $\hat{p}(\cdot)$ is a monotone function of v.

By Assumption **1, V is a finite collection of subintervals. For ease of exposition, we prove the claim when there is only one subinterval $[\underline{v}, \overline{v}]$. The proof is almost identical with a finite number but it is more cumbersome. Consider a sequence of partitions $V^n = \{v_0^n = \underline{v} < v_1^n < \cdots < v_n^n = \overline{v}\}$ for $n = 1, 2, \cdots$ with $\min_{0 \leq i \leq n-1} v_{i+1}^n - v_i^n \to 0$ and $V^{n+1} \subset V^n$. We define

$$\begin{aligned} f_i^n &= \mu_v\left(\left[v_{i-1}^n, v_i^n\right)\right), 1 \leq i \leq n-1 \\ f_n^n &= \mu_v\left(\left[v_{n-1}^n, \overline{v}\right]\right) \end{aligned}$$

$$v_i^n = \begin{cases} \frac{\int v \mathbf{1}\left[v \in \left[v_{i-1}^n, v_i^n\right)\right] d\mu_v}{f_i^n} & f_i^n > 0, i \leq n-1 \\ \frac{\int v \mathbf{1}\left[v \in \left[v_{n-1}^n, \overline{v}\right)\right] d\mu_v}{f_n^n} & f_n^n > 0, i = n \\ \frac{v_{i-1}^n + v_i^n}{2} & f_i^n = 0, i \geq 1 \end{cases}$$

$$\hat{p}^n\left(v_i^n\right) = \begin{cases} \frac{\int \hat{p}(v) \mathbf{1}\left[v \in \left[v_{i-1}^n, v_i^n\right)\right] d\mu_v}{f_i^n} & f_i^n > 0, i \leq n-1 \\ \frac{\int \hat{p}(v) \mathbf{1}\left[v \in \left[v_{n-1}^n, \overline{v}\right)\right] d\mu_v}{f_n^n} & f_n^n > 0, i = n \\ \frac{\hat{p}\left(v_{i-1}^n\right) + \hat{p}\left(v_i^n\right)}{2} & f_i^n = 0, i \geq 1 \end{cases}$$

In words, the above constructs a discretization of the buyer values v and the DM's interim prices $\hat{p}(v)$. Since $\hat{p}^n(v)$ is an increasing function of v and by construction, $\hat{p}^n \succcurlyeq_{\text{SOSD}} v^n$ and V^n is finite, we can apply the result from the first part. That is, an information structure (S^n, π^n) exists where $\pi^n : V^n \to \Delta(S^n)$ such that

$$\hat{p}^n\left(v_i^n\right) = \sum_{s \in S^n} \pi^n\left(\{s\} | v_i^n\right) \mathbb{E}\left[\overline{v} | s\right]$$

Note that each (S^n, π^n) induces a distribution over posterior beliefs of the buyers given by $\tau^n \in \Delta(\Delta(V^n))$. Since any probability measure in $\Delta(V^n)$ can be embedded in $\Delta(V)$. This is because for any $\mu \in \Delta(V^n)$, we can construct $\hat{\mu} \in \Delta(\Theta)$ defined by $\hat{\mu}(A) = \sum_{i=1}^n \mu_i \mathbf{1}\left[v_i^n \in A\right]$ where A is an arbitrary Borel subset of V. Similarly, we can find $\hat{\tau}^n \in \Delta(\Delta(V))$ which is equivalent to τ^n.

Now consider the probability measure ζ^n representing the joint distribution of v^n and posterior mean $\mathbb{E}_\mu[v] = \int v d\mu$ for any $\mu \in \text{Supp}(\hat{\tau}^n)$ induced by $\hat{\tau}^n$. Note that $\zeta^n \in \Delta(V \times V)$. By an application of Reisz Representation theorem (see Theorem 14.12 in Aliprantis and Border (2013)), $\Delta(V \times V)$ is compact according to the weak-* topology.[14] This implies that the sequence $\{\zeta^n\}$ must have a convergent subsequence whose limit is given by $\zeta \in \Delta(V \times V)$. Let $\mathcal{G}^n$ be the σ-field generated by the sets $\left\{\left[v_i^n, v_{i+1}^n\right)\right\}_{i \leq n-1} \cup \left\{\left[v_{n-1}^n, \overline{v}\right]\right\}$ and let $\mathcal{F}^n = \mathcal{G}^n \times \{\emptyset, \Delta(V)\}$. In words, $\mathcal{F}^n$ conveys the information that $v \in \left[v_i^n, v_{i+1}^n\right)$ or $v \in \left[v_{n-1}^n, \overline{v}\right]$. Note that $\mathcal{F}^n \subset \mathcal{F}^{n+1}$ because $V^n \subset V^{n+1}$. Moreover,

$$\mathbb{E}[\zeta^n | \mathcal{F}^n] = (v^n, \hat{p}^n)$$

where in the above, $(v^n, \hat{p}^n)$ is the random variable with values $(v_i^n, \hat{p}_i^n)$ with probability f_i^n. Note that the above holds by the construction of τ^n and ζ^n. As a result

$$\begin{aligned}
\mathbb{E}\left[\zeta^{n+1} | \mathcal{F}^n\right] &= \mathbb{E}\left[\mathbb{E}\left[\zeta^{n+1} | \mathcal{F}^{n+1}\right] | \mathcal{F}^n\right] \\
&= \mathbb{E}\left[\left(v^{n+1}, \hat{p}^{n+1}\right) | \mathcal{F}^n\right] \\
&= (v^n, \hat{p}^n)
\end{aligned}$$

where the last equality follows because $\mathbb{E}[\hat{p}(v) | \mathcal{F}^n] = \hat{p}^n, \mathbb{E}[v | \mathcal{F}^n] = v^n$ given the definition of $\hat{p}^n$ and v^n above. All of this implies that $\mathcal{F}^n$ is a filtration and $(\zeta^n, \mathcal{F}^n)$ forms a bounded martingale – for a definition see Doob (1994). Hence by Doob's martingale convergence theorem – see Theorem XI.14 in Doob (1994), we must have that

$$\lim_{n \to \infty} \mathbb{E}[\zeta^n | \mathcal{F}^n] = \mathbb{E}[\zeta | \mathcal{F}]$$

Therefore, $\mathbb{E}_\zeta\left[\int v d\mu | v\right] = \hat{p}(v)$. This concludes the proof.

[14] A rough argument for sequential compactness of $\Delta(V \times V)$ is as follows: Note that $C(V \times V)$, the space of all continuous functions on $V \times V$, is separable since V is a compact, metrizable, and Hausdorff space (see Reisz's Theorem in Royden and Fitzpatrick (1988) – section12.3, page 251.) This implies that there exists a countable subset $\{\alpha_i\}_{i=1}^\infty$ of $C(V \times V)$ which is dense in $C(V \times V)$ according to sup-norm. Thus, for any sequence of measures $\{\mu_m\}_{m=1}^\infty$ in $\Delta(V \times V)$, for a given i, the sequence $\left\{\int \alpha_i(v) d\mu_m\right\}_{m=1}^\infty$ must have a convergent subsequent. Iterating repeatedly, as we increase i, we can find a subsequence $\{\mu_{m_k}\}_{k=1}^\infty$ where $\left\{\int \alpha_i(v) d\mu_{m_k}\right\}$ converges. We define $\zeta(\alpha_i) = \lim_{k \to \infty} \int \alpha_i(v) d\mu_{m_k}$. Since $\{\alpha_i\}$ is dense in $C(V \times V)$, then $\zeta(\alpha) = \lim_{k \to \infty} \int \alpha(v) d\mu_{m_k}$ must exists for all $\alpha \in C(V \times V)$ and can be similarly defined. It is easy to show that $\zeta(\alpha)$ is a linear functional over $C(V \times V)$ and thus a member of its dual, $C(V)^*$. Hence, by Riesz Representation Theorem, there must exist a measure $\hat{\zeta} \in \Delta(V \times V)$ where $\zeta(\alpha) = \int \alpha d\hat{\zeta}$. This implies that μ_{m_k} converges to $\hat{\zeta}$ according to the weak-* topology and hence, $\Delta(V \times V)$ is sequentially compact.

A.2 Proof of Proposition 1

Proof. We show the claim by first showing that for all y, either the monotonicity constraint is binding or the majorization constraint, equation 6 , is binding at the optimum. Suppose to the contrary that this does not hold. Note that a change in $p(y)$ for a measure zero of y's, does not affect the objective, and the majorization constraint. This implies that in order to achieve a contradiction, we need to rule out an interval in which neither majorization nor monotonicity constraint is binding. Suppose that there exists an interval $I = [y_1, y_2]$ for which majorization and monotonicity are slack. Note that under the validity of the first order approach and given any effort profile $a(\theta)$, the optimal rating system must be a solution to the following planning problem:

$$\max_{p(\cdot)} \sum_{\theta e \Theta} f(\theta) \lambda(\theta) \left[\int_0^1 p(y) g(y|q(\theta)) dy - c(a(\theta), \theta) \right] \tag{P1}$$

subject to

$$\int_0^1 p(y) g_q(y|a(\theta)) dy = c_a(a(\theta), \theta)$$

$$\sum_{\theta \in \Theta} f(\theta) \int_0^y [p(y') - y'] g(y'|a(\theta)) dy' \geq 0, \forall y \in [0,1]$$

$$\sum_{\theta \in \Theta} f(\theta) \int_0^1 [p(y) - y] g(y|a(\theta)) dy = 0$$

$$p(y) - p(y') \geq 0, \forall y \geq y'$$

By combining the Theorems 1 in section 9.3 and 9.4 of Luenberger (1997) – together with the fact that we have finitely many types and thus finitely many linear equality constraints, there must exist Lagrange multipliers $\gamma(\theta)$ – for the incentive compatibility constraint – so that $p(y)$ satisfies

$$p \in \arg\max_{\hat{p}} \int_0^1 \hat{p}(y) \sum_{\theta} f(\theta) [\lambda(\theta) g(y|a(\theta)) + \gamma(\theta) g_a(y|a(\theta))] dy \tag{16}$$

subject to $\hat{p} \succcurlyeq_{\text{SOSD}} y$, and $\hat{p}(\cdot)$ is monotone. Let us define $h(y) = \sum_{\theta} f(\theta) g(y|a(\theta))$ and

$$\alpha(y) = \frac{\sum_{\theta} f(\theta) [\lambda(\theta) g(y|a(\theta)) + \gamma(\theta) g_a(y|a(\theta))]}{h(y)}$$

The Lagrangian associated with (16) is given by

$$\begin{aligned}\mathcal{L} &= \int_0^1 \hat{p}(y)\,\alpha(y)\,h(y)\,dy + \\ &\int_0^1 \int_0^y (\hat{p}(y') - y')\,h(y')\,dy' dM(y) \\ &- m \int_0^1 (\hat{p}(y') - y')\,h(y')\,dy' \\ &= \int_0^1 \hat{p}(y)\,\alpha(y)\,h(y)\,dy + \\ &\int_0^1 (\hat{p}(y) - y)\,[M(1) - M(y)]\,h(y)\,dy \\ &- m \int_0^1 (\hat{p}(y') - y')\,h(y')\,dy'\end{aligned}$$

where $M(y)$ is an increasing function y. Moreover,

$$\int_0^1 \int_0^y (\hat{p}(y') - y') \sum_\theta f(\theta)\,g(y'|a(\theta))\,dy' dM(y) = 0$$

From the result in **KleinerMoldovanuStrack, we know that there exists an $p_e(y)$ extreme point of the set $\{p : p \succcurlyeq_{\text{SOSD}} y, p : \text{monotone}\}$ that maximizes the objective in (16) and a collection of disjoint intervals $\left[\underline{y}_i, \overline{y}_i\right)$ exists such that

$$p_e(y) = \begin{cases} y & y \notin \bigcup_i \left[\underline{y}_i, \overline{y}_i\right) \\ \frac{\int_{\underline{y}_i}^{\overline{y}_i} y \sum_\theta f(\theta) g(y|a(\theta)) dy}{\int_{\underline{y}_i}^{\overline{y}_i} \sum_\theta f(\theta) g(y|a(\theta)) dy} & y \in \left[\underline{y}_i, \overline{y}_i\right) \end{cases}$$

Note that optimality conditions implied by the Lagrangian are that if $y \in \left(\underline{y}_i, \overline{y}_i\right)$ then

$$\alpha(y) - m + M(1) - M(y) = 0.$$

In other words, $\alpha(y)$ must be weakly increasing. Moreover, if $(z_1, z_2) \subset [0,1] \setminus \bigcup_i \left[\underline{y}_i, \overline{y}_i\right)$, then $M(y)$ has to be constant.

If the solution of the optimization problem (P1) is not a extreme point of $\{p : p \succcurlyeq_{\text{SOSD}} y, p : \text{monotone}\}$, then by Krein-Milman (see ***??), it must be a convex combination of the extreme points of the set $\{p : p \succcurlyeq_{\text{SOSD}} y, p : \text{monotone}\}$. Hence, there must exist another extreme point $\tilde{p}_e(y)$ that also achieves the optimum in (16). If $p_e \neq \tilde{p}_e$, there must exist $y \in (0,1)$ so that $p_e(y) = y$ for an interval around y and $\tilde{p}_e(y)$ is constant for an interval around y. By optimality, it must be that

$$\alpha(y') - m + M(1) - M(y') = 0, M(y') = M(y)$$

for $y' \in I$, an interval around y. This means that there must exist a constant, $c = m - M(1) + M(y)$ so that for all $y' \in I$

$$\sum_{\theta} f(\theta)\left[\lambda(\theta) g(y'|a(\theta)) + \gamma(\theta) g_a(y'|a(\theta))\right] = c \sum_{\theta} f(\theta) g(y'|a(\theta))$$

or

$$\sum_{\theta} f(\theta)\left[(\lambda(\theta) - c) g(y'|a(\theta)) + \gamma(\theta) g_a(y'|a(\theta))\right] = 0$$

which then implies that $\{g(y|a(\theta)), g_a(y|a(\theta))\}_{\theta\in\Theta}$ are linearly dependent over I' which is in contradiction with our assumption. This concludes the proof. □

A.3 Proof of Proposition 2

We first prove the following Lemma:

Lemma 4. *Consider the optimization problem* $\max_p \int_0^1 p(y)\alpha(y)h(y)\,dy$ *subject to* $\{p : p \succcurlyeq_{SOSD} y, p : \text{monotone}\}$ *Suppose that* $\alpha(x)$ *is continuously differentiable and that its derivative changes sign* $k < \infty$ *times, i.e., we can partition* $[0,1]$ *into* k *intervals where in each interval* $\alpha'(x)$ *has the same sign but not in two consecutive intervals. Then, an optimal information structure is an alternating partition (between full revelation and pooling) with at most* k *intervals.*

Proof. As in proof of Proposition 1, we know that $\int_0^1 p(y)\,dH$ is maximized at extreme point p_e of the set $\{p : p \succcurlyeq_{\text{SOSD}} y, p : \text{monotone}\}$ and thus a collection of disjoint intervals $\left[\underline{y}_i, \overline{y}_i\right)$ exists such that

$$p_e(y) = \begin{cases} y & y \notin \bigcup_i \left[\underline{y}_i, \overline{y}_i\right) \\ \frac{\int_{\underline{y}_i}^{\overline{y}_i} y \sum_\theta f(\theta) g(y|a(\theta)) dy}{\int_{\underline{y}_i}^{\overline{y}_i} \sum_\theta f(\theta) g(y|a(\theta)) dy} & y \in \left[\underline{y}_i, \overline{y}_i\right) \end{cases}$$

As we have shown in proof of Proposition **1, $\alpha(y)$ must be increasing over any subinterval $I \subset [0,1] \setminus \bigcup_i \left[\underline{y}_i, \overline{y}_i\right)$. Thus, in order to show the result, it suffices to show that we cannot have $\overline{y}_i = \underline{y}_j$ for some $i \neq j$. This would mean that p_e is associated with an alternating partition and that there is at most k intervals.

Suppose to the contrary that $\overline{y}_i = \underline{y}_j$ for some $i \neq j$. There are three possibilities:

1. $\frac{\int_{\underline{y}_i}^{\overline{y}_i} \alpha(y)h(y)dy}{\int_{\underline{y}_i}^{\overline{y}_i} h(y)dy} > \frac{\int_{\underline{y}_j}^{\overline{y}_j} \alpha(y)h(y)dy}{\int_{\underline{y}_j}^{\overline{y}_j} h(y)dy}$. In this case, if we consider $\tilde{p}_e$ which pools the entire interval

$\left[\underline{y}_i, \overline{y}_j\right)$ and is otherwise the same as p_e, then

$$\int_{\underline{y}_i}^{\overline{y}_j} \tilde{p}_e(y)\,\alpha(y)\,h(y)\,dy = \int_{\underline{y}_i}^{\overline{y}_j} \alpha(y)\,h(y)\,dy \frac{\int_{\underline{y}_i}^{\overline{y}_j} y h(y)\,dy}{\int_{\underline{y}_i}^{\overline{y}_j} h(y)\,dy}$$
$$= \int_{\underline{y}_i}^{\overline{y}_j} \alpha(y)\,h(y)\,dy \mathbb{E}\left[y|\underline{y}_i \le y < \overline{y}_j\right]$$

If we define $a = \frac{H(\overline{y}_i)-H(\underline{y}_i)}{H(\overline{y}_j)-H(\underline{y}_i)}$, $z_1 = \mathbb{E}\left[\alpha(y)\,|\underline{y}_i \le y < \overline{y}_i\right]$, $z_2 = \mathbb{E}\left[\alpha(y)\,|\underline{y}_j \le y < \overline{y}_j\right]$ and recall that $p_e\left(\underline{y}_i\right) = \mathbb{E}\left[y|\underline{y}_i \le y < \overline{y}_i\right], p_e\left(\underline{y}_j\right) = \mathbb{E}\left[y|\underline{y}_j \le y < \overline{y}_j\right]$, then we can write

$$\int_{\underline{y}_i}^{\overline{y}_j} \tilde{p}_e(y)\,\alpha(y)\,h(y)\,dy = \left[H(\overline{y}_j) - H\left(\underline{y}_i\right)\right] \mathbb{E}\left[\alpha(y)\,|\underline{y}_i \le y < \overline{y}_j\right] \mathbb{E}\left[y|\underline{y}_i \le y < \overline{y}_j\right]$$
$$= \left[H(\overline{y}_j) - H\left(\underline{y}_i\right)\right] (a z_1 + (1-a) z_2) \left(a p_e\left(\underline{y}_i\right) + (1-a)\, p_e\left(\underline{y}_j\right)\right)$$
$$> \left[H(\overline{y}_j) - H\left(\underline{y}_i\right)\right] \left(a z_1 p_e\left(\underline{y}_i\right) + (1-a)\, z_2 p_e\left(\underline{y}_j\right)\right)$$
$$= \int_{\underline{y}_i}^{\overline{y}_j} p_e(y)\,\alpha(y)\,h(y)\,dy$$

where the inequality follows because $z_1 > z_2$ and $p_e\left(\underline{y}_i\right) < p_e\left(\underline{y}_j\right)$ and Chebyshev's sum inequality – see Hardy et al. (1934), Theorem 43. The above then implies that $\tilde{p}_e$ delivers a higher value of $\int p\alpha dH$ which is a contradiction.

2. $\frac{\int_{\underline{y}_i}^{\overline{y}_i} \alpha(y)h(y)dy}{\int_{\underline{y}_i}^{\overline{y}_i} h(y)dy} < \frac{\int_{\underline{y}_j}^{\overline{y}_j} \alpha(y)h(y)dy}{\int_{\underline{y}_j}^{\overline{y}_j} h(y)dy}$. Note that since p_e is optimal, we must have that

$$\alpha\left(\underline{y}_j\right) \ge \mathbb{E}\left[\alpha(y)\,|\underline{y}_j \le y < \overline{y}_j\right]$$

because otherwise, there exist $\hat{y} \in \left(\underline{y}_j, \overline{y}_j\right)$ such that $\mathbb{E}\left[\alpha(y)\,|\underline{y}_j \le y < \hat{y}\right] < \mathbb{E}\left[\alpha(y)\,|\hat{y} \le y < \overline{y}_j\right]$. Then, a similar argument as above shows that separating $\left[\underline{y}_j, \hat{y}\right)$ from $\left[\hat{y}, \overline{y}_j\right)$ increases the objective, which is a contradiction. Similarly, we can show that $\alpha(\overline{y}_j) \le \mathbb{E}\left[\alpha(y)\,|\underline{y}_j \le y < \overline{y}_j\right]$. Similarly,

$$\alpha\left(\underline{y}_i\right) \ge \mathbb{E}\left[\alpha(y)\,|\underline{y}_i \le y < \overline{y}_i\right] \ge \alpha(\overline{y}_i)$$

and since α is continuous at $\overline{y}_i = \underline{y}_j$, we must have

$$\alpha\left(\underline{y}_i\right) \geq \mathbb{E}\left[\alpha\left(y\right)|\underline{y}_i \leq y < \overline{y}_i\right] \geq \alpha\left(\overline{y}_i\right) \geq \mathbb{E}\left[\alpha\left(y\right)|\underline{y}_j \leq y < \overline{y}_j\right] \geq \alpha\left(\overline{y}_j\right).$$

This is a contradiction.

3. Finally if $\frac{\int_{\underline{y}_i}^{\overline{y}_i}\alpha(y)h(y)dy}{\int_{\underline{y}_i}^{\overline{y}_i}h(y)dy} = \frac{\int_{\underline{y}_j}^{\overline{y}_j}\alpha(y)h(y)dy}{\int_{\underline{y}_j}^{\overline{y}_j}h(y)dy}$, then pooling the entire interval $\left[\underline{y}_i, \overline{y}_j\right)$ keeps the value unchanged. This proves the lemma.

□

Now, we can use Lemma 4 to prove Proposition 2.

Proof. Under the first order approach, the optimal interim prices must maximize $\int_0^1 p\left(y\right)\alpha\left(y\right)h\left(y\right)dy$ where $h\left(y\right) = \sum_{i=1}^2 f_i g\left(y|a_i\right)$ and $\alpha\left(y\right)h\left(y\right) = g\left(y|a_1\right) + \sum_i \gamma_i g_a\left(y|a_i\right)$. Note that we can show that γ_2 is positive. We do this by imposing the incentive constraints for both types as an inequality of the form $\int_0^1 p\left(y\right)g_a\left(y|a_i\right)dy \geq c_a\left(a_i, \theta_i\right)$. If this inequality is slack at the optimum for type 2, then at the optimum we can raise a_2 and redistribute the proceeds across all realizations of y, i.e., increase all $p\left(y\right)$ by the same amount. This raises the payoff of type 1 because $a_2 \geq a_1$ and monotone likelihood ratio implies that such an increase shifts the distribution to the right. This, however, does not affect the incentives of type 1 – since it is a uniform increase in $p\left(\cdot\right)$ – while does not violate the incentive constraint of type 2 since the inequality is slack. Hence, type 2 incentive constraint is binding at the optimum of the relaxed planning problem which means that γ_2 is positive. Moreover, a similar argument implies that γ_1 is positive.

Given the definitions in Assumption 3, we can write

$$\alpha\left(y\right) = \frac{g\left(y|a_1\right) + \sum_i \gamma_i g_a\left(y|a_i\right)}{\sum_i f_i g\left(y|a_i\right)} = \frac{1 + \gamma_1 \phi\left(\hat{y}^{-1}\left(y\right)\right) + \gamma_2 \psi\left(\hat{y}^{-1}\left(y\right)\right)}{f_1 + f_2 \hat{y}^{-1}\left(y\right)}$$

where in the above $\hat{y}^{-1}$ is the inverse of the function $\hat{y}\left(z\right)$ defined in Assumption 3 which is strictly increasing by part 1 of Assumption 3. Since the RHS of the above is only a function of $\hat{y}^{-1}\left(y\right)$ or z, we can define $\hat{\alpha}\left(\hat{y}^{-1}\left(y\right)\right) = \alpha\left(y\right)$. Since $\hat{y}$ is increasing by Assumption 3, $\hat{\alpha}$ inherits the monotonicity properties of α. After some algebra, we have

$$\left(\left(f_1 + f_2 z\right)^2 \hat{\alpha}'\left(z\right)\right)' = \left(f_1 + f_2 z\right)\left[\gamma_1 \phi''\left(z\right) + \gamma_2 \psi''\left(z\right)\right]$$

Since $\gamma_2, \gamma_1 \geq 0$, Assumption 3 implies that there is a cutoff value of z – possibly at the corners – where for values of z below the above is negative while for high values of z it is positive. This means that if $\hat{\alpha}'\left(\min z\right)$ is negative then it changes sign only once but if it is positive, it changes sign at most twice. This combined with Lemma 4 establishes the claim in the Proposition. □

B Online Appendix

B.1 Separable Distributions

Consider the problem in this section and assume that $g(y|a)$, the density of y given a satisfies the following separability

$$g(y|a) = 1 + \beta(a)\, m(y) \tag{17}$$

where $m(y)$ is an increasing function that satisfies $\int_0^1 m(y)\, dy = 0$ and $\beta(a)$ is increasing and concave. Note that under this specification, the marginal benefit of effort is given by

$$\int_0^1 p(y)\, g_a(y|a)\, dy = \beta'(a) \int_0^1 p(y)\, m(y)\, dy$$

Hence, if effort profile $a(\theta)$ is optimal, then

$$\beta'(a(\theta)) \int_0^1 p(y)\, m(y)\, dy = c_a(a(\theta), \theta)$$

Hence, we must have

$$\frac{c_a(a(\theta), \theta)}{\beta'(a(\theta))} = \frac{c_a(a(\theta'), \theta')}{\beta'(a(\theta'))}$$

Thus, if we choose the effort level of the lowest type $\underline{a}$, the above determines the effort level for all the other types. Let us refer to the solution of the above as $\hat{a}(\underline{a}, \theta)$. Hence, the problem of optimal rating design is given by

$$\max_{\underline{a}, p(\cdot)} \int_0^1 p(y)\, g(y|\underline{a})\, dy - c(\underline{a}, \underline{\theta})$$

subject to

$$\int_0^1 p(y)\, g_a(y)\, dy = c_a(\underline{a}, \underline{\theta}) \tag{18}$$

$$\int_0^y p(\hat{y}) \int g(\hat{y}|\hat{a}(\underline{a}, \theta))\, dF d\hat{y} \geq \int_0^y \hat{y} \int g(\hat{y}|\hat{a}(\underline{a}, \theta))\, dF d\hat{y}, \forall y \in (0,1)$$

$$\int_0^1 p(\hat{y}) \int g(\hat{y}|\hat{a}(\underline{a}, \theta))\, dF d\hat{y} = \int_0^1 \hat{y} \int g(\hat{y}|\hat{a}(\underline{a}, \theta))\, dF d\hat{y}$$

$$p(y) \geq p(y'), \forall y \geq y'$$

Similar to the analysis in subsection 4, given $\underline{a}$, the problem of solving for optimal interim prices is to maximize $\int p(y) \frac{g(y|\underline{a}) + \gamma g_a(y|\underline{a})}{h(y)} h(y)\, dy$ subject to majorization and monotonicity where $h(y) =$

$\int g\left(y|\hat{a}\left(\underline{a},\theta\right)\right)dF\left(\theta\right)$. Note that in this formulation γ is the Lagrange multiplier associated with the incentive constraint

Given the separability assumption on $g\left(\cdot|\cdot\right)$, we can show that the function $\frac{g(y|\underline{a})+\gamma g_a(y|\underline{a})}{h(y)}$ is either decreasing in y – when γ is low enough – or increasing – for high γ. Thus the solution of the above problem is either full pooling or full information. Since full pooling leads to marginal benefit of effort being 0, we have the following proposition:

Proposition 7. *Suppose that $g\left(\cdot|a\right)$ satisfies (17).Then optimal monopoly rating system is full disclosure.*

This result can be understood by considering the two effects identified before: redistributive and incentive. Given our specification of the distribution, the forces cannot be balanced. Since the redistributive force cannot dominate as it leads to no effort being taken by the DM, full disclosure should be optimal.