Midterm 1

15-317: Constructive Logic

October 3, 2013

Name: Andrew ID:

Instructions

- This exam is closed-book, but one two-sided sheet of notes is permitted. The last page of the exam recaps some rules you may find useful.
- There are 4 problems on 13 pages. Not all problems are the same size or difficulty. You have 80 minutes to complete the exam.
- When writing proofs, remember to label each inference with the rule used and any variables or parameters discharged (e.g., $\supset I^u$). Be sure to use the correct notation when substituting proofs for assumptions (e.g., in reductions). When writing natural deduction proofs, you may omit the judgment *true*.
- You may find it helpful to construct your proofs on scratch paper (such as the back of a page) before writing it clearly in the space provided.
- Stay cool and good luck!

	Max	Score
1	75	
2	40	
3	20	
4	15	
Total:	150	

Please keep in mind that this is a sample solution, *not a model solution. Problems admit multiple correct answers, and the answer the instructor thought of may not necessarily be the best or most elegant.*

Problems

1. **A "new" connective**

In this section, we consider the $\times$ connective, which is defined by the following natural deduction rules.

$$\frac{A \text{ true} \quad B \text{ true}}{A \times B \text{ true}} \times I \qquad \frac{A \times B \text{ true} \qquad \begin{array}{c} \overline{A \text{ true}}^{\,u} \quad \overline{B \text{ true}}^{\,v} \\ \vdots \\ C \text{ true} \end{array}}{C \text{ true}} \times E^{u,v}$$

(a) (10 points) Show local soundness for the $\times$ connective by demonstrating a local reduction.

Solution:

$$\frac{\dfrac{\begin{array}{c}\mathcal{D}_1 \\ A \text{ true}\end{array} \quad \begin{array}{c}\mathcal{D}_2 \\ B \text{ true}\end{array}}{A \times B \text{ true}} \times I \qquad \begin{array}{c} \overline{A \text{ true}}^{\,u} \quad \overline{B \text{ true}}^{\,v} \\ \mathcal{E} \\ C \text{ true} \end{array}}{C \text{ true}} \times E^{u,v} \quad \Longrightarrow_R \quad \begin{array}{c} \dfrac{\mathcal{D}_1}{A \text{ true}}\,u \quad \dfrac{\mathcal{D}_2}{B \text{ true}}\,v \\ \mathcal{E} \\ C \text{ true} \end{array}$$

(b) (10 points) Show local completeness for the $\times$ connective by demonstrating a local expansion.

Solution:

$$\begin{array}{c}\mathcal{D} \\ A \times B \text{ true}\end{array} \Longrightarrow_E \frac{\begin{array}{c}\mathcal{D} \\ A \times B \text{ true}\end{array} \qquad \dfrac{\overline{A \text{ true}}^{\,u} \quad \overline{B \text{ true}}^{\,v}}{A \times B \text{ true}} \times I}{A \times B \text{ true}} \times E^{u,v}$$

(c) (5 points) Give sequent calculus rules for the $\times$ connective in the same style as the sequent calculus system we've been studying (the rules are listed in Figure 1). There should be one right rule ($\times R$) and one left rule ($\times L$).

Solution:

$$\frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \times B} \times R \qquad \frac{\Gamma, A \times B, A, B \Longrightarrow C}{\Gamma, A \times B, \Longrightarrow C} \times L$$

Using our natural deduction notation with explicit contexts, we have these natural deduction rules for the $\times$ connective:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \times B} \times I \qquad \frac{\Gamma \vdash A \times B \quad \Gamma, A, B \vdash C}{\Gamma \vdash C} \times E$$

(d) (10 points) Recall the soundness theorem for sequent calculus, which says that if $\Gamma \Longrightarrow A$, then $\Gamma \vdash A$. Give the case of soundness corresponding to the $\times L$ case. As in your homework, you may use the following lemmas:

- **Weakening** For all Γ, A, C, if $\Gamma \vdash C$, then $\Gamma, A \vdash C$.
- **Substitution** For all Γ, A, C, if $\Gamma \vdash A$ and $\Gamma, A \vdash C$, then $\Gamma \vdash C$.

Solution: We have

$$\dfrac{\overset{\mathcal{D}}{\Gamma, A_1 \times A_2, A_1, A_2 \Longrightarrow A}}{\Gamma, A_1 \times A_2 \Longrightarrow A} \times L$$

and construct

$$\dfrac{\dfrac{}{\Gamma, A_1 \times A_2 \vdash A_1 \times A_2}\, hyp \qquad \overset{\textit{by i.h. on } \mathcal{D}}{\Gamma, A_1 \times A_2, A_1, A_2 \vdash A}}{\Gamma, A_1 \times A_2 \vdash A} \times E$$

(e) (10 points) Recall the completeness theorem for sequent calculus, which says that if $\Gamma \vdash A$, then $\Gamma \Longrightarrow A$. Give the case of completeness corresponding to the $\times E$ case. As in your homework, you may use the following lemmas:

- **Weakening** For all Γ, A, C, if $\Gamma \Longrightarrow C$, then $\Gamma, A \Longrightarrow C$.
- **Identity** For all Γ, A: $\Gamma, A \Longrightarrow A$.
- **Cut** For all Γ, A, C, if $\Gamma \Longrightarrow A$ and $\Gamma, A \Longrightarrow C$, then $\Gamma \Longrightarrow C$.

Solution: We have

$$\dfrac{\overset{\mathcal{D}}{\Gamma \vdash A \times B} \quad \overset{\mathcal{E}}{\Gamma, A, B \vdash C}}{\Gamma \vdash C} \times E$$

and construct

$$\dfrac{\overset{\textit{by i.h. on } \mathcal{D}}{\Gamma \Longrightarrow A \times B} \quad \dfrac{\dfrac{\overset{\textit{by i.h. on } \mathcal{E}}{\Gamma, A, B \Longrightarrow C}}{\Gamma, A \times B, A, B \Longrightarrow C} \textit{weaken}}{\Gamma, A \times B \Longrightarrow C} \times L}{\Gamma \Longrightarrow C} \textit{cut}$$

You may have noticed that this connective is "remarkably similar" to $\wedge$. It can be shown to be equivalent.

(f) (10 points) Show that $A \times B \vdash A \wedge B$.

Solution:

$$\dfrac{\dfrac{}{A \times B \vdash A \times B}\,hyp \quad \dfrac{\dfrac{}{A, B \vdash A}\,hyp \quad \dfrac{}{A, B \vdash B}\,hyp}{A, B \vdash A \wedge B}\,\wedge I}{A \times B \vdash A \wedge B}\,\times E$$

(g) (10 points) Show that $A \wedge B \vdash A \times B$.

Solution:

$$\dfrac{\dfrac{\dfrac{}{A \wedge B \vdash A \wedge B}\,hyp}{A \wedge B \vdash A}\,\wedge E_L \quad \dfrac{\dfrac{}{A \wedge B \vdash A \wedge B}\,hyp}{A \wedge B \vdash B}\,\wedge E_R}{A \wedge B \vdash A \times B}\,\times I$$

(h) (10 points) Come up with proof terms for introducing and eliminating $\times$. Present versions of the natural deduction rules (without explicit contexts) $\times E$ and $\times I$ that assign your proof terms, and show the reduction rule for these proof terms.

Solution:

$$\frac{M : A \quad N : B}{M \times N : A \times B}\ \times I \qquad \frac{M : A \times B \quad \begin{array}{c} \overline{u : A}\ u \quad \overline{v : B}\ v \\ \vdots \\ N : C \end{array}}{\mathbf{let}\, u \times v = M\ \mathbf{in}\ N : C}\ \times E^{u,v}$$

$$\mathbf{let}\, u \times v = M_1 \times M_2\ \mathbf{in}\ N \Longrightarrow_R [M_1/u][M_2/v]N$$

2. **Natural Numbers and Proof Terms**

   Recall the rules for natural number arithmetic and induction with proof terms (recapped in Figure 2; we have added the rule of reflexivity for your convenience).

   (a) (25 points) Give a natural deduction proof of $\forall n. \neg n = 0 \supset \exists m. n = \mathsf{s}m$. Omit typing annotations in quantifications (as we just did) for brevity's sake.

Solution:

$$
\dfrac{
\dfrac{
\dfrac{}{a : \mathsf{nat}}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{\neg 0 = 0}\,u
\qquad
\dfrac{}{0 = 0}\,{=_N I_0}
}{\bot}\,{\supset}E
}{\exists m. 0 = \mathsf{s}m}\,{\bot}E
}{\neg 0 = 0 \supset \exists m. 0 = \mathsf{s}m}\,{\supset}I^u
\qquad
\dfrac{
\dfrac{
\dfrac{}{a' : \mathsf{nat}}
\qquad
\dfrac{
\dfrac{
\dfrac{}{a' : \mathsf{nat}}
}{\mathsf{s}a' : \mathsf{nat}}\,\mathsf{nat}I_{\mathsf{s}}
}{\mathsf{s}a' = \mathsf{s}a'}\,\mathsf{refl}
}{\exists m. \mathsf{s}a' = \mathsf{s}m}\,\exists I
}{\neg \mathsf{s}a' = 0 \supset \exists m. \mathsf{s}a' = \mathsf{s}m}\,{\supset}I^w
}{\neg a = 0 \supset \exists m. a = \mathsf{s}m}\,\mathsf{nat}E^{a',v}
}{\forall n. \neg n = 0 \supset \exists m. n = \mathsf{s}m}\,\forall I^a
$$

   (b) (15 points) Give the corresponding proof term. Represent proofs of equality by "_", since they have no computational meaning.

Solution:

$$\lambda a : \mathsf{nat}. R(a, \lambda u. \mathsf{abort}^{\exists m. 0 = \mathsf{s}m}(u\ \_), a'.v.\lambda w.\langle a', \_\rangle)$$

(c) **For bonus credit** With one more proof step you can prove $\neg 0 = 0 \supset \exists m.0 = sm$. Show the resulting proof term (don't reduce!), then show the reduction steps that get you to a term that can no longer be reduced. (Reduction rules are in Figure 4.)

Solution:

$$(\lambda a : \mathsf{nat}.R(a, \lambda u.\mathsf{abort}^{\exists m.0=\mathsf{s}m}(u\ \_), a'.v.\lambda w.\langle a', \_\rangle)0)$$
$$\Longrightarrow_R R(0, \lambda u.\mathsf{abort}^{\exists m.0=\mathsf{s}m}(u\ \_), a'.v.\lambda w.\langle a', \_\rangle)$$
$$\Longrightarrow_R \lambda u.\mathsf{abort}(u\ \_)$$

3. **Classical Logic**

   Recall the rules for classical logic (recapped in Figure 5).

   Here are some familiar tautologies of classical propositional logic. Prove each one using *constructive* natural deduction if possible. If it's not possible, prove it using classical reasoning, that is, using any of the rules *proof by contradiction*, *law of excluded middle*, *double-negation elimination*, or *Peirce's law*. Your classical proof(s) may use either the familiar definition of $\neg A$ as $A \supset \bot$ or the judgmental definition of Figure 5.

   (a) (10 points) $(A \lor B) \supset (\neg A \supset B)$

   Solution: Constructive deduction:

$$\dfrac{\dfrac{\dfrac{}{A \lor B}\,u \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\neg A}\,x \quad \dfrac{}{A}\,v}{\bot}\,{\supset}E}{B}\,{\bot}E}{\neg A \supset B}\,{\supset}I^{x} \qquad \dfrac{\dfrac{}{B}\,w}{\neg A \supset B}\,{\supset}I^{y}}{\neg A \supset B}\,{\lor}E^{v,w}}{(A \lor B) \supset (\neg A \supset B)}\,{\supset}I^{u}$$

   (b) (10 points) $(\neg A \supset B) \supset (A \lor B)$

   Solution: Classical deduction:

$$\dfrac{\dfrac{\dfrac{}{A \lor \neg A}\,LEM \qquad \dfrac{\dfrac{}{A}\,v}{A \lor B}\,{\lor}I_L \qquad \dfrac{\dfrac{\dfrac{}{\neg A \supset B}\,u \quad \dfrac{}{\neg A}\,w}{B}\,{\supset}E}{A \lor B}\,{\lor}I_R}{A \lor B}\,{\lor}E^{v,w}}{(\neg A \supset B) \supset (A \lor B)}\,{\supset}I^{u}$$

4. (15 points) **Mistakes Were Made**

Consider the following purported proof:

$$
\dfrac{
  \dfrac{
    \dfrac{}{(\forall x:\tau.A(x)) \supset \exists y:\tau.B(y)}\,u
    \qquad
    \dfrac{\dfrac{}{A(a)}\,v}{\forall x:\tau.A(x)}\,\forall I^{a}
  }{\exists y:\tau.B(y)}\,{\supset}E
  \qquad
  \dfrac{
    \dfrac{}{a:\tau}
    \qquad
    \dfrac{\dfrac{}{B(a)}\,w}{A(a)\supset B(a)}\,{\supset}I^{v}
  }{\exists z:\tau.A(z)\supset B(z)}\,\exists I
}{
  \dfrac{\exists z:\tau.A(z)\supset B(z)}{((\forall x:\tau.A(x))\supset \exists y:\tau.B(y)) \supset \exists z:\tau.A(z)\supset B(z)}\,{\supset}I^{u}
}\,\exists E^{a,w}
$$

This proof is incorrect. Circle the label(s) of the rule(s) that are applied incorrectly. Explain what is wrong with each.

Solution: The $\forall$ introduction is incorrect, as it uses the parameter a introduced by the $\exists$ elimination and the assumption $v : A(a)$ introduced by the $\supset$ introduction that concludes $A(a) \supset B(a)$.

Useful Rules

$$\frac{\Gamma \Longrightarrow A \quad \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \wedge B} \wedge R \qquad \frac{\Gamma, A \wedge B, A \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C} \wedge L_1 \qquad \frac{\Gamma, A \wedge B, B \Longrightarrow C}{\Gamma, A \wedge B \Longrightarrow C} \wedge L_2$$

$$\frac{}{\Gamma \Longrightarrow \top} \top R$$

$$\frac{\Gamma, A \Longrightarrow B}{\Gamma \Longrightarrow A \supset B} \supset R \qquad \frac{\Gamma, A \supset B \Longrightarrow A \quad \Gamma, A \supset B, B \Longrightarrow C}{\Gamma, A \supset B \Longrightarrow C} \supset L$$

$$\frac{\Gamma \Longrightarrow A}{\Gamma \Longrightarrow A \vee B} \vee R_1 \qquad \frac{\Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \vee B} \vee R_2 \qquad \frac{\Gamma, A \vee B, A \Longrightarrow C \quad \Gamma, A \vee B, B \Longrightarrow C}{\Gamma, A \vee B \Longrightarrow C} \vee L$$

$$\frac{}{\Gamma, \bot \Longrightarrow C} \bot L$$

$$\frac{}{\Gamma, P \Longrightarrow P} \text{init } (P \text{ atomic})$$

Figure 1: Rules for the sequent calculus.

$$\frac{n : \mathsf{nat} \quad M_0 : C(0)\ \mathit{true} \quad \begin{array}{c} \overline{x : \mathsf{nat}} \quad \overline{u : C(x)\ \mathit{true}}^{\,u} \\ \vdots \\ M_s : C(\mathsf{s}\,x)\ \mathit{true} \end{array}}{R(n, M_0, x.\,u.\,M_s) : C(n)\ \mathit{true}} \mathsf{nat}E^{x,u}$$

$$\frac{}{0 = 0} =_N I_0 \qquad \frac{n = m}{\mathsf{s}\,n = \mathsf{s}\,m} =_N I_{\mathsf{s}} \qquad \frac{0 = \mathsf{s}\,n}{J} =_N E_{0\mathsf{s}} \qquad \frac{\mathsf{s}\,n = 0}{J} =_N E_{\mathsf{s}0} \qquad \frac{\mathsf{s}\,m = \mathsf{s}\,n}{m = n} =_N E_{\mathsf{ss}} \qquad \frac{n : \mathsf{nat}}{n = n} \mathsf{refl}$$

Figure 2: Rules for natural numbers and induction.

Constructors | Destructors

$$\frac{M : A \quad N : B}{\langle M, N\rangle : A \wedge B}\ \wedge I$$

$$\frac{M : A \wedge B}{\textbf{fst}\, M : A}\ \wedge E_L$$

$$\frac{M : A \wedge B}{\textbf{snd}\, M : B}\ \wedge E_R$$

$$\frac{}{\langle\,\rangle : \top}\ \top I$$

no destructor for $\top$

$$\frac{\begin{array}{c}\overline{u : A}\ u \\ \vdots \\ M : B\end{array}}{\lambda u{:}A.\ M : A \supset B}\ \supset I^u$$

$$\frac{M : A \supset B \quad N : A}{M\, N : B}\ \supset E$$

$$\frac{M : A}{\textbf{inl}^B\, M : A \vee B}\ \vee I_L$$

$$\frac{N : B}{\textbf{inr}^A\, N : A \vee B}\ \vee I_R$$

$$\frac{M : A \vee B \quad \begin{array}{c}\overline{u : A}\ u \\ \vdots \\ N : C\end{array} \quad \begin{array}{c}\overline{w : B}\ w \\ \vdots \\ O : C\end{array}}{\textbf{case}\, M\ \textbf{of inl}\, u \Rightarrow N \mid \textbf{inr}\, w \Rightarrow O : C}\ \vee E^{u,w}$$

no constructor for $\bot$

$$\frac{M : \bot}{\textbf{abort}^C\, M : C}\ \bot E$$

$$\frac{\begin{array}{c}\overline{a : \tau} \\ \vdots \\ M : A(a)\end{array}}{\lambda a{:}\tau.\ M : \forall x{:}\tau.\ A(x)}\ \forall I^a$$

$$\frac{M : \forall x{:}\tau.\ A(x) \quad t : \tau}{M\, t : A(t)}\ \forall E$$

$$\frac{t : \tau \quad M : A(t)}{\langle t, M\rangle : \exists x{:}\tau.\ A(x)}\ \exists I$$

$$\frac{M : \exists x{:}\tau.\ A(x) \quad \begin{array}{c}\overline{a : \tau} \quad \overline{u : A(a)}\ u \\ \vdots \\ N : C\end{array}}{\textbf{let}\langle a, u\rangle = M\ \textbf{in}\ N : C}\ \exists E^{a,u}$$

Figure 3: Proof terms for natural deduction.

$$
\begin{array}{lcl}
\mathbf{fst}\,\langle M, N\rangle & \Longrightarrow_R & M \\
\mathbf{snd}\,\langle M, N\rangle & \Longrightarrow_R & N \\
(\lambda u{:}A.\ M)\,N & \Longrightarrow_R & [N/u]M \\
\mathbf{case}\ \mathbf{inl}^B\,M\ \mathbf{of}\ \mathbf{inl}\,u \Rightarrow N \mid \mathbf{inr}\,w \Rightarrow O & \Longrightarrow_R & [M/u]N \\
\mathbf{case}\ \mathbf{inr}^A\,M\ \mathbf{of}\ \mathbf{inl}\,u \Rightarrow N \mid \mathbf{inr}\,w \Rightarrow O & \Longrightarrow_R & [M/w]O \\
R(0, M_0, x.\,u.\,M_s) & \Longrightarrow_R & M_0 \\
R(\mathsf{s}\,n', M_0, x.\,u.\,M_s) & \Longrightarrow_R & [R(n', M_0, x.\,u.\,M_s)/u][n'/x]\,M_s
\end{array}
$$

Figure 4: Rules for term reduction.

$$
A\ \mathit{false} := \begin{array}{c} A\,\mathit{true} \\ \vdots \\ \# \end{array}
\qquad
\frac{A\ \mathit{false} \quad A\,\mathit{true}}{J}\ \mathit{contra}
\qquad
\frac{\begin{array}{c} \overline{A\ \mathit{false}}^{\,k} \\ \vdots \\ \# \end{array}}{A\,\mathit{true}}\ \mathit{PBC}^k
$$

$$
\frac{\begin{array}{c} \overline{A\,\mathit{true}}^{\,u} \\ \vdots \\ \# \end{array}}{\neg A\,\mathit{true}}\ \neg I^u
\qquad
\frac{\neg A\,\mathit{true} \qquad \begin{array}{c} \overline{A\ \mathit{false}}^{\,k} \\ \vdots \\ J \end{array}}{J}\ \neg E^k
$$

$$
\frac{\neg\neg A}{A}\ \mathit{DNE}
\qquad
\frac{}{A \lor \neg A}\ \mathit{LEM}
\qquad
\frac{}{((A \supset B) \supset A) \supset A}\ \mathit{PL}
$$

Figure 5: Rules (including derived rules) for classical natural deduction.