Markets with Within-Type Adverse Selection*

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Abstract

We study bilateral trade with a seller owning multiple units of a good, where each unit is of binary quality. The seller privately knows her "type" — defined by the number of lemons that she owns — and which units in her endowments are the lemons ("*within-type adverse selection*"). We characterize the set of informationally constrained Pareto optimal allocations and show that every such allocation must involve a trade characterized by a threshold λ^*, with types having less (more) than λ^* units of lemons selling only their lemons (selling their entire endowment). We provide conditions for a distribution shift that give Pareto-improving allocations.

Keywords: Multiple-unit lemon market; Within-type adverse selection
JEL Classification: D21, D82, D86

1 Introduction

Since the seminal work of Akerlof (1970), it has been well-accepted that adverse selection is a key factor that contributes to market failures. Akerlof illustrated the effects of adverse selection in a model in which sellers each own an indivisible good of privately known quality and attempt to sell their good to uninformed buyers. Subsequent works have extended the analysis to situations in which the seller owns multiple units of a good of homogenous quality and can sell any part of her endowment (i.e., a divisible good). In this paper, we further

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extend this literature by studying a setting in which the seller not only has multiple units of a good but also each unit is of potentially *different* quality, which is indistinguishable by the buyer ex ante.

Such a setting describes many wholesale trades, such as the trade of used cars between car dealers and car rental companies and the liquidation of assets after a business shutdown. In patent trades, the project owner (such as a startup company) also often owns multiple projects that look similar to outsiders, and the project owner has private information about each project's potential. In the financial market, banks securitize their loans and sell them to institutional investors. The quality of each loan varies, and the banks also often have private information about the quality.

A distinctive feature in such lemon markets is the presence of "*within-type adverse selection*," where, fixing the seller's endowment realization (or type), the buyer also cannot distinguish the lemons from the high-quality units in the seller's endowment. Therefore, at any per-unit price, the buyer is more likely to be supplied with the seller's lemons.[1] This problem is potentially mitigated using a bundled trade, where the seller is given only the options to either trade her entire endowment or nothing at all, because a bundled trade prevents the seller from self-selecting only her lemons for trade.[2] However, the *overall* quality of the seller's endowment is also her private information ("*across-type adverse selection*"). Therefore, a given bundle price is also more likely to attract a seller who owns more lemons but completely drive out a seller with more high-quality units, and this makes it even more difficult for a trade of high-quality units to occur.

In this paper, we study a bilateral trade model with a buyer (he) facing a seller (she) who owns multiple units of a good. Each unit can be of either high or low quality. The two players' valuations of each unit are correlated, but the quality of each unit is privately known to only the seller. We characterize all the Pareto optimal allocations that satisfy the incentive constraints arising from the seller's private information about the quality. We show that *every* such constrained Pareto optimal allocation must involve a trade that has a "*threshold property*," where, if the seller has more than some (endogenously determined)

[1]For example, Calem et al. (2011) document evidence that banks cherry-picked their riskier loans to securitize during the subprime mortgage crisis.

[2]A common example of a bundled trade in practice is an "output contract," which is often used in trade with smaller farms to prevent farmers from keeping their better products or side-selling them to other buyers. In the entertainment industry, telecommunication companies also often purchase all of the programs produced by a studio each year. Patent trades also often have such a "bulk purchase" feature. For instance, in 2012, Microsoft purchased 800 patents and the licenses to more than 300 patents from AOL Inc. for more than $1 billion. Similarly, when Google purchased Motorola's mobile business arm for $12.5 billion, the sale included the transfer of all of Motorola's more than 22000 patents (see Sandhu et al. (2013) for details).

threshold (denoted by λ^*) units of lemons, she trades her entire endowment, whereas if the seller has less than λ^* units of lemons, she trades all of her lemons but none of her high-quality units.

Let us use a few examples to briefly describe some key issues and results in our model. Let H and L denote a high-quality and a low-quality unit, respectively. The seller's opportunity costs for H and L are c_H and c_L, respectively, with $c_H > c_L$, and the buyer's valuations are such that there are gains from a trade for both an H and an L. To make the explanation more concrete, we assume here that the seller has a finite number of units (whereas the main model considers a continuum of units) and the buyer makes a take-it-or-leave-it offer to the seller (whereas the main analysis studies a mechanism design problem to derive the set of constrained Pareto optimal allocations.).

Suppose first that the seller has two units and her "type" is either $\{H, H\}$ or $\{L, L\}$, each occurring with equal probability.[3] In this case, the best that the buyer can do is either offer a per-unit price of c_L (and buy from only type $\{L, L\}$) or c_H (and buy from both types but give some rent to type $\{L, L\}$ in the process).

Suppose now that the quality distribution changes, with the seller having exactly 1 H and 1 L with a probability of one. Note that the expected quality of a given unit is unchanged from above, and the buyer still cannot distinguish H from L in the economy ex ante. However, the buyer can now offer a two-unit bundle price of $c_H + c_L$ for the seller's entire endowment and obtain the first best utility. This simple example illustrates how, fixing the expected quality of each unit of good in the economy, the equilibrium efficiency also depends on how quality is distributed across different types of sellers.

Next, suppose that the seller has three units instead, and her type is either $\bar{\theta} = \{L, H, H\}$ or $\underline{\theta} = \{L, L, H\}$. Now, regardless of the probabilities of types $\bar{\theta}$ and $\underline{\theta}$, there is not a full-bundle price that allows the buyer to obtain the first best utility. If the buyer offers a high three-unit bundle price of $c_L + 2c_H$, both types of sellers will accept the offer, and type $\underline{\theta}$ earns a rent of $c_H - c_L$. Suppose that this purchasing strategy is suboptimal for the buyer because his expected utility is higher from offering a lower three-unit bundle price of $2c_L + c_H$, which allows him to buy from only type $\underline{\theta}$ but without giving her any rent. Observe that the buyer can further increase his expected utility by complementing this low bundle price of $2c_L + c_H$ with a low per-unit price offer of c_L, which will induce type $\bar{\theta}$ to sell her one unit of L, generating an additional surplus for the buyer. However, it is not immediately clear

[3]Note that in our main analysis, to simplify the language, we refer to the seller's "type" as simply the quantity of L in her endowment.

if the buyer can improve his utility even further by offering some two-unit bundle option to also buy one H from type $\bar{\theta}$.

Our key result — that trade in any constrained Pareto optimum must have the "threshold property" — implies that the answer is no. The threshold property implies that each type of seller either sells her entire endowment or sells all of her Ls but none of her Hs. Thus, if the buyer finds it worthwhile to offer a two-unit bundle to entice type $\bar{\theta}$ to sell an H, he must also find it worthwhile to entice type $\bar{\theta}$ to sell her entire endowment, which has been assumed not to be the case. This means that the buyer's optimal purchasing strategy is to offer a menu that consists of the low three-unit bundle price (meant for $\underline{\theta}$) and the low per-unit price (meant for $\bar{\theta}$).

In the classic lemon market models — for example, Akerlof (1970), where the seller owns an indivisible unit of a good, and Attar et al. (2011), where the seller owns a divisible unit with homogenous quality —, when quality is binary, the equilibrium trade always takes one of the following two forms: either all of the units in the economy are traded, or only the Ls and none of the Hs are traded. By contrast, in the example here, whether an H unit is traded also depends on which type of seller owns it — an H unit is traded if it belongs to type $\underline{\theta}$ but is not traded if it belongs to type $\bar{\theta}$.

The remainder of this paper proceeds as follows. We first discuss the related literature in the next subsection. Next, we present our model in Section 2 and provide the main analysis in Sections 3 and 4. We then describe a few extensions in Section 5 and, finally, we conclude in Section 6. Unless stated otherwise, all of the proofs are provided in Appendix A, and an Online Appendix provides additional details.

1.1 Related literature

Within the adverse selection literature, our paper is most closely related to works on the lemon market with a divisible good. For example, Rothschild and Stiglitz (1976), Attar et al. (2011, 2014, 2017), and Ales and Maziero (2016) consider competition on the buyer's side; Stiglitz (1977) and Chade and Schlee (2012, 2020) consider the monopsony case; and Gerardi et al. (2022) consider a dynamic model. Our main point of departure is that we consider *heterogeneous* quality within the seller's endowment; this leads to within-type adverse selection, which is absent in the aforementioned papers.[4]

[4] Nguyen (2022) empirically studies the social health insurance market in Vietnam and shows that bundling health insurance at the household level mitigates inefficiency due to an analogous form of within-type adverse selection, where each household privately knows the health risks of its members and self-selects only the high risk members into insurance.

Huangfu and Liu (2022) study a bargaining problem with adverse selection where there are two goods, a seller owns one indivisible unit of each good, and the quality of her two goods are correlated. Crocker and Snow (2011) study a similar setup framed as an insurance screening problem in which the insuree has multiple perils with correlated risk type. Our setting is substantially different from theirs because we have only one good, although the seller has multiple units of this good with potentially different quality levels.

Our paper is also related to Samuelson (1984), who considers the bargaining problem with one-sided incomplete information, with the seller having an indivisible unit of a good. Because of the linearity in the preferences in Samuelson's model, the buyer cannot benefit from offering a menu of contracts to screen the seller. Our preference specification is similar to Samuelson's; however, because our seller's endowment is divisible and contains portions with different quality levels, single-crossing is satisfied in our setup. Another notable difference is that in our model, trade in the Pareto optimal allocations, including the seller-optimal and the buyer-optimal ones, all feature the threshold property, whereas in Samuelson's analysis, the trade pattern in the buyer-optimal equilibrium can be quite different from that in the seller-optimal equilibrium.

One of the reasons that the threshold property arises in our model is due to within-type adverse selection; thus, one has to run down the seller's endowment of lemons using low prices for the early units. Such a trade pattern is reminiscent of that in dynamic lemon markets (e.g., Daley and Green (2012), Fuchs et al. (2016), Kim (2017), Kaya and Kim (2018) and Gerardi et al. (2022)), where the low-quality sellers are "skimmed off" over time with low prices in the early stages.[5] However, there are two substantial differences between our paper and the aforementioned literature. First, in a dynamic lemon market, the buyer's cost of offering a low price for the early units is a delay in trade with the high-quality seller, whereas the associated cost in our model is the potential of completely driving a seller with many high-quality units out of the market. Second, in a dynamic lemon market, the buyer usually has no commitment to future prices, whereas the prices of each unit in our model are determined from the start.[6]

Finally, our paper is also related to the literature on commodity bundling (e.g., Adams

[5]The basic trade pattern is as follows: the buyer offers a low price in the early stages, and a low-quality seller randomizes between accepting and rejecting/waiting, whereas a high-quality seller always waits. Over time, the buyer becomes more confident of the quality and more willing to offer a high price.

[6]This distinction on the commitment to prices is of particular relevance to Gerardi et al. (2022), who consider the case in which the seller's good is divisible and the buyer can make offers to buy small portions of it over time. They show that when there is diminishing marginal utility from consumption, the lack of commitment to future contracts harms the buyer à la Coasian dynamics.

and Yellen, 1976; McAfee et al., 1989; Fang and Norman, 2006; Armstrong, 2013; Chen and Riordan, 2013; Chen and Li, 2018), whereby a multiproduct seller finds it worthwhile to offer a discount price for a predetermined basket of *distinct* goods. In this literature, bundling refers to a trade of (possibly multiple units of) at least two distinct goods. By contrast, in our model, there are multiple units of only *one* good, and bundling simply refers to trading high quantities instead of individual units. Thus, the logic behind the gains from bundling is quite different.

2 Model

A buyer (he) faces a seller (she), who is endowed with a continuum of a good with a total measure of one. Each marginal unit of the good is of either quality L or quality H. For brevity, we call an L-quality (H-quality) unit L (H). The seller's endowment can consist of both Ls and Hs, and the quality of each marginal unit is the seller's private information. The buyer's valuations of L and H are v_L and v_H, respectively, and the seller's opportunity costs of L and H are c_L and c_H, respectively, with $c_L < c_H$. Let the trade surplus for L and H be denoted by $s_L := v_L - c_L$ and $s_H := v_H - c_H$, respectively. We assume that $s_H, s_L > 0$, implying a trade of any unit is always socially efficient.

A trade contract is described by (q, t), where $q \in [0, 1]$ is the quantity supplied by the seller to the buyer, and $t \in \mathbb{R}$ is the monetary transfer from the buyer to the seller. If a trade (q, t) occurs with the seller fulfilling the quantity obligation with x_L units of L and x_H units of H, where $x_L + x_H = q$, then the buyer's utility is $x_L v_L + x_H v_H - t$, and the seller's utility is $t - x_L c_L - x_H c_H$. In the absence of a trade, both of the players' outside options are zero. Because $c_H > c_L$, the seller will first run down her Ls before providing any H.[7] Let $[x]^+ := \max\{x, 0\}$. Thus, the cost for a seller with λ units of L to supply q units is $C(q, \lambda)$, and the buyer's valuation of these q units is $V(q, \lambda)$, where

$$C(q, \lambda) := qc_L + [q - \lambda]^+ (c_H - c_L), \tag{1}$$

$$V(q, \lambda) := qv_L + [q - \lambda]^+ (v_H - v_L). \tag{2}$$

Therefore, the seller's private information is fully captured by λ, which is henceforth referred to as the *seller's type*. Under a trade (q, t) between the buyer and the type-λ seller, the

[7] We assume that after agreeing to trade q units, the choice of which q units to trade is determined by the seller.

utilities of the buyer and the seller are $V(q,\lambda)-t$ and $t-C(q,\lambda)$, respectively. Let F denote the commonly known distribution of λ. We maintain the following assumption throughout:

Assumption 1. *F is continuous and admits a density f that is strictly positive over $(0,1)$. The hazard rate $f/(1-F)$ is nondecreasing.*

Assumption 1 is a common assumption in the mechanism design literature and is satisfied by many common distributions, including the uniform distribution, Beta distribution, and the truncated normal and logistic distributions.

Remark 1. The crucial distinction between the two players in the model is that only one of them has private information. Our convention is that the player with (without) private information is labeled the "seller" ("buyer"). These labels are not important because we do not restrict the signs of v, c and t. For example, in the insurance market, the roles of the "buyer" and the "seller" are different from their literal meanings — the "buyer" is the insurance company, the "seller" is the insuree, and the "good" is the insuree's risk. In this case, the valuation (v) and the cost (c) of the good are both negative; therefore, the transfer t is negative, representing an insurance premium paid by the insuree to the insurance company.

2.1 Single-Unit Seller Benchmark.

To compare our results with the classic lemon market models, we first provide a "*single-unit seller benchmark*" that fixes the quantity and quality of the units in the economy, but each unit is now owned by a different seller. Specifically, in this benchmark economy, instead of facing one seller, the buyer faces a continuum of sellers of a total measure of one, but each seller owns only a marginal unit of the good. Thus, the total measure of goods in the economy is still one. The probability of facing a seller with an L is $E[\lambda] := \int_0^1 \lambda f(\lambda)\, d\lambda$, and the buyer's expected valuation of each marginal unit is $E[v] := E[\lambda]\, v_L + (1-E[\lambda])\, v_H$.

Say that there is *severe (mild) adverse selection* if $E[v] < (\geq)\, c_H$. Regardless of how the trade contracts are determined, there are only two possible equilibrium outcomes in this benchmark economy, as follows: either (i) all the Ls but none of the Hs in the economy are traded, or (ii) every unit in the economy is traded. In particular, under severe adverse selection, the trade outcome is always (i), whereas outcome (ii) can arise only under mild adverse selection. Akerlof (1970) establishes this result in a market equilibrium where the players are price-takers, and Attar et al. (2011) provide a strategic foundation for this result in an environment in which the seller has a divisible unit and there is nonexclusive competition on the buyer's side.

3 Constrained Pareto Optimal Allocations

We study the set of Pareto optimal allocations that satisfy the players' incentive and participation constraints in our model. To define the problem, let $\{q(\lambda), t(\lambda)\}_{\lambda\in[0,1]}$ denote a direct mechanism.[8] Given a direct mechanism, let $U^B(\lambda) = V(q(\lambda), \lambda) - t(\lambda)$ and $U^S(\lambda) = t(\lambda) - C(q(\lambda), \lambda)$. For a given $b \geq 0$, define program ($\mathcal{P}$) as follows:

$$\max_{\{q(\lambda),t(\lambda)\}_{\lambda\in[0,1]}} \int_0^1 U^S(\lambda) f(\lambda) d\lambda, \qquad \text{s.t.} \qquad (IC^S), (IR^S) \text{ and } (IR^B) \quad , \tag{$\mathcal{P}$}$$

where

$$U^S(\lambda) \geq t(\lambda') - C(q(\lambda'), \lambda) \qquad \forall \lambda, \lambda', \tag{IC^S}$$

$$U^S(\lambda) \geq 0 \qquad \forall \lambda, \tag{IR^S}$$

$$\int_0^1 U^B(\lambda) f(\lambda) d\lambda \geq b. \tag{IR^B}$$

Program ($\mathcal{P}$) finds the mechanism that maximizes the seller's expected utility, subject to the mechanism satisfying every type of seller's truth-telling and participation constraints and also providing the buyer an expected utility of at least b. By varying b, the player's utilities in the solutions to program ($\mathcal{P}$) characterize the informationally constrained Pareto optimal allocations, which are hereafter referred to (for short) as the *second best (SB) allocations*.

3.1 Threshold Property of Second Best Allocations

Let $\mathcal{I}$ be the indicator function, where $\mathcal{I}(x) = 1$ if x holds, and $\mathcal{I}(x) = 0$ otherwise.

Lemma 1. *The solution to program ($\mathcal{P}$) must satisfy the following conditions:*

$$U^S(\lambda) = U^S(0) + (c_H - c_L) \int_0^\lambda \mathcal{I}(q(l) > l) dl \quad \forall \lambda, \tag{3}$$

$$q(\cdot) \text{ is nondecreasing.} \tag{4}$$

[8] By the revelation principle, it is without loss of generality to consider only direct mechanisms. We consider only deterministic mechanisms for now. We discuss stochastic mechanisms in Section 5.4.

Constraint (IC^S) holds whenever these two conditions are satisfied.

Using standard envelope theorem arguments, conditions (3) and (4) are jointly sufficient conditions for the seller's truth-telling constraint (IC^S). However, condition (4) is not a necessary condition — i.e., a nonmonotonic $q(\cdot)$ can still be incentive-compatible because single-crossing holds only weakly here. Nevertheless, we can show that an incentive-compatible but nonmonotonic $q(\cdot)$ must be strictly suboptimal for program ($\mathcal{P}$). Therefore, we can still restrict our attention to only monotonic $q(\cdot)$.

From equation (3), $U^S(\lambda)$ increases with λ; thus, the seller's participation constraint (IR^S) holds for all types if it holds for type $\lambda = 0$. Moreover, equation (3) implies that

$$t(\lambda) = U^S(0) + (c_H - c_L)\int_0^\lambda \mathcal{I}(q(l) > l)\, dl + C(q(\lambda), \lambda). \tag{5}$$

By substituting this expression of $t(\lambda)$ into the objective function of program ($\mathcal{P}$) and the buyer's expected utility in (IR^B), we can show that (IR^B) must bind at the optimum because if it does not bind, we can shift some utility from the buyer to the seller by increasing $U^S(0)$ via increasing the transfer schedule uniformly.[9]

Putting these observations together, program ($\mathcal{P}$) becomes program ($\tilde{\mathcal{P}}$), as follows:

$$\max_{q(\cdot), U^S(0)} \quad \int_0^1 S(q(\lambda), \lambda) f(\lambda)\, d\lambda - b \qquad \text{s.t.}$$

$$q(\cdot) \text{ is nondecreasing}, \quad \underbrace{U^S(0) \geq 0}_{\left(\tilde{IR}^S\right)}, \quad \text{and} \quad \underbrace{\int_0^1 \psi^B(q(\lambda), \lambda)\, d\lambda - U^S(0) = b}_{\left(\tilde{IR}^B\right)}, \tag{$\tilde{\mathcal{P}}$}$$

where

$$\begin{aligned}
S(q, \lambda) &:= V(q, \lambda) - C(q, \lambda) \\
\psi^S(q, \lambda) &:= (c_H - c_L)\,\mathcal{I}(q > \lambda)\,[1 - F(\lambda)] \\
\psi^B(q, \lambda) &:= S(q, \lambda) f(\lambda) - \psi^S(q, \lambda).
\end{aligned}$$

In words, $S(q, \lambda)$ is the total surplus from trading q units with type λ. Fixing the type-0 seller's utility, which is denoted by $U^S(0)$, $\psi^S(q, \lambda)$ is the type-λ seller's virtual information rent beyond $U^S(0)$, and $\psi^B(q, \lambda)$ is the buyer's share of the trade surplus when trading with

[9]See Lemma 4 in Appendix A for the formal argument.

type λ.[10] Thus, program ($\tilde{\mathcal{P}}$) is equivalent to maximizing the total expected surplus among all nondecreasing quantity schedule $q(\cdot)$, subject to the participation constraint of the type-0 seller and the buyer receiving *exactly* an expected utility of b.

The solution to program ($\tilde{\mathcal{P}}$) gives the optimal quantity schedule for program ($\mathcal{P}$). Let

$$\bar{b} := \max_{\text{nondecreasing } q(\cdot)} \int_0^1 \psi^B(q(\lambda), \lambda)\, d\lambda \tag{6}$$

If $b > \bar{b}$, it is impossible to simultaneously satisfy both constraints ($\tilde{IR}^S$) and ($\tilde{IR}^B$).

Definition 1. $q(\cdot)$ is a "*threshold schedule*" if there exists a threshold $x \in [0,1]$ such that $q(\lambda) = \lambda$ for all $\lambda < x$ and $q(\lambda) = 1$ for all $\lambda \geq x$.

Proposition 1. *For all $b \in \left[0, \bar{b}\right]$, a solution to program ($\tilde{\mathcal{P}}$) exists. If $q(\cdot)$ is a solution to program ($\tilde{\mathcal{P}}$), then $q(\cdot)$ must be a threshold schedule.*

Under a threshold schedule, there is never a partial trade of the seller's endowment of a particular quality — if the seller is of type $\lambda < x$, she trades all of her Ls but none of her Hs; if she is of type $\lambda \geq x$, she trades her entire endowment. Proposition 1 implies that any SB allocation must involve a trade with such a property. This result is useful because each threshold schedule is fully characterized by its scalar threshold; thus, the SB allocation can be obtained by optimizing over the set of possible thresholds, $[0,1]$.

3.1.1 Sketch of proof of Proposition 1

We explain the main arguments for Proposition 1 next. A reader who prefers to skip these details can jump to Subsection 3.2, where we use the threshold property of the optimal schedule to derive the set of SB allocations.

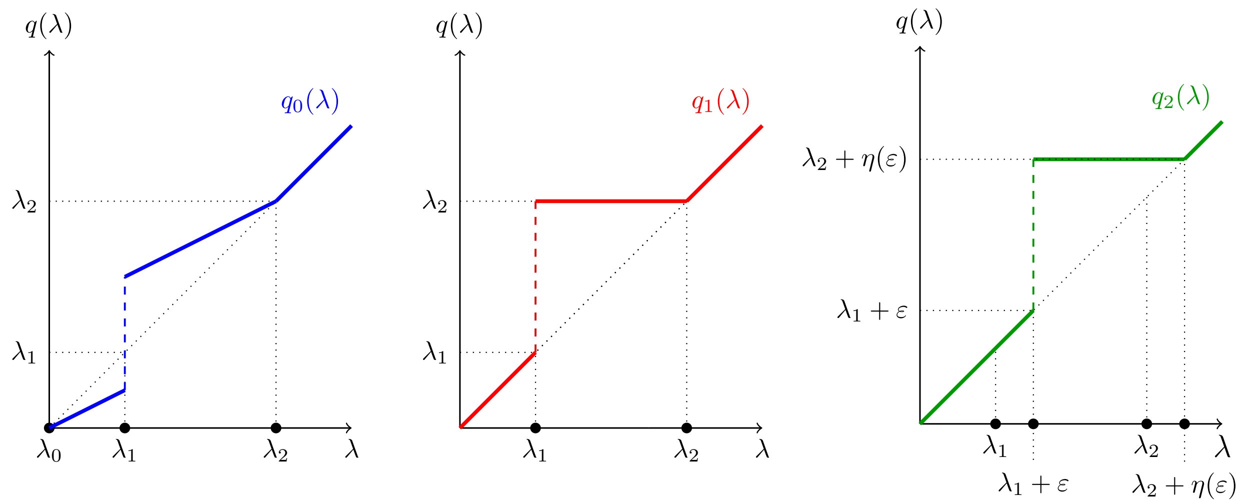
First, observe that both $S(q, \lambda)$ and $\psi^B(q, \lambda)$ are increasing in q when $q \neq \lambda$.[11] This implies that if $q(\cdot)$ is a solution to program ($\tilde{\mathcal{P}}$), it must have the following two properties: (i) $q(\lambda)$ cannot be smaller than λ, and (ii) if $q(\lambda) > \lambda$ over some interval (λ_1, λ_2), then $q(\cdot)$ must exhibit bunching in (λ_1, λ_2). To see these two points visually, consider quantity schedule $q_0(\cdot)$ in the left panel (in blue) of Figure 1. By increasing the quantities for each $\lambda \in (\lambda_0, \lambda_1)$ to $q(\lambda) = \lambda$ and for each $\lambda \in (\lambda_1, \lambda_2)$ to $q(\lambda) = \lambda_2$ — see schedule $q_1(\cdot)$ in the

[10] Note that this means that the seller's expected utility can also be expressed as $\int_0^1 \psi^S(q(\lambda), \lambda)\, d\lambda + U^S(0)$, and the buyer's expected utility is the left-hand side of constraint ($\tilde{IR}^B$).

[11] $S(q, \lambda)$ always increases with q because s_L and s_H are both positive. $\psi^S(q, \lambda)$ is constant over $q \in [0, \lambda)$ and $q \in (\lambda, 1]$ but has a discrete jump at $q = \lambda$. Thus $\psi^B(q, \lambda)$ increases with q when $q \neq \lambda$, but it has a discrete jump at $q = \lambda$ that is possibly downward.

Figure 1: On the Suboptimality of Nonthreshold Schedules

(The diagonal dotted lines are the 45-degree lines.)



$q_0(\cdot)$ is suboptimal relative to $q_1(\cdot)$ because there are types in which $q_0(\lambda) < \lambda$.
When $\lambda_2 < 1$, it is possible to construct schedule $q_2(\cdot)$ that dominates $q_1(\cdot)$, where $\eta(\cdot)$ is defined in equations (8) and (9).

middle panel (in red) — the schedule is still nondecreasing, but both the objective value and the buyer's surplus on the left-hand side of constraint ($\tilde{IR}^B$) become higher because of the properties of S and ψ^B noted above. By increasing $U^S(0)$ accordingly, constraint ($\tilde{IR}^B$) will also hold for schedule $q_1(\cdot)$, which thus makes it a feasible and better schedule than $q_0(\cdot)$.

Next, we claim that schedule $q_1(\cdot)$ is strictly suboptimal if $\lambda_2 < 1$, which will then imply that the solution $q(\cdot)$ must be a threshold schedule. This result relies on the following statistical property of F, which is proven in Appendix A:

Lemma 2. *When $f/(1-F)$ is nondecreasing, the following property holds: for any $\lambda_1 < \lambda_2$, if $\varepsilon_1, \varepsilon_2 > 0$ are such that $\int_{\lambda_1+\varepsilon_1}^{\lambda_2+\varepsilon_2} 1 - F(\lambda)\, d\lambda = \int_{\lambda_1}^{\lambda_2} 1 - F(\lambda)\, d\lambda$, then*

$$\int_{\lambda_1+\varepsilon_1}^{\lambda_2+\varepsilon_2} \Big[(\lambda_2+\varepsilon_2) - \lambda\Big] f(\lambda)\, d\lambda \;>\; \int_{\lambda_1}^{\lambda_2} [\lambda_2 - \lambda]\, f(\lambda)\, d\lambda. \tag{7}$$

To explain why Lemma 2 implies that $\lambda_2 < 1$ is always suboptimal, first note that under schedule $q_1(\cdot)$, each of types $\lambda \in [\lambda_1, \lambda_2)$ sells $(\lambda_2 - \lambda)$ units of H and derives some information rent. Let the expected virtual rent given to the types in $[\lambda_1, \lambda_2)$ be denoted by

$$R = \int_{\lambda_1}^{\lambda_2} \psi^S(q_1(\lambda), \lambda)\, d\lambda = (c_H - c_L) \int_{\lambda_1}^{\lambda_2} 1 - F(\lambda)\, d\lambda. \tag{8}$$

Instead of obtaining Hs from the types in $[\lambda_1, \lambda_2)$, consider now obtaining Hs from the types in $[\lambda_1 + \varepsilon, \lambda_2')$ for some $\varepsilon > 0$. (Note that this is possible only if $\lambda_2 < 1$.) If we restrict that the rent provided to these types is R, then λ_2' is some $\lambda_2 + \eta(\varepsilon)$ characterized (implicitly) by

$$R \;=\; (c_H - c_L)\int_{\lambda_1+\varepsilon}^{\lambda_2+\eta(\varepsilon)} 1 - F(\lambda)\, d\lambda. \tag{9}$$

This is illustrated by quantity schedule $q_2(\cdot)$ in the right panel (in green) of Figure 1. In terms of the surplus generated, the upside of schedule $q_2(\cdot)$ is that each λ within the interval selling some Hs now sells $\lambda_2 + \eta(\varepsilon) - \lambda$ units of H, which is an increase of $\eta(\varepsilon)$ units per type relative to under $q_1(\cdot)$. However, the downside is that the types in $[\lambda_1, \lambda_1 + \varepsilon)$ no longer sell their Hs and are replaced by the types in $[\lambda_2, \lambda_2 + \eta(\varepsilon))$, who have fewer units of H each. The resulting net change in the expected surplus (i.e., objective value of program ($\tilde{\mathcal{P}}$)) is

$$\begin{aligned}\delta_\varepsilon =& \int_0^1 S(q_2(\lambda), \lambda) f(\lambda)\, d\lambda - \int_0^1 S(q_1(\lambda), \lambda) f(\lambda)\, d\lambda \\ =& s_H \left[\int_{\lambda_1+\varepsilon}^{\lambda_2+\eta(\varepsilon)} (\lambda_2 + \eta(\varepsilon) - \lambda) f(\lambda)\, d\lambda - \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \lambda) f(\lambda)\, d\lambda\right]\end{aligned} \tag{10}$$

From equations (8) and (9), Lemma 2 implies that δ_ε is strictly positive. Note that $\int_0^1 \psi^B(q_2(\lambda), \lambda) - \psi^B(q_1(\lambda), \lambda)\, d\lambda = \delta_\varepsilon$ as well. Thus, if $q_1(\cdot)$ satisfies constraint ($\tilde{IR}^B$), we can increase the value of $U^S(0)$ by δ_ε; in turn, $q_2(\cdot)$ will also satisfy constraint ($\tilde{IR}^B$) under this new value of $U^S(0)$. This shows that $q_2(\cdot)$ is feasible but attains a higher objective value than $q_1(\cdot)$. Thus, $q_1(\cdot)$ is strictly suboptimal if $\lambda_2 < 1$.

Summarizing the previous argument, to extract Hs from the seller, we must give her information rent. The quantity of H (and hence surplus) that we can extract from each type depends on the type interval chosen to extract the Hs from. Fixing the rent to allocate, Lemma 2 implies that when $f/(1-F)$ is nondecreasing, the surplus generated is always higher when extracting the Hs from (and giving this rent to) the higher types. This property leads to the threshold property of the solution quantity schedule.

Remark 2. The role of the monotone hazard rate property here is quite different from that in standard mechanism design problems. In standard mechanism design problems, the monotone hazard rate property ensures that the point-wise optimum of the virtual surplus is monotonic in the type, hence making it the solution to the mechanism design problem. By contrast, even when the hazard rate is monotonic here, the point-wise optimum is still generally *not* monotonic and is hence not a solution because it violates incentive-compatibility. Instead, as explained above, the monotone hazard rate property ensures that the solution quantity schedule has the threshold property, and we can then use this property to solve for the solution in the next subsection.

3.2 The Set of Second Best Allocations

By Proposition 1, for program ($\tilde{\mathcal{P}}$), we can optimize over only threshold schedules, which are parametrized by their (scalar) thresholds. For brevity, call the threshold schedule with threshold x the threshold-x schedule. When $q(\cdot)$ is the threshold-x schedule, the values of $\int_0^1 S(q(\lambda),\lambda) f(\lambda)\, d\lambda$ and $\int_0^1 \psi^S(q(\lambda),\lambda)\, d\lambda$ are, respectively,

$$\hat{S}(x) := E[\lambda]\, s_L + \int_x^1 (1-\lambda)\, s_H f(\lambda)\, d\lambda, \tag{11}$$

$$\hat{\psi}^S(x) := \int_x^1 (c_H - c_L)[1 - F(\lambda)]\, d\lambda. \tag{12}$$

Restricting attention to only threshold schedules, program ($\tilde{\mathcal{P}}$) becomes

$$\max_{x\in[0,1],\ U^S(0)\geq 0} \quad \hat{S}(x) - b \qquad \text{s.t.} \qquad \hat{S}(x) - \hat{\psi}^S(x) - U^S(0) = b. \tag{13}$$

$\hat{S}(x)$ is strictly decreasing in x. Therefore, if $\hat{S}(0) - \hat{\psi}^S(0) \geq b$, the solution to program (13) is $x = 0$ and $U^S(0) = \hat{S}(0) - \hat{\psi}^S(0) - b$. If $\hat{S}(0) - \hat{\psi}^S(0) < b$ instead, the optimal threshold x for program (13) is the smallest x satisfying the program's constraint. This implies that $U^S(0)$ must be zero. Thus, in this case, the optimal x is

$$\lambda^*(b) := \min \Lambda(b)\ , \qquad \text{where} \qquad \Lambda(b) := \left\{ z \middle| z \in [0,1] \text{ and } \hat{S}(z) - \hat{\psi}^S(z) = b \right\}. \tag{14}$$

Note that because $\bar{b}$ is also equal to $\max_{x\in[0,1]} \hat{S}(x) - \hat{\psi}^S(x)$, the set $\Lambda(b)$ is nonempty for all $b \in \left[\hat{S}(0) - \hat{\psi}^S(0), \bar{b}\right]$, meaning that $\lambda^*(b)$ is well-defined. The solution to program (13) gives the optimal $q(\cdot)$ for program ($\mathcal{P}$), and the optimal $t(\cdot)$ is obtained from equation (5).

Proposition 2. *For all $b \in \left[0, \bar{b}\right]$, program ($\mathcal{P}$) has a unique solution.*

- *(A1): If $b \leq E[v] - c_H$, the solution is $(q^*(\lambda), t^*(\lambda)) = (1, E[v] - b)$ for all $\lambda \in [0,1]$. The seller's expected utility is $\hat{S}(0) - b$.*

- *(A2): If $b > E[v] - c_H$, the solution is*

$$(q^*(\lambda), t^*(\lambda)) = \begin{cases} (\lambda, \lambda c_L) & , \ \forall \lambda < \lambda^*(b) \\ (1, C(1, \lambda^*(b))) & , \ \forall \lambda \geq \lambda^*(b) \end{cases}.$$

The seller's expected utility is $\hat{\psi}^S(\lambda^*(b))$, and $\lambda^*(b)$ is strictly positive and strictly increasing in b.

Proposition 2 characterizes the set of SB allocations and the unique direct mechanism that attains each SB allocation. The argument for the proposition follows from the discussion above and also noting that $\hat{S}(0) - \hat{\psi}^S(0) = E[v] - c_H$.

To highlight some general features of the SB allocations, we categorize them into two classes — A1 and A2. In an A1 SB allocation, trade occurs under the threshold-0 schedule — i.e., every unit is traded. This is because the type-0 seller's participation constraint is slack (i.e., $U^S(0) > 0$). Thus, from constraint $\left(\tilde{IR}_B\right)$ in program $(\tilde{\mathcal{P}})$, utility can be transferred between the buyer and the seller without any efficiency loss. Therefore, the sum of the two players' utilities is always the *first best surplus* of $\hat{S}(0)$, which is the surplus derived when all the units are traded. An A1 SB allocation can exist only if there is mild adverse selection, but mild adverse selection does not guarantee that all the SB allocations are of class A1.[12]

By contrast, in an A2 SB allocation, trade occurs under the threshold-$\lambda^*(b)$ schedule, where $\lambda^*(b)$ is strictly positive. Here, the type-0 seller's participation constraint binds and transferring utility from the seller to the buyer requires distortion away from the first best trade. This distortion worsens with the buyer's required expected utility b because $\lambda^*(b)$ increases with b and the total expected surplus decreases with the threshold. When there is severe adverse selection, all the SB allocations are of class A2, but there can also be A2 SB allocations under mild adverse selection.

As for the mechanism, each SB allocation involves a trade described by some threshold-x schedule, where each type $\lambda < x$ sells all of her Ls (and none of her Hs) for the price of λc_L, and each type $\lambda \geq x$ sells her entire endowment at the price of the cost of type x's endowment, $C(1,x)$. The following are two possible ways of implementing this mechanism: the first is "mixed bundling," where the seller is offered the options to either sell her entire endowment at a full bundle price of $C(1,x)$ or sell à la carte at a marginal price of c_L. The second way is "marginal pricing with a switch," where the seller is offered a marginal price of c_L for the first x units and upon selling x units, the seller is offered a marginal price of c_H for all her subsequent units. When the transfers are negative (e.g., in insurance), this latter way is simply a quantity discount.

The buyer-optimal (seller-optimal) SB allocation is the allocation of program $(\mathcal{P})$ under $b = \bar{b}$ $(b = 0)$. Let the optimal threshold of the buyer-optimal and seller-optimal programs be denoted by λ^{B*} and λ^{S*}, respectively. Proposition 2 implies that $\lambda^{S*} \leq \lambda^{B*}$, with the

[12]Recall that there is mild (severe) adverse selection if $E[v] - c_H \geq (<)\, 0$, and $E[v] - c_H = \hat{S}(0) - \hat{\psi}^S(0)$.

inequality holding strictly unless $\lambda^{S*} = 1$ or $\lambda^{B*} = 0$. Moreover, if the threshold-x schedule is the trade schedule for some SB allocation, it must be the case that $x \in \left[\lambda^{S*}, \lambda^{B*}\right]$. This implies the following efficiency ranking:

Corollary 1. *The buyer-optimal (seller-optimal) SB allocation always attains the lowest (highest) total surplus among all the SB allocations.*

The observation that every optimal threshold x must be in the set $\left[\lambda^{S*}, \lambda^{B*}\right]$ also implies that in *any* SB allocation, a type $\lambda < \lambda^{S*}$ always sells only her Ls and none of her Hs, whereas a type $\lambda \geq \lambda^{B*}$ always sells her entire endowment. By contrast, the trade for a type in $\left(\lambda^{S*}, \lambda^{B*}\right)$ — assuming that this set of types is nonempty — depends on the particular SB allocation. For example, in the seller-optimal SB allocation, every such type continues to sell her entire endowment, whereas in the buyer-optimal SB allocation, every such type sells only her Ls.

As an example, Figure 2 plots the set of SB allocations for the case where F is the uniform distribution. The details of the associated closed-form solution are found in Appendix B. When adverse selection is very mild ($s_H \geq c_H - c_L$), all the SB allocations are of class A1; thus, the slope of the frontier is always -1. When adverse selection is only moderately mild ($s_H + s_L \geq c_H - c_L > s_H$), both classes of SB allocations are possible. The slope of the frontier for the A2 allocations is always steeper than -1, and, for the specific case of the uniform distribution, this slope is the constant $-\frac{c_H - c_L}{c_H - c_L - s_H}$.[13]

4 Pareto-improving Distribution Shifts

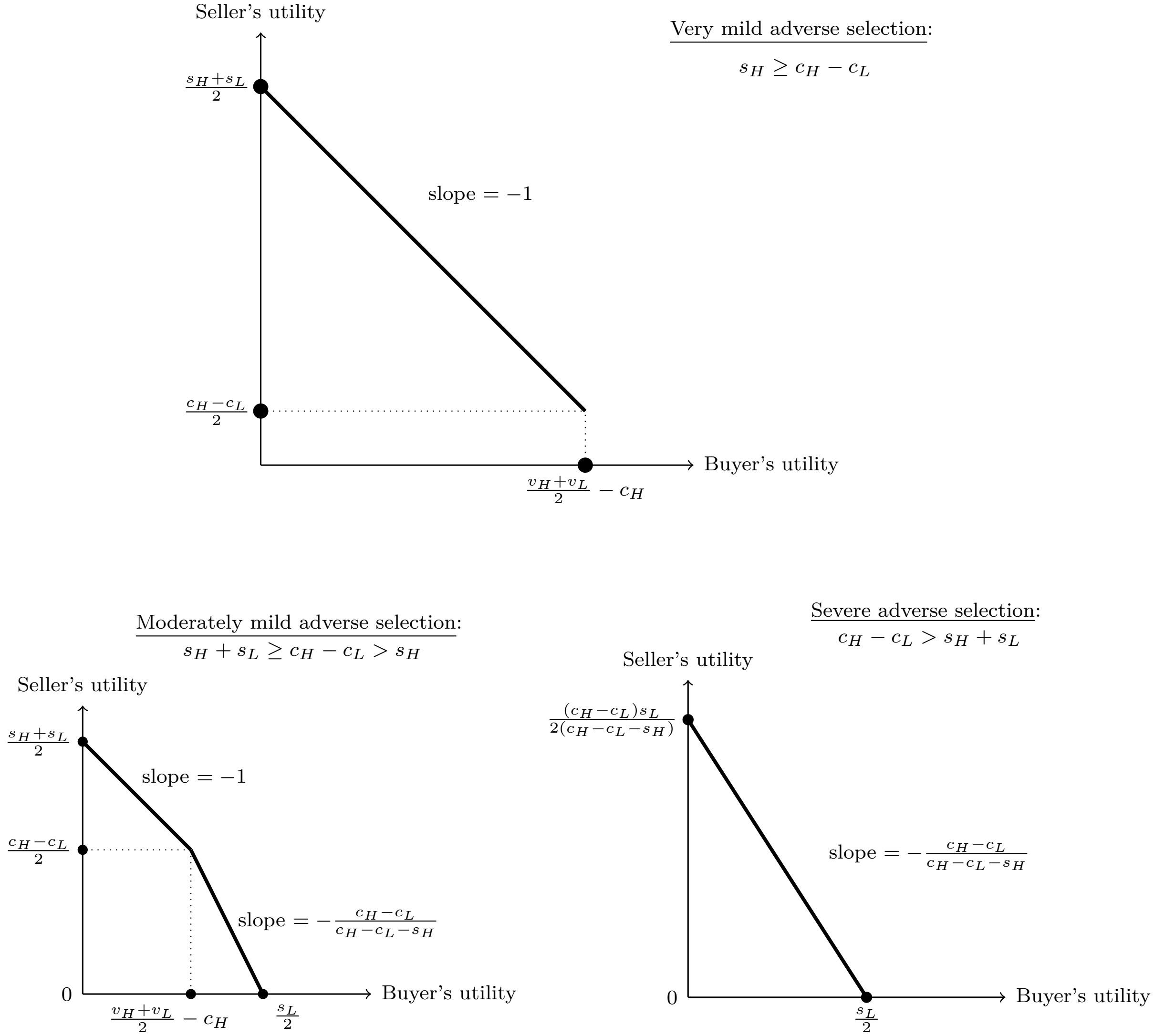
Fix some SB allocation of distribution F and let it be denoted by A_F. Let G be another distribution of λ and assume that G has the same mean as F and satisfies Assumption 1.[14] In this section, we derive conditions for G such that there is an SB allocation of distribution G that Pareto dominates A_F. Let $\hat{\lambda}_F$ denote the threshold of the threshold quantity schedule associated with allocation A_F. To ensure that the analysis is nonvacuous, we restrict our attention to only A_F in which $\hat{\lambda}_F \in (0, 1)$.[15]

[13] Note that the linear frontier for the A2 SB allocations in Figure 2 is specific to the uniform distribution. The frontier is generally nonlinear and Online Appendix D.1 provides a sufficient condition on F for the frontier to be concave.

[14] i.e., $\int_0^1 \lambda dG(\lambda) = \int_0^1 \lambda dF(\lambda)$, G has a density that has full support over $(0, 1)$, and the hazard rate of G is nondecreasing.

[15] When G and F have the same mean, the first best surplus values of the two distributions are the same. If $\hat{\lambda}_F = 0$, the total expected surplus in A_F is the first best surplus; thus, it is impossible for A_F to be Pareto

Figure 2: Constrained Pareto Frontier under the Uniform Distribution



For $x \in [0,1]$, define $\gamma(x)$ implicitly by

$$\int_{\gamma(x)}^{1} 1 - G(\lambda)\, d\lambda = \int_{x}^{1} 1 - F(\lambda)\, d\lambda. \tag{15}$$

$\gamma(x)$ is well-defined because G is strictly increasing.

Proposition 3. *There exists a SB allocation of distribution G that Pareto dominates A_F if and only if*[16]

$$\left(1 - \gamma\left(\hat{\lambda}_F\right)\right)\left[1 - G\left(\gamma\left(\hat{\lambda}_F\right)\right)\right] \; > \; \left(1 - \hat{\lambda}_F\right)\left[1 - F\left(\hat{\lambda}_F\right)\right]. \tag{16}$$

Let $\hat{q}_F^{\gamma}$ denote the threshold schedule with threshold $\gamma\left(\hat{\lambda}_F\right)$. To briefly explain the role of each condition in Proposition 3, first note that from equations (12) and (15), the seller's expected utility under schedule $\hat{q}_F^{\gamma}$ in distribution G is the same as her expected utility in allocation A_F. Next, the inequality in (16) is a condition that implies that the total expected surplus under schedule $\hat{q}_F^{\gamma}$ in distribution G is strictly higher than the total expected surplus in allocation A_F.[17] Therefore, if (16) holds, schedule $\hat{q}_F^{\gamma}$ in distribution G gives an allocation that Pareto dominates A_F.

A class of G that is of particular interest to us is when G is obtained from a sequence of mean-preserving contractions (MPC) on F. To elaborate on it, recall that there are two forms of adverse selection in our model — within-type and across-type. Intuitively, within-type adverse selection is absent for types $\lambda = 0$ and $\lambda = 1$ because the quality of each of their respective endowments is homogenous, but the across-type adverse selection problem is most severe between these two types because the difference in the overall quality of their endowments is the greatest. By contrast, if F is degenerate at the mean type (assumed to be in $(0,1)$), there is no across-type adverse selection and the adverse selection problem is due entirely to the seller's private information regarding which units are the Ls in her endowment — i.e., within-type. Therefore, a MPC provides a natural way to shift adverse selection from across-type to within-type, while maintaining the expected quality, and it is hence of interest to explore how such a shift can affect the set of SB allocations.

From Rothschild and Stiglitz (1970), we know that if G is obtained from a sequence of

dominated by any SB allocation of G. If $\hat{\lambda}_F = 1$, then in A_F, the seller's expected utility is zero and the buyer's expected utility is $E[\lambda]\, s_L$, meaning that every SB allocation of G either leads to the same expected utilities as A_F or Pareto dominates A_F. Therefore, in both cases, the analysis is trivial.

[16] We thank a referee for pointing out the necessity part of this condition.

[17] This point follows from doing an integration by parts on the surplus function $\hat{S}$ in equation (11), and the details are provided in equations (27) to (30) in Appendix A.

MPCs on F, then G second-order stochastically dominates F ("$G \succeq_{SOSD} F$" for short), which is equivalent to the following condition:[18]

$$\gamma(\lambda) \leq \lambda \quad \forall \lambda \in (0,1). \tag{17}$$

The following property is readily verified:

Lemma 3. *If $G\left(\hat{\lambda}_F\right) < F\left(\hat{\lambda}_F\right)$, then $\gamma\left(\hat{\lambda}_F\right) \leq \hat{\lambda}_F$ implies that inequality (16) holds.*

Therefore, if G is obtained from a sequence of MPCs on F and $G\left(\hat{\lambda}_F\right) < F\left(\hat{\lambda}_F\right)$, Proposition 3 implies that A_F is always Pareto dominated by some SB allocation of G.

More generally, say that "G dominates F" if *every* SB allocation of F is Pareto dominated by *some* SB allocation of G. From the preceding discussion, given any G that is obtained from a sequence of MPCs on F, the condition to check whether G dominates F is narrowed down to verifying inequality (16) for only the types in which $G(\lambda) \geq F(\lambda)$. For some distributions, this condition is readily checked. Example 1 below provides an example when F is the uniform distribution.

Example 1. Suppose that F is the uniform distribution.[19] Let G be another distribution that has the same mean as F and $G \succeq_{SOSD} F$ — i.e., G is obtained from a sequence of MPCs on F. If, additionally, G intersects F exactly once in $(0,1)$ — with the intersecting type denoted by λ' — and is concave over $(\lambda', 1)$, then the inequality in (16) holds for all $\lambda \in (0,1)$.[20] Thus, G dominates F. In this example, $G(\lambda)$ is greater than $F(\lambda)$ only for $\lambda \geq \lambda'$, and the inequality in (16) is readily verified to hold for such λs when G is concave and F is uniform. The proof is in Appendix A.

5 Extensions

Our characterization results rest on Proposition 1, which shows that the quantity schedule in any SB allocation must be a threshold schedule. In Online Appendix C, we extend our

[18] The commonly used definition for $G \succeq_{SOSD} F$ is $\int_0^x G(\lambda)\,d\lambda \leq \int_0^x F(\lambda)\,d\lambda$ for all $x \in (0,1)$. When G has the same mean as F, we have $\int_0^1 G(\lambda)\,d\lambda = \int_0^1 F(\lambda)\,d\lambda$. Thus, this implies that $G \succeq_{SOSD} F$ if $\int_x^1 1 - G(\lambda)\,d\lambda \leq \int_x^1 1 - F(\lambda)\,d\lambda$ for all $x \in (0,1)$. Since $1 - G(\lambda)$ is strictly positive for all $\lambda \in (0,1)$, the last inequality is equivalent to $\gamma(x) \leq x$ for all $x \in (0,1)$.

[19] As noted above, to ensure that the example is nonvacuous, we assume that the associated optimal threshold of every SB allocation of F is strictly in the set $(0,1)$.

[20] An example of such a distribution G is $G(\lambda) = 0.5 + 0.5\,(2\lambda - 1)^{\frac{1}{k}}$ for any positive and odd integer k.

baseline model in a few directions and provide sufficient conditions for an analogous version of Proposition 1 to hold in these extensions. We briefly describe these extensions below.

5.1 Nonbinary Quality

The following is a way to extend our model to incorporate more than two quality levels. Let $k \geq 2$ be the number of quality levels, where k is a finite integer. For $i \in \{1, \ldots, k\}$, let v_i and c_i denote the buyer's valuation and the seller's cost of a marginal unit of quality i, with $v_i < v_{i'}$ and $c_i < c_{i'}$ for all any $i < i'$. Let $\tau : \{0, 1, \ldots, k\} \times [0, 1] \to [0, 1]$ be a commonly known function. The seller's privately known type is summarized by a scalar $\theta \in [0, 1]$, where $\tau(i|\theta)$ is the quantity of units with a quality level less than or equal to i in type-θ seller's endowment.[21] As the seller will supply her lower quality units first, the cost for a type-θ seller to supply q units and the buyer's valuation for these q units are (with an abuse of notation), respectively,

$$C(q, \theta) = qc_1 + \sum_{i=1}^{k-1} [q - \tau(i|\theta)]^+ (c_{i+1} - c_i) \tag{18}$$

$$V(q, \theta) = qv_1 + \sum_{i=1}^{k-1} [q - \tau(i|\theta)]^+ (v_{i+1} - v_i) \tag{19}$$

In this setup, the main program is still program ($\mathcal{P}$), except that the type λ is replaced by θ. From equation (18), the seller's utility satisfies (weak) single-crossing (as in the baseline model) if for every i, $\tau(i|\theta)$ is increasing in θ.[22] With single-crossing, the argument used to transform program ($\mathcal{P}$) into program ($\tilde{\mathcal{P}}$) in the baseline model applies here.

A threshold schedule in the current setup is a schedule $q(\theta)$ in which there are thresholds $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_k$ such that if $\theta \in [\theta_{i-1}, \theta_i)$, $q(\theta) = \tau(i|\theta)$, with the convention that $\theta_0 = 0$. In words, if $\theta \in [\theta_{i-1}, \theta_i)$, then type θ sells all of her units that are of quality i or lower (and nothing else). Thus, there is never a partial trade of the seller's endowment of a particular quality. In Online Appendix C.1, we provide an illustration for the $k = 3$ case and a sufficient condition (see Condition 1) for the distribution of θ and the τ-function such that the quantity schedule of the solution to program ($\mathcal{P}$) in this setup with three quality levels is always a threshold schedule, as just defined.

[21] Thus, for type θ, the quantity of quality i is $\tau(i|\theta) - \tau(i-1|\theta)$, with the convention that $\tau(0|\theta) = 0$. This implies that for all θ, $\tau(i|\theta)$ must be nondecreasing in i and $\tau(k|\theta) = 1$. Therefore, $\tau(\cdot|\theta)$ is also the CDF of the quality of type-θ seller's endowment.

[22] Per the previous footnote, when interpreting $\tau(\cdot|\theta)$ as a CDF, single-crossing is satisfied if $\tau(\cdot|\theta)$ first-order stochastically dominates $\tau(\cdot|\theta')$ for any $\theta < \theta'$.

5.2 Asymmetric Information on Endowment Size

In Online Appendix C.2, we extend our baseline model to allow the seller's endowment size to also be her private information. The setup is as follows: let $n \in [0, 1]$ denote the endowment size. The seller now has a two-dimensional type (n, λ), with $\lambda \leq n$. Type (n, λ) has λ units of Ls and $(n - \lambda)$ units of Hs. A trade contract is still (q, t), and type (n, λ) can accept the contract only if $q \leq n$. After accepting the contract, the seller's utility is still $t - C(q, \lambda)$, and the buyer's utility is still $V(q, \lambda) - t$, with C and V defined in equations (1) and (2).

A threshold schedule in this setup is a schedule $q(n, \lambda)$ in which there is a threshold x such that if $\lambda \leq x$, type (λ, n) sells only her Ls and none of her Hs (i.e., $q(n, \lambda) = \lambda$), and if $\lambda > x$, type (λ, n) sells her entire endowment (i.e., $q(n, \lambda) = n$). In Appendix C.2, we provide a sufficient condition for the type distribution (see Condition 2) such that the quantity schedule of the solution to program ($\mathcal{P}$) in this current problem with asymmetric information on the endowment size must be a threshold schedule of the form just described.

5.3 Diminishing Marginal Utility

Our baseline model assumes that the buyer's utility for each marginal unit depends on only the quality but not the quantity. In Online Appendix C.3, we allow the buyer to have diminishing marginal utility. Specifically, when the buyer receives q units consisting of x_L units of L and x_H units of H, his total valuation for these $q = x_L + x_H$ units is $\nu_L(x_L) + \nu_H(x_H)$. We assume that for $i \in \{L, H\}$, $\nu_i(0) = 0$, ν_i is strictly increasing and strictly concave (i.e., diminishing marginal utility), and $\nu_i'(1) > c_i$, where the last assumption implies that a trade of any marginal unit is still always socially efficient. The specification for the seller remains unchanged. Thus, under a trade (q, t) with type-λ seller, the buyer's utility is (with an abuse of notation) $V(q, \lambda) - t$, where, now, $V(q, \lambda) = \nu_L(q)$ if $q \leq \lambda$, and $V(q, \lambda) = \nu_L(\lambda) + \nu_H(q - \lambda)$ if $q > \lambda$. The type-λ seller's utility is $t - C(q, \lambda)$, where the definition of C is still the same as that in equation (1). We provide a sufficient condition (see Condition 3 in Appendix C.3) for the type distribution F together with the ν_H function — without any restriction on ν_L — such that the quantity schedule of the solution to program ($\mathcal{P}$) in this setup (with diminishing marginal utility) is still always a threshold schedule, defined in Definition 1. We also show (see Lemma 10 in Appendix C.3) that Condition 3 is always satisfied when F is the uniform distribution.

5.4 Stochastic Mechanism

In Online Appendix C.4, we consider the use of stochastic mechanisms to solve program ($\mathcal{P}$).[23] Because the utility functions of both the seller and the buyer are linear in the transfers, it is sufficient to allow for stochasticity only in the quantity. Let a stochastic contract be a double (α, t), where t is still the transfer from the buyer to the seller, and α is the distribution of the quantity that the seller must supply to the buyer. Under a contract (α, t) between the buyer and the type-λ seller, the buyer's and the seller's expected utility are $\int_0^1 V(q, \lambda)\, d\alpha(q) - t$ and $t - \int_0^1 C(q, \lambda)\, d\alpha(q)$, respectively, where C and V are defined in equations (1) and (2). Thus, a direct stochastic mechanism is a menu of stochastic contracts $\{\alpha(\cdot|\lambda), t(\lambda)\}_{\lambda \in [0,1]}$.

We show that if $(1-\lambda) f(\lambda) / [1 - F(\lambda)]$ is nondecreasing in λ, the solution to program ($\mathcal{P}$) is still always the deterministic mechanism considered in our baseline analysis. This condition on the type distribution is strictly stronger than Assumption 1, but it is still satisfied by many distributions.[24] For example, it is satisfied by the uniform distribution, any power distribution $F(\lambda) = \lambda^z$ with $z \geq 1$, and the $Beta(a, b)$ distribution with shape parameters $a \geq 1$ and $b > 0$.

6 Conclusion

In this paper, we studied a model of trade with adverse selection in which a seller owns multiple units that are of potentially different quality levels. In our model, the seller not only privately knows the overall quality of her endowment (across-type adverse selection) but also the quality of each of her units (within-type adverse selection). We characterize the informationally constrained Pareto optimal allocations when the quality is binary. A key feature in all of these second best allocations is that trade always occurs via a "threshold schedule," where the types higher than some threshold sell their entire endowments, whereas the types lower than the threshold sell only their lemons (and none of their Hs).

In our model, because of within-type adverse selection, the seller's endowment of Ls must first be run down before she can be expected to supply any Hs. This feature is distinct from existing models of the lemon market and raises a few issues that could be worthwhile for

[23] Strausz (2006) shows that in screening problems, a sufficient condition for the suboptimality of a stochastic mechanism is that the optimal deterministic mechanism does not have any contract pooling. This condition is violated whenever the optimal threshold is less than 1.

[24] This condition has also been used in some dynamic mechanism design problems. See, for example, Boleslavsky and Said (2013).

further research, particularly in a market setting with multiple buyers behaving strategically.

First, our analysis is based on an exclusive bilateral trade — i.e., a seller is not allowed to sell to multiple buyers. Attar et al. (2011) have shown that when the seller's endowment is divisible but homogenous, nonexclusive competition can vastly alter the equilibrium outcome. With within-type adverse selection, nonexclusivity raises the additional problem of the buyers wanting to "free-ride" on the other buyers to run down the seller's endowment of Ls. In turn, this raises the questions of whether information on the (bilateral) trade contracts offered/accepted by a seller should be made public and whether the contracts should be allowed to be conditioned on the other contracts that the seller has entered.

Relatedly, if the seller's endowment is sold over time, information on past trade behaviors can have a very different effect under within-type adverse selection. When the quality within the seller's endowment is homogenous (as is the case in most existing models), information that a seller has previously sold some units at low prices signals to the market that the rest of her endowment is also of low quality. In contrast, with within-type adverse selection, knowing that the seller has already sold some units can increase the market's expected valuation of the units left in the seller's endowment because the seller is expected to sell her lemons first. How this feature affects the equilibrium trade dynamics is then unclear. We leave the study of these issues to future research.

References

Adams, W. J. and J. L. Yellen (1976). Commodity bundling and the burden of monopoly. *The Quarterly Journal of Economics 90*(3), 475–498.

Akerlof, G. A. (1970). The market for "lemon": Quality uncertainty and the market mechanism. *The Quarterly Journal of Economics 84*(3), 488–500.

Ales, L. and P. Maziero (2016). Non-exclusive dynamic contracts, competition, and the limits of insurance. *Journal of Economic Theory 166*, 362–395.

Armstrong, M. (2013). A more general theory of commodity bundling. *Journal of Economic Theory 148*(2), 448–472.

Attar, A., T. Mariotti, and F. Salanié (2011). Nonexclusive competition in the market for lemons. *Econometrica 79*(6), 1869–1918.

Attar, A., T. Mariotti, and F. Salanié (2014). Nonexclusive competition under adverse selection. *Theoretical Economics 9*(1), 1–40.

Attar, A., T. Mariotti, and F. Salanié (2017). Multiple contracting in insurance markets. *Working Paper*.

Boleslavsky, R. and M. Said (2013). Progressive screening: Long-term contracting with a privately known stochastic process. *Review of Economic Studies 80*(1), 1–34.

Calem, P., C. Henderson, and J. Liles (2011). "cherry picking" in subprime mortgage securitizations: Which subprime mortgage loans were sold by depository institutions prior to the crisis of 2007? *Journal of Housing Economics 20*(2), 120–140.

Chade, H. and E. Schlee (2012). Optimal insurance with adverse selection. *Theoretical Economics 7*(3), 571–607.

Chade, H. and E. E. Schlee (2020). Insurance as a lemons market: Coverage denials and pooling. *Journal of Economic Theory 189*, 105085.

Chen, Y. and J. Li (2018). Bundled procurement. *Journal of Public Economics 159*, 116–127.

Chen, Y. and M. H. Riordan (2013). Profitability of product bundling. *International Economic Review 54*(1), 35–57.

Crocker, K. J. and A. Snow (2011). Multidimensional screening in insurance markets with adverse selection. *Journal of Risk and Insurance 78*(2), 287–307.

Daley, B. and B. Green (2012). Waiting for news in the market for lemons. *Econometrica 80*(4), 1433–1504.

Fang, H. and P. Norman (2006). To bundle or not to bundle. *The RAND Journal of Economics 37*(4), 946–963.

Fuchs, W., A. Ory, and A. Skrzypacz (2016). Transparency and distressed sales under asymmetric information. *Theoretical Economics 11*(3), 1103–1144.

Gerardi, D., L. Maestri, and I. Monzón (2022). Bargaining over a divisible good in the market for lemons. *American Economic Review 112*(5), 1591–1620.

Huangfu, B. and H. Liu (2022). Information spillover in multi-good adverse selection. *American Economic Journal:Microeconomics*, Forthcoming.

Kaya, A. and K. Kim (2018). Trading dynamics with private buyer signals in the market for lemons. *The Review of Economic Studies 85*(4), 2318–2352.

Kim, K. (2017). Information about sellers' past behavior in the market for lemons. *Journal of Economic Theory 169*, 365–399.

McAfee, R. P., J. McMillan, and M. D. Whinston (1989). Multiproduct monopoly, commodity bundling, and correlation of values. *The Quarterly Journal of Economics 104*(2), 371–383.

Nguyen, A. (2022). Household bundling of health insurance. *Working Paper*.

Rothschild, M. and J. E. Stiglitz (1970). Increasing risk: I. a definition. *Journal of Economic Theory 2*(3), 225–243.

Rothschild, M. and J. E. Stiglitz (1976). Equilibrium in competitive insurance markets: An essay on the economics of imperfect information. *The Quarterly Journal of Economics 90*(4), 629–649.

Samuelson, W. (1984). Bargaining under asymmetric information. *Econometrica: Journal of the Econometric Society*, 995–1005.

Sandhu, C., J. Osterman, and K. Krpata (2013, June). Strategic patent acquisition: Evaluating and acquiring patent portfolios. Technical report, Practical Law The Journal.

Stiglitz, J. E. (1977). Monopoly, non-linear pricing and imperfect information: The insurance market. *The Review of Economic Studies 44*(3), 407–430.

Strausz, R. (2006). Deterministic versus stochastic mechanisms in principal–agent models. *Journal of Economic Theory 128*(1), 306–314.

A Appendix: Proofs

Proof of Lemma 1

Proof. Let $u^B(q,t;\lambda) := V(q,\lambda) - t$ and $u^S(q,t;\lambda) := t - C(q,\lambda)$. Suppose that $q(\cdot)$ and $t(\cdot)$ are jointly a solution to program ($\mathcal{P}$). Since they satisfy constraint (IC^S), by the envelope theorem, $\frac{d}{d\lambda}u^S(q(\lambda), t(\lambda); \lambda) = -\frac{\partial C(q(\lambda),\lambda)}{\partial\lambda} = \mathcal{I}(q(\lambda) > \lambda)(c_H - c_L)$. Thus, the first condition in Lemma 1 is a necessary condition for constraint (IC^S). By standard arguments, if this condition holds and $q(\cdot)$ is also nondecreasing, constraint (IC^S) holds.

Suppose that $\lambda_2 > \lambda_1$ but $q(\lambda_2) < q(\lambda_1)$. Constraint ($IC^S$) implies that

$$q(\lambda_1) \leq \lambda_1 \quad \text{or} \quad q(\lambda_2) \geq \lambda_2. \tag{20}$$

To see why, suppose, for a contradiction, that statement (20) is false. The negation of statement (20) is $q(\lambda_1) > \lambda_1$ *and* $q(\lambda_2) < \lambda_2$. Constraints (IC^S) for λ_1 and λ_2 imply that

$$\begin{aligned} & C(q(\lambda_2), \lambda_1) + C(q(\lambda_1), \lambda_2) - C(q(\lambda_1), \lambda_1) \geq C(q(\lambda_2), \lambda_2) \\ \iff \quad & [q(\lambda_2) - \lambda_1]^+ + [q(\lambda_1) - \lambda_2]^+ - [q(\lambda_1) - \lambda_1]^+ \ \geq \ [q(\lambda_2) - \lambda_2]^+ \end{aligned}$$

The right-hand side (RHS) is zero. Since $q(\lambda_1) > \lambda_1$, at least one of $[q(\lambda_2) - \lambda_1]^+$ or $[q(\lambda_1) - \lambda_2]^+$ must be strictly positive. If $[q(\lambda_1) - \lambda_2]^+ = 0$ and only $[q(\lambda_2) - \lambda_1]^+ > 0$, then the left-hand side (LHS) is $q(\lambda_2) - q(\lambda_1) < 0$. Thus, $[q(\lambda_1) - \lambda_2]^+$ must be strictly positive. Therefore, every term on the LHS is strictly positive. Thus, the LHS is $(q(\lambda_2) - \lambda_1) + (q(\lambda_1) - \lambda_2) - (q(\lambda_1) - \lambda_1) = q(\lambda_2) - \lambda_2$, but this is strictly negative — contradiction. Therefore, statement (20) must hold.

Given that statement (20) holds, we first consider the case of $q(\lambda_1) \leq \lambda_1$. With $q(\lambda_2) < q(\lambda_1)$, it must imply that $t(\lambda_1) - t(\lambda_2) = c_L [q(\lambda_1) - q(\lambda_2)]$; if not, one of the two types has a profitable deviation from taking the other type's contract. This implies that $u^S(q(\lambda_2), t(\lambda_2); \lambda_2) = u^S(q(\lambda_1), t(\lambda_1); \lambda_2)$. In this case, replacing type λ_2's contract with $(q(\lambda_1), t(\lambda_1))$ maintains incentive-compatibility for all types because this gives type λ_2 the same utility as before, and because $(q(\lambda_1), t(\lambda_1))$ was already part of the mechanism, it does not affect the incentive-compatibility of the other types. With this replacement, the buyer's utility from λ_2 *strictly* increases by $[q(\lambda_1) - q(\lambda_2)]\, s_L$. This implies that we can then increase $t(\lambda)$ by some small $\varepsilon > 0$ for all λ without violating the buyer's participation constraint (IR^B), and this change increases the objective value of program ($\mathcal{P}$). This contradicts the assumption that $q(\cdot)$ and $t(\cdot)$ are jointly a solution to program ($\mathcal{P}$). Therefore, $q(\lambda_1) \leq \lambda_1$ cannot hold.

The preceding paragraph implies that $q(\lambda_2) \geq \lambda_2$ must hold, which implies that $t(\lambda_1) - t(\lambda_2) = c_H [q(\lambda_1) - q(\lambda_2)]$. By the same argument, by replacing λ_2's contract with λ_1's contract, incentive-compatibility of all types is maintained, but the buyer's surplus from λ_2 increases by $[q(\lambda_1) - q(\lambda_2)]\, s_H$. In turn, we can raise $t(\cdot)$ uniformly and increase the objective value, and this contradicts the assumption that $q(\cdot)$ and $t(\cdot)$ are jointly a solution to program ($\mathcal{P}$). Therefore, $q(\lambda_2) \geq \lambda_2$ also cannot hold.

From the two preceding paragraphs, statement (20) cannot hold, and we thus have a contradiction to $\lambda_2 > \lambda_1$ and $q(\lambda_2) < q(\lambda_1)$. □

Derivation of Program ($\tilde{\mathcal{P}}$)

Lemma 4. *Under the solution to program ($\mathcal{P}$), (IR^B) must bind. Moreover, if $q(\cdot)$ is part of a solution to program ($\tilde{\mathcal{P}}$), $q(\cdot)$ is also part of a solution to program ($\mathcal{P}$).*

Proof. Using the expression of $t(\lambda)$ in equation (5) and doing an integration by parts,

$$\begin{aligned}\int_0^1 t(\lambda) f(\lambda) d\lambda &= U^S(0) + (c_H - c_L)\int_0^1 \mathcal{I}(q(\lambda) > \lambda)[1 - F(\lambda)] d\lambda + \int_0^1 C(q(\lambda), \lambda) f(\lambda) d\lambda. \\ &= U^S(0) + \int_0^1 \psi^S(q(\lambda), \lambda) d\lambda + \int_0^1 C(q(\lambda), \lambda) f(\lambda) d\lambda.\end{aligned}$$

Therefore, the objective function of program ($\mathcal{P}$) becomes

$$\int_0^1 t(\lambda) f(\lambda) d\lambda - \int_0^1 C(q(\lambda), \lambda) f(\lambda) d\lambda \ = U^S(0) + \int_0^1 \psi^S(q(\lambda), \lambda) d\lambda, \tag{21}$$

and constraint (IR^B) becomes

$$\begin{aligned} & \int_0^1 V(q(\lambda), \lambda) f(\lambda) d\lambda - \int_0^1 t(\lambda) f(\lambda) d\lambda \ \geq \ b \\ \iff & \int_0^1 [V(q(\lambda), \lambda) - C(q(\lambda, \lambda))] f(\lambda) d\lambda - \int_0^1 \psi^S(q(\lambda), \lambda) d\lambda - U^S(0) \ \geq \ b \\ \iff & \int_0^1 \psi^B(q(\lambda), \lambda) d\lambda - U^S(0) \ \geq \ b. \end{aligned} \tag{22}$$

If constraint (IR^B), as expressed in equation (22), does not bind, we can raise $U^S(0)$ to increase the objective value in equation (21). Thus, constraint (IR^B) must bind, implying that

$$\begin{aligned} U^S(0) &= \int_0^1 \psi^B(q(\lambda), \lambda) d\lambda - b \\ &= \int_0^1 S(q, \lambda) f(\lambda) d\lambda - \int_0^1 \psi^S(q, \lambda) d\lambda - b. \end{aligned}$$

Substituting this expression of $U^S(0)$ into the objective function in equation (21), the objective function becomes $\int_0^1 S(q, \lambda) f(\lambda) d\lambda - b$. Therefore, together with Lemma 1, we obtain program ($\tilde{\mathcal{P}}$). □

Proof of Proposition 1

Fix some $b \in \left[0, \bar{b}\right]$. We first prove some important properties of the solution to program ($\tilde{\mathcal{P}}$). Throughout, we consider only schedules $q(\cdot)$ that are right-continuous — i.e., for any x, $\lim_{\lambda \downarrow x^+} q(\lambda) = q(x)$. This is without loss of generality because $q(\cdot)$ must be a bounded and nondecreasing function over $[0,1]$. Therefore, the set of discontinuous points is countable and has a Lebesgue measure of zero. Since F has no atom anywhere, the discontinuous points of $q(\cdot)$ do not affect the values of the objective function and the constraints in program ($\tilde{\mathcal{P}}$).

Lemma 5. *Suppose that $q^*(\cdot)$ is a solution to program ($\tilde{\mathcal{P}}$).*

1. *It holds that $q^*(\lambda) \geq \lambda$ for all $\lambda \in [0,1]$.*
2. *Let $\mathcal{X}$ be the set of λ such that $q^*(\lambda) > \lambda$.*
   (a) *If $\mathcal{X}$ is empty, then $q^*(\cdot)$ is a threshold schedule, with $q^*(\lambda) = \lambda$ for all λ.*
   (b) *Suppose that $\mathcal{X}$ is nonempty. Let λ_1 denote the smallest λ in $\mathcal{X}$, and let λ_2 denote the smallest $\lambda \in (\lambda_1, 1]$ such that $q^*(\lambda) = \lambda$. It holds that $q^*(\lambda) = \lambda_2$ for all $\lambda \in [\lambda_1, \lambda_2]$.*

Proof. For any λ, $S(q,\lambda)$ and $\psi^B(q,\lambda)$ are both strictly increasing in q for any $q \in [0,\lambda)$. Thus, point 1 follows. Point 2a is a corollary of point 1.

For point 2b, we first note that if $\mathcal{X}$ is nonempty, the smallest element of $\mathcal{X}$ must exist because $q^*(\cdot)$ is right-continuous. Similarly, λ_2 must also exist because $q^*(1) = 1$ (from point 1) and $q^*(\cdot)$ is right-continuous. Since $q^*(\cdot)$ is nondecreasing, we know that $q^*(\lambda) \leq \lambda_2$ for all $\lambda \in [\lambda_1, \lambda_2]$. Moreover, for all $\lambda \in [\lambda_1, \lambda_2)$, $\lambda_2 > \lambda$ implies that $S(\lambda_2, \lambda) \geq S(q^*(\lambda), \lambda)$ and $\psi^B(\lambda_2, \lambda) \geq \psi^B(q^*(\lambda_2), \lambda)$, where the inequality holds strictly if $q^*(\lambda) < \lambda_2$. Therefore, $q^*(\lambda)$ must be λ_2 for all $\lambda \in [\lambda_1, \lambda_2]$. □

Corollary 2. *Suppose that $q^*(\cdot)$ is a solution to program ($\tilde{\mathcal{P}}$). Let $\mathcal{X}$ and λ_2 be as defined in Lemma 5. If $\mathcal{X}$ is nonempty, $q^*(\cdot)$ is a threshold schedule if and only if $\lambda_2 = 1$.*

The proof of Proposition 1 is as follows:

Proof. Let $q^*(\cdot)$ be an optimal schedule for program ($\tilde{\mathcal{P}}$), assumed (without loss of generality) to be right-continuous, and let u^* denote the corresponding optimal value of $U^S(0)$. Let λ_1 and λ_2 be as defined in Lemma 5. We know that $q^*(\lambda) = \lambda_2$ for all $\lambda \in [\lambda_1, \lambda_2]$. By Corollary 2, we only have to prove that $\lambda_2 = 1$.

Suppose, for a contradiction, that $\lambda_2 < 1$. This implies that there exists $\lambda_3 > \lambda_2$ such that $q^*(\lambda) = \lambda$ for all $\lambda \in [\lambda_2, \lambda_3]$. For some small $\varepsilon > 0$, define $\eta(\varepsilon)$ implicitly by

$$\int_{\lambda_1+\varepsilon}^{\lambda_2+\eta(\varepsilon)} 1 - F(\lambda)\, d\lambda = \int_{\lambda_1}^{\lambda_2} 1 - F(\lambda)\, d\lambda. \tag{23}$$

We choose ε to be small enough such that $\lambda_2 + \eta(\varepsilon) \leq \lambda_3$. Next, define schedule $\hat{q}_\varepsilon(\cdot)$ as follows (see the schedule on the right panel of Figure 1 for an illustration):

$$\hat{q}_\varepsilon(\lambda) = \begin{cases} q^*(\lambda) & , \text{ if } \lambda \notin [\lambda_1, \lambda_3) \\ \lambda & , \text{ if } \lambda \in [\lambda_1, \lambda_1 + \varepsilon) \\ \lambda_2 + \eta(\varepsilon) & , \text{ if } \lambda \in [\lambda_1 + \varepsilon, \lambda_2 + \eta(\varepsilon)] \\ \lambda & , \text{ if } \lambda \in [\lambda_2 + \eta(\varepsilon), \lambda_3) \end{cases}. \tag{24}$$

Observe that

$$\begin{aligned}\int_0^1 \psi^B(\hat{q}_\varepsilon(\lambda), \lambda)\, d\lambda - \int_0^1 \psi^B(q^*(\lambda), \lambda)\, d\lambda &= \int_{\lambda_1}^{\lambda_3} \psi^B(\hat{q}_\varepsilon(\lambda), \lambda)\, d\lambda - \int_{\lambda_1}^{\lambda_3} \psi^B(q^*(\lambda), \lambda)\, d\lambda \\ &= \int_{\lambda_1}^{\lambda_3} [S(\hat{q}_\varepsilon(\lambda), \lambda) - S(q^*(\lambda), \lambda)]\, f(\lambda)\, d\lambda,\end{aligned}$$

because

$$\begin{aligned}&\int_{\lambda_1}^{\lambda_3} \psi^S(\hat{q}_\varepsilon(\lambda), \lambda)\, d\lambda - \int_{\lambda_1}^{\lambda_3} \psi^S(q^*(\lambda), \lambda)\, d\lambda \\ =&\int_{\lambda_1+\varepsilon}^{\lambda_2+\eta(\varepsilon)} (c_H - c_L)[1 - F(\lambda)]\, d\lambda - \int_{\lambda_1}^{\lambda_2} (c_H - c_L)[1 - F(\lambda)]\, d\lambda,\end{aligned}$$

which is zero from equation (23). Moreover,

$$\begin{aligned}\delta_\varepsilon :=& \int_{\lambda_1}^{\lambda_3} S[(\hat{q}_\varepsilon(\lambda), \lambda) - S(q^*(\lambda), \lambda)]\, f(\lambda)\, d\lambda \\ =& \int_{\lambda_1+\varepsilon}^{\lambda_2+\eta(\varepsilon)} (\lambda_2 + \eta(\varepsilon) - \lambda)\, s_H f(\lambda)\, d\lambda - \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \lambda)\, s_H f(\lambda)\, d\lambda \ > 0,\end{aligned} \tag{25}$$

where the inequality follows from Lemma 2, which is proved below. Since $\hat{q}_\varepsilon$ is also nondecreasing, the schedule $\hat{q}_\varepsilon(\cdot)$ together with $U^S(0) = u^* + \delta_\varepsilon$ also satisfy all the constraints for program ($\tilde{\mathcal{P}}$), and the resulting objective value increases by $\delta_\varepsilon > 0$. This contradicts the optimality of q^*. $\square$

Proof of Lemma 2

Proof. We first show the following property:

$$\text{If } f/(1-F) \text{ is nondecreasing, then } \lambda' > \lambda \implies \frac{F(\lambda') - F(\lambda)}{[1-F(\lambda')](\lambda' - \lambda)} > \frac{f(\lambda)}{1-F(\lambda)}. \quad (26)$$

Fix any $\lambda \in (0,1)$. For $\lambda' \in (\lambda, 1)$, let $A(\lambda') := \frac{F(\lambda')-F(\lambda)}{[1-F(\lambda')](\lambda'-\lambda)}$. Observe that $\lim_{\lambda' \downarrow \lambda} A(\lambda') = \frac{f(\lambda)}{1-F(\lambda)}$ and $\lim_{\lambda' \uparrow 1} A(\lambda') = \infty$. Suppose, for a contradiction, that there exist $\lambda' > \lambda$ in which $A(\lambda') \leq \frac{f(\lambda)}{1-F(\lambda)}$. Since $A(\cdot)$ is continuous, by the mean value theorem, there must exist $\tilde{\lambda} \in (\lambda, 1)$ such that $A'\left(\tilde{\lambda}\right) = 0$ and $A\left(\tilde{\lambda}\right) \leq \frac{f(\lambda)}{1-F(\lambda)}$. By some algebra, $A'\left(\tilde{\lambda}\right) = \frac{\left(\tilde{\lambda}-\lambda\right) f\left(\tilde{\lambda}\right)[1-F(\lambda)] - \left[F\left(\tilde{\lambda}\right)-F(\lambda)\right]\left[1-F\left(\tilde{\lambda}\right)\right]}{\left[1-F\left(\tilde{\lambda}\right)\right]^2 \left(\tilde{\lambda}-\lambda\right)^2}$. Therefore, $A'\left(\tilde{\lambda}\right) = 0$ implies that

$$\frac{f\left(\tilde{\lambda}\right)}{1-F\left(\tilde{\lambda}\right)} = \frac{F\left(\tilde{\lambda}\right) - F(\lambda)}{\left(\tilde{\lambda}-\lambda\right)[1-F(\lambda)]} = \underbrace{A\left(\tilde{\lambda}\right)}_{\leq \frac{f(\lambda)}{1-F(\lambda)}} \underbrace{\left(\frac{1-F\left(\tilde{\lambda}\right)}{1-F(\lambda)}\right)}_{<1} < K = \frac{f(\lambda)}{1-F(\lambda)}.$$

However, $\tilde{\lambda} > \lambda$ implies that $\frac{f\left(\tilde{\lambda}\right)}{1-F\left(\tilde{\lambda}\right)} \geq \frac{f(\lambda)}{1-F(\lambda)}$ — contradiction.

We prove Lemma 2 now. Fix any $\lambda_1, \lambda_2 > 0$. For $x \geq \lambda_1$, define $\phi(x)$ to be such that $\int_x^{\phi(x)} 1 - F(\lambda)\, d\lambda = \int_{\lambda_1}^{\lambda_2} 1 - F(\lambda)\, d\lambda$. $\phi(x)$ is strictly increasing in x and, by the implicit function theorem, differentiable at x , with $\phi'(x) = \frac{1-F(x)}{1-F(\phi(x))}$. Let

$$D(x) := \int_x^{\phi(x)} (\phi(x) - \lambda) f(\lambda)\, d\lambda - \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \lambda) f(\lambda)\, d\lambda.$$

Our goal is to show that $D(x) > 0$ for all $x \in (\lambda_1, \lambda_1 + \delta)$ for some $\delta > 0$. Note that $D(x)$ is also differentiable at x, with

$$\begin{aligned} D'(x) &= [F(\phi(x)) - F(x)] \frac{1-F(x)}{1-F(\phi(x))} - [\phi(x) - x] f(x) \\ &= \left[\frac{F(\phi(x)) - F(x)}{[1-F(\phi(x))][\phi(x) - x]} - \frac{f(x)}{1-F(x)}\right] (1-F(x))(\phi(x) - x). \end{aligned}$$

Since $\phi(x) > x$, property (26) implies that $D'(x) > 0$. Since $D(x)$ is continuous and $D(\lambda_1) = 0$, $D'(x) > 0$ for all $x \in (\lambda_1, \lambda_1 + \delta)$ implies that $D(x) > 0$ for all $x \in (\lambda_1, \lambda_1 + \delta)$. $\square$

Proof of Proposition 2

Proof. For class A1, from the discussion in the main text, q^* is the threshold-0 schedule, and $U^S(0) = \hat{S}(0) - \hat{\psi}^S(0) - b = E[v] - c_H - b$. Therefore, for all λ, $q^*(\lambda) = 1$ and $\int_0^\lambda \mathcal{I}(q^*(l) > l)\, dl = \lambda$, which means that

$$t(\lambda) = (E[v] - c_H - b) + (c_H - c_L)\lambda + C(1, \lambda) = E[v] - b.$$

For class A2, from the discussion in the main text, q^* is the threshold-$\lambda^*(b)$ schedule, and $U^S(0) = 0$. For $\lambda < \lambda^*(b)$, $q^*(\lambda) = \lambda$ and $\int_0^\lambda \mathcal{I}(q^*(l) > l)\, dl = 0$. Therefore, $t(\lambda) = C(\lambda, \lambda) = \lambda c_L$. For $\lambda \geq \lambda^*(b)$, $q^*(\lambda) = 1$ and $\int_0^\lambda \mathcal{I}(q^*(l) > l)\, dl = \lambda - \lambda^*(b)$. Therefore,

$$t(\lambda) = (c_H - c_L)[\lambda - \lambda^*(b)] + C(1, \lambda) = C(1, \lambda^*(b)).$$

Finally, to show that $\lambda^*(b)$ is strictly increasing in b, let $\hat{\psi}^B(x) := \hat{S}(x) - \hat{\psi}^S(x)$. Suppose, for a contradiction, that there exist $b' > b$ but $\lambda^*(b') \leq \lambda^*(b)$. Note that $\hat{\psi}^B(\lambda^*(b')) = b' > b$. By the intermediate value theorem, since $\hat{\psi}^B$ is continuous and $\hat{\psi}^B(0) < b < \hat{\psi}^B(\lambda^*(b'))$, there exists $z \in (0, \lambda^*(b'))$ such that $\hat{\psi}^B(z) = b$. Because $z < \lambda^*(b') \leq \lambda^*(b)$, this contradicts $\lambda^*(b)$ being the smallest x in which $\hat{\psi}^B(x) = b$. $\square$

Proof of Proposition 3

Proof. Let the distribution involved for $\hat{S}$ and $\hat{\psi}^S$ be indicated by a subscript.[25] In allocation A_F, since the optimal threshold $\hat{\lambda}_F$ is positive, the value of $U^S(0)$ is zero. This implies that in A_F, the seller's expected utility is $u_F^S := \hat{\psi}_F^S\left(\hat{\lambda}_F\right)$ and the buyer's expected utility is $u_F^B := \hat{S}_F\left(\hat{\lambda}_F\right) - \hat{\psi}_F^S\left(\hat{\lambda}_F\right)$. For distribution $J \in \{F, G\}$, let $\bar{J}(\lambda) = (1-\lambda)[1 - J(\lambda)]$. By an integration by parts,

$$\begin{aligned}
\hat{S}_J(x) &= \int_0^1 \lambda s_L dJ(\lambda) + \int_x^1 (1-\lambda) s_H dJ(\lambda) && (27)\\
&= \int_0^1 \lambda s_L dJ(\lambda) + s_H\left[-(1-x)J(x) + \int_x^1 J(\lambda)\, d\lambda\right] && (28)\\
&= \int_0^1 \lambda s_L dJ(\lambda) + s_H\left[(1-x)(1-J(x)) - \int_x^1 1 - J(\lambda)\, d\lambda\right], && (29)\\
&= \int_0^1 \lambda s_L dJ(\lambda) + s_H \bar{J}(x) - \left(\frac{s_H}{c_H - c_L}\right)\hat{\psi}_J^S(x) && (30)
\end{aligned}$$

[25]For $J \in \{F, G\}$, $\hat{S}_J(x) = \int_0^1 \lambda s_L dJ(\lambda) + \int_x^1 (1-\lambda) s_H dJ(\lambda)$, $\hat{\psi}_J^S(x) = \int_x^1 (c_H - c_L)[1 - J(\lambda)]\, d\lambda$.

We first prove the "only if" direction. Let $\tilde{A}_G$ be the SB allocation of G that Pareto dominates A_F, let $\tilde{u}_G^B$ and $\tilde{u}_G^S$ denote the buyer's and the seller's expected utility under $\tilde{A}_G$, respectively, and let $\tilde{\lambda}_G$ denote the threshold that attains $\tilde{A}_G$. Under our assumption that $\hat{\lambda}_F \in (0,1)$, we know that $u_F^B > E[v] - c_H$. Since $\tilde{u}_G^B > u_F^B$, $\tilde{u}_G^B > E[v] - c_H$ as well; thus, $\tilde{u}_G^S = \hat{\psi}_G^S\left(\tilde{\lambda}_G\right)$. Let $\hat{\lambda}_G = \gamma\left(\hat{\lambda}_F\right)$; thus, $\hat{\psi}_G^S\left(\hat{\lambda}_G\right) = \hat{\psi}_F^S\left(\hat{\lambda}_F\right)$. It is straightforward to see that the threshold-$\hat{\lambda}_G$ schedule is also associated with a SB allocation under distribution G. Let this allocation be denoted by A_G and let u_G^B and u_G^S denote the buyer's and the seller's expected utility under A_G, respectively. By a similar argument as above, $u_G^B > E[v] - c_H$ and hence, $u_G^S = \hat{\psi}_G^S\left(\hat{\lambda}_G\right) = \hat{\psi}_F^S\left(\hat{\lambda}_F\right) = u_F^S < \tilde{u}_G^S$. This implies that $u_G^B > \tilde{u}_G^B \geq u_F^B$. Thus, $\hat{S}_G\left(\hat{\lambda}_G\right) = u_G^B + u_G^S > u_F^B + u_F^S = \hat{S}_F\left(\hat{\lambda}_F\right)$. Using equation (30), since $\hat{\psi}_G^S\left(\hat{\lambda}_G\right) = \hat{\psi}_F^S\left(\hat{\lambda}_F\right)$, $\hat{S}_G\left(\hat{\lambda}_G\right) > \hat{S}_F\left(\hat{\lambda}_F\right)$ implies that $\bar{G}\left(\hat{\lambda}_G\right) > \bar{F}\left(\hat{\lambda}_F\right)$.

Next, we prove the "if" direction. Let $\hat{\lambda}_G = \gamma\left(\hat{\lambda}_F\right)$ again. Condition (16) states that $\bar{G}\left(\hat{\lambda}_G\right) > \bar{F}\left(\hat{\lambda}_F\right)$. Therefore, using equation (30), since $\hat{\psi}_G^S\left(\hat{\lambda}_G\right) = \hat{\psi}_F^S\left(\hat{\lambda}_F\right)$, we have $\hat{S}_G\left(\hat{\lambda}_G\right) - \hat{S}_F\left(\hat{\lambda}_F\right) = \bar{s}_H\left[\bar{G}\left(\hat{\lambda}_G\right) - \bar{F}\left(\hat{\lambda}_F\right)\right] > 0$, which means that

$$\hat{S}_G\left(\hat{\lambda}_G\right) - \hat{\psi}_G^S\left(\hat{\lambda}_G\right) > \hat{S}_F\left(\hat{\lambda}_F\right) - \hat{\psi}_F^S\left(\hat{\lambda}_F\right) = u_F^B \tag{31}$$

Consider the program

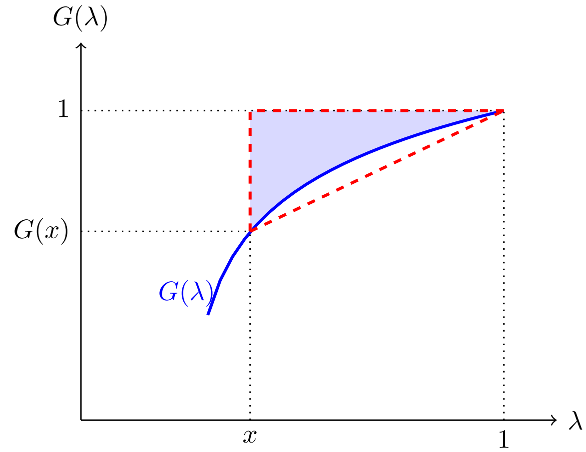
$$\max_{x\in[0,1],\ U^S(0)\geq 0} \quad \hat{S}_G(x) - b \quad \text{s.t.} \quad \hat{S}_G(x) - \hat{\psi}_G^S(x) - U^S(0) = u_F^B + \varepsilon,$$

where ε is chosen such that $\hat{S}_G\left(\hat{\lambda}_G\right) - \hat{\psi}_G^S\left(\hat{\lambda}_G\right) = u_F^B + \varepsilon$. Equation (31) implies that $\varepsilon > 0$, meaning that the buyer's expected utility of the SB allocation of the program above is strictly higher than u_F^B. By construction, the threshold $\hat{\lambda}_G$ together with $U^S(0) = 0$ are jointly a feasible solution, meaning that the seller's expected utility (i.e., the value of the program above) is at least $\hat{\psi}_G^S\left(\hat{\lambda}_G\right) = \hat{\psi}_F^S\left(\hat{\lambda}_F\right) = u_F^S$. Thus, the allocation from the solution Pareto dominates A_F. □

Proof of Lemma 3

Proof. Following the notations for $\bar{G}$ and $\bar{F}$ in the previous proof, observe that the inequality in (16) is equivalent to $\gamma\left(\hat{\lambda}_F\right) < \bar{G}^{-1}\left(\bar{F}\left(\hat{\lambda}_F\right)\right)$, where $\bar{G}^{-1}$ is the inverse of $\bar{G}$. Moreover, since $\bar{G}$ is strictly decreasing, $G\left(\hat{\lambda}_F\right) \leq F\left(\hat{\lambda}_F\right) \implies \bar{G}\left(\hat{\lambda}_F\right) \geq \bar{F}\left(\hat{\lambda}_F\right) \implies \hat{\lambda}_F \leq \bar{G}^{-1}\left(\bar{F}\left(\hat{\lambda}_F\right)\right)$. The property thus follows. □

Figure 3: On $\int_x^1 1 - G(\lambda)\, d\lambda < \frac{1}{2}\bar{G}(x)$ when G is concave.



$\int_x^1 1 - G(\lambda)\, d\lambda$ is the (blue) shaded area.
$\frac{1}{2}\bar{G}(x) = \frac{1}{2}(1-x)(1-G(x))$ is the area of the dashed triangle (in red).

Proof of Example 1

Proof. As noted in the main text, we only have to check that condition (16) holds for $x \in (\lambda', 1)$, where G is concave. When G is concave at x, $\int_x^1 1 - G(\lambda)\, d\lambda < \frac{1}{2}\bar{G}(x)$ — see Figure 3 for a straightforward illustration of this property.[26] When F is the uniform distribution, $\int_x^1 1 - F(\lambda)\, d\lambda = \frac{1}{2}\bar{F}(x)$. Combining these two observations, we have $\frac{1}{2}\bar{F}(x) = \int_x^1 1 - F(\lambda)\, d\lambda = \int_{\gamma(x)}^1 1 - G(\lambda)\, d\lambda < \frac{1}{2}\bar{G}(\gamma(x))$, and $\bar{F}(x) < \bar{G}(\gamma(x)) \implies \gamma(x) < \bar{G}^{-1}\left(\bar{F}(x)\right)$ — i.e., condition (16). $\square$

B Example: SB Allocations for Uniform Distribution

In this section, we provide a detailed characterization of the set of SB allocations when F is the uniform distribution. When $F(\lambda) = \lambda$, there is mild (severe) adverse selection if $s_H + s_L \geq (<)\, c_H - c_L$. Let $\hat{\psi}^B(x) := \hat{S}(x) - \hat{\psi}^S(x)$. Thus,

$$\hat{S}(x) = 0.5 s_L + 0.5(1-x)^2 s_H \quad \text{and} \quad \hat{\psi}^S(x) = 0.5(1-x)^2 \Delta c, \quad \text{where} \quad \Delta c = c_H - c_L \quad ;$$
$$\hat{\psi}^B(x) = 0.5 s_L + 0.5(1-x)^2 (s_H - \Delta c) \tag{32}$$

[26] Formally, since G is strictly increasing and concave when $\lambda \geq x$, it implies that for any $\lambda \in (x, 1)$, $\frac{1-G(\lambda)}{1-\lambda} < \frac{1-G(x)}{1-x} \iff \frac{1-G(x)}{1-x}(1-\lambda) > 1 - G(\lambda)$. Thus, $\int_x^1 1 - G(\lambda)\, d\lambda < \frac{1-G(x)}{1-x}\int_x^1 (1-\lambda)\, d\lambda = \frac{1}{2}\bar{G}(x)$.

Therefore, for $x \in (0,1)$, $\frac{d}{dx}\hat{\psi}^B(x) \geq (\leq) 0$ if $s_H - \Delta c \leq (\geq) 0$. We break things down into three possible cases.

Very mild adverse selection: $s_H \geq \Delta c$.

Since $s_H \geq \Delta c$, $\hat{\psi}^B(x)$ is decreasing in x. Therefore, $\bar{b} = \hat{\psi}^B(0) = E[v] - c_H = 0.5(v_H + v_L) - c_H$. By Proposition 2, all the SB allocations are of class A1, where the seller's expected utility is $\hat{S}(0) - b = 0.5(s_H + s_L) - b$. Specifically, when $b = 0$ and $\bar{b}$, the seller's expected utilities are $\hat{S}(0) = 0.5(s_H + s_L)$ and $\hat{S}(0) - (E[v] - c_H) = 0.5\Delta c$.

Moderately mild adverse selection: $s_H + s_L \geq \Delta c > s_H$.

Since $s_H < \Delta c$, $\hat{\psi}^B(x)$ is increasing in x. This means that $\bar{b} = \hat{\psi}^B(1) = 0.5 s_L$, and $\hat{\psi}^B(1) > \hat{\psi}^B(0) \implies 0.5 s_L = \bar{b} > E[v] - c_H$. For $b \leq E[v] - c_H$, the SB allocation is of class A1, which has been characterized above. For $b \in (E[v] - c_H, 0.5 s_L]$, the SB allocation is of A2, where $\lambda^*(b)$ is characterized by $\hat{\psi}^B(\lambda^*(b)) = b$. From equation (32) above, this implies that

$$(1 - \lambda^*(b))^2 = \tfrac{0.5 s_L - b}{0.5(\Delta c - s_H)} \qquad \Longleftrightarrow \qquad \lambda^*(b) = 1 - \sqrt{\tfrac{0.5 s_L - b}{0.5(\Delta c - s_H)}}$$
$$\hat{\psi}^S(\lambda^*(b)) \;=\; 0.5(1 - \lambda^*(b))^2 \Delta c \;=\; \left(\tfrac{0.5 s_L - b}{\Delta c - s_H}\right)\Delta c,$$

and $\frac{d}{db}\hat{\psi}^S(\lambda^*(b)) = -\frac{\Delta c}{\Delta c - s_H}$. Specifically, when $b = E[v] - c_H$, $\lambda^*(b) = 0$ and $\hat{\psi}^S(\lambda^*(b)) = 0.5\Delta c$; when $b = \bar{b} = 0.5 s_L$, $\lambda^*\left(\bar{b}\right) = 1$ and $\hat{\psi}^S\left(\lambda^*\left(\bar{b}\right)\right) = 0$.

Severe adverse selection: $s_H + s_L < \Delta c$.

As in the case with moderately mild adverse selection, $\hat{\psi}^B(x)$ is increasing in x, meaning that $\bar{b} = \hat{\psi}^B(1) = 0.5 s_L$. Since $E[v] - c_H < 0$, all SB allocations must be of A2, where $\lambda^*(b)$ and $\hat{\psi}^S(\lambda^*(b))$ have been characterized above. In particular, $\hat{\psi}^S\left(\lambda^*\left(\bar{b}\right)\right) = 0$ and $\hat{\psi}^S(\lambda^*(0)) = \frac{0.5 s_L}{\Delta c - s_H}\Delta c$.

Online Appendix for "Markets with Within-Type Adverse Selection"

C Online Appendix: Extensions

C.1 Three Quality Levels

Consider the setup described in Section 5.1 for $k = 3$. To streamline the exposition with the baseline model, let the three quality levels be L, M and H, with $v_L < v_M < v_H$ and $c_L < c_M < c_H$. Let $s_M = v_M - c_M$ and assume that $s_M > 0$. To shorten the notation slightly, let $\tau(1|\theta) = \tau_L(\theta)$ and $\tau(2|\theta) = \tau_M(\theta)$. Therefore, type-$\theta$ seller has $\tau_L(\theta)$ units of L, $\tau_M(\theta) - \tau_L(\theta)$ units of M, and $1 - \tau_M(\theta)$ units of H, and

$$\begin{aligned} C(q,\theta) =& qc_L + [q - \tau_L(\theta)]^+ (c_M - c_L) + [q - \tau_M(\theta)]^+ (c_H - c_M), \\ V(q,\theta) =& qv_L + [q - \tau_L(\theta)]^+ (v_M - v_L) + [q - \tau_M(\theta)]^+ (v_H - v_M). \end{aligned}$$

With an abuse of notation, let F denote the distribution of θ and f its density, assumed to be strictly positive over $\theta \in (0,1)$. To simplify the exposition, we also assume that τ_L and τ_H are both differentiable, and $\tau_L'(\theta), \tau_M'(\theta) > 0$ for all θ (i.e., single-crossing).

Our main program is program ($\mathcal{P}$), with the type λ replaced by θ. Note that

$$\frac{\partial C(q,\theta)}{\partial \theta} = \mathcal{I}(q > \tau_L(\theta))\, \tau_L'(\theta)(c_M - c_L) + \mathcal{I}(q > \tau_M(\theta))\, \tau_M'(\theta)(c_H - c_M).$$

Thus, by the envelope theorem,

$$\begin{aligned} U^S(\theta) = U^S(0) &+ (c_M - c_L)\int_0^\theta \tau_L'(l)\,\mathcal{I}(q(l) > \tau_L(l))\,dl \\ &+ (c_H - c_M)\int_0^\theta \tau_M'(l)\,\mathcal{I}(q(l) > \tau_M(l))\,dl. \end{aligned}$$

Following the same argument as that for the baseline model, program ($\tilde{\mathcal{P}}$) in the current setup is

$$\begin{gathered} \max_{q(\cdot),U^S(0)} \quad \textstyle\int_0^1 S(q(\theta),\theta) f(\theta)\,d\theta - b \qquad \text{s.t.} \\ q(\cdot) \text{ is nondecreasing}, \quad U^S(0) \geq 0, \quad \text{and} \quad \textstyle\int_0^1 \psi^B(q(\theta),\theta)\,d\theta - U^S(0) = b \end{gathered} \tag{33}$$

where

$$
\begin{aligned}
S(q,\theta) :=& V(q,\theta) - C(q,\theta), \\
\psi^S(q,\theta) :=& \left[(c_M - c_L)\,\tau_L'(\theta)\,\mathcal{I}(q > \tau_L(\theta)) + (c_H - c_M)\,\tau_M'(\theta)\,\mathcal{I}(q > \tau_M(\theta))\right][1 - F(\theta)], \\
\psi^B(q,\theta) :=& S(q,\theta)\,f(\theta) - \psi^S(q,\theta).
\end{aligned}
$$

A threshold schedule here is a schedule $q(\cdot)$ in which there exist θ_M and θ_H, with $\theta_M \le \theta_H$, such that $q(\theta) = \tau_L(\theta)$ if $\theta < \theta_M$, $q(\theta) = \tau_M(\theta)$ if $\theta \in [\theta_M, \theta_H)$, and $q(\theta) = 1$ if $\theta \ge \theta_H$.

Lemma 6. *If $q(\cdot)$ is a solution to program (33), it must satisfy the following properties:*

1. *$q(\theta) \ge \tau_L(\theta)$ for all θ.*
2. *If there exists an interval $X = \left[\underline{\theta}, \bar{\theta}\right]$ such that $q(\theta) > \tau_M(\theta)$ for all $\theta \in X$, then there is some $\hat{q}$ such that $q(\theta) = \hat{q}$ for all $\theta \in X$.*
3. *If there exists an interval $X = \left[\underline{\theta}, \bar{\theta}\right]$ such that $\tau_L(\theta) < q(\theta) \le \tau_M(\theta)$ for all $\theta \in X$, then there is some $\hat{q}$ such that $q(\theta) = \min\{\tau_M(\theta), \hat{q}\}$ for all $\theta \in X$.*

Proof. Lemma 6 is the extension of Lemma 5 and follows from $S(q,\theta)$ and $\psi^B(q,\theta)$ being strictly increasing in q when $q < \tau_L(\theta)$, $q \in (\tau_L(\theta), \tau_M(\theta))$ and $q \in (\tau_M(\theta), 1)$. □

Next, let $q^*(\cdot)$ denote a solution to program (33). Let χ be the set of θ in which $q^*(\theta) \notin \{\tau_L(\theta), \tau_M(\theta), 1\}$. Note that if χ is empty, then q^* is a threshold schedule. We will show that χ is empty if the following condition holds:

Condition 1. $f/[1-F]$ is nondecreasing, τ_L and τ_M are weakly concave, and $\frac{\tau_M(\theta) - \tau_L(\theta)}{\tau_L'(\theta)}\left(\frac{f(\theta)}{1-F(\theta)}\right)$ is nondecreasing in θ.

The following is an example that satisfies Condition 1: F is log-concave (e.g., uniform distribution), and τ_M and τ_L are affine functions, with τ_M weakly steeper than τ_L.

We will prove that χ is empty under Condition 1 now. As before, without loss of generality, we assume that q^* is right-continuous. Let θ_1 be the infimum of χ. Henceforth, let $q^*(\theta_1)$ be denoted by q_1^*. Because of point 1 of Lemma 6 and that q^* is right-continuous, it must be the case that $q_1^* \in (\tau_L(\theta), 1)$. We break things down into two cases:

- Case 1: $\tau_L(\theta_1) < q_1^* \le \tau_M(\theta_1)$.
- Case 2: $\tau_M(\theta_1) < q_1^* < 1$.

Consider Case 1 first. By point 3 of Lemma 6, there exists $\theta_2 < 1$ such $q^*(\theta) = \min\{\tau_M(\theta), \tau_L(\theta_2)\}$ for all $\theta \in [\theta_1, \theta_2]$ and there exists θ_3 such that $q^*(\theta) = \tau_L(\theta)$ for all $\theta \in [\theta_2, \theta_3]$. For some small $\varepsilon > 0$, define $\eta^1(\varepsilon)$ implicitly by

$$\int_{\theta_1+\varepsilon}^{\theta_2+\eta^1(\varepsilon)} \tau_L'(\theta)[1-F(\theta)]\,d\theta = \int_{\theta_1}^{\theta_2} \tau_L'(\theta)[1-F(\theta)]\,d\theta. \tag{34}$$

Pick the ε to be small enough such that $\theta_2 + \eta^1(\varepsilon) < \theta_3$ and $\tau_L(\theta_2 + \eta^1(\varepsilon)) < \tau_M(\theta_1 + \varepsilon)$. Let $\hat{q}_\varepsilon^1$ be the schedule in which

$$\hat{q}_\varepsilon^1(\theta) = \begin{cases} q^*(\theta) & \text{if } \theta \notin [\theta_1, \theta_3) \\ \tau_L(\theta) & \text{if } \theta \in [\theta_1, \theta_1+\varepsilon) \\ \min\{\tau_M(\theta), \tau_L(\theta_2 + \eta^1(\varepsilon))\} & \text{if } \theta \in [\theta_1+\varepsilon, \theta_2+\eta^1(\varepsilon)) \\ \tau_L(\theta) & \text{if } \theta \in [\theta_2+\eta^1(\varepsilon), \theta_3) \end{cases}$$

Thus,

$$\begin{aligned}
\delta_\varepsilon^1 =& \int_0^1 \left[S\left(\hat{q}_\varepsilon^1(\theta), \theta\right) - S(q^*(\theta), \theta)\right] f(\theta)\,d\theta \\
=& s_M \int_{\theta_1+\varepsilon}^{\theta_2+\eta^1(\varepsilon)} \left[\min\left\{\tau_M(\theta), \tau_L\left(\theta_2+\eta^1(\varepsilon)\right)\right\} - \tau_L(\theta)\right] f(\theta)\,d\theta \\
&- s_M \int_{\theta_1}^{\theta_2} \left[\min\{\tau_M(\theta), \tau_L(\theta_2)\} - \tau_L(\theta)\right] f(\theta)\,d\theta
\end{aligned}$$

By equation (34) and Lemma 7 below, we have $\delta_\varepsilon^1 > 0$. Equation (34) implies that $\int_0^1 \psi^S(\hat{q}_\varepsilon^1(\theta), \theta)\,d\theta = \int_0^1 \psi^S(q^*(\theta), \theta)\,d\theta$. This implies that $\int_0^1 \psi^B(\hat{q}_\varepsilon^1(\theta), \theta)\,d\theta - \int_0^1 \psi^B(q^*(\theta), \theta)\,d\theta = \delta_\varepsilon^1 > 0$. Thus, $\hat{q}_\varepsilon^1$ is feasible, but $\hat{q}_\varepsilon^1$ achieves a higher objective value than q^*, which is a contradiction. This rules out Case 1.

Next, consider Case 2. Let θ_2 be the largest θ in which $q^*(\theta) = q_1^*$. Lemma 6 implies that either $\tau_M(\theta_2) = q_1^*$ (Case 2A) or $\tau_L(\theta_2) = q_1^*$ (Case 2B).

Consider Case 2A first. Under Case 2A, there must exist θ_3 such that $q^*(\theta) = \tau_M(\theta)$ for all $\theta \in [\theta_2, \theta_3]$. For some small $\varepsilon > 0$, define $\eta^{2A}(\varepsilon)$ implicitly by

$$\int_{\theta_1+\varepsilon}^{\theta_2+\eta^{2A}(\varepsilon)} \tau_M'(\theta)[1-F(\theta)]\,d\theta = \int_{\theta_1}^{\theta_2} \tau_M'(\theta)[1-F(\theta)]\,d\theta. \tag{35}$$

Pick the ε to be small enough such that $\theta_2 + \eta^{2A}(\varepsilon) < \theta_3$. Let $\hat{q}^{2A}_{\varepsilon}$ be the schedule in which

$$\hat{q}^{2A}_{\varepsilon}(\theta) = \begin{cases} q^*(\theta) & \text{if } \theta \notin [\theta_1, \theta_3) \\ \tau_M(\theta) & \text{if } \theta \in [\theta_1, \theta_1 + \varepsilon) \\ \tau_M\left(\theta_2 + \eta^{2A}(\varepsilon)\right) & \text{if } \theta \in [\theta_1 + \varepsilon, \theta_2 + \eta^{2A}(\varepsilon)) \\ \tau_M(\theta) & \text{if } \theta \in [\theta_2 + \eta^{2A}(\varepsilon), \theta_3) \end{cases}$$

Thus,

$$\begin{aligned} \delta^{2A}_{\varepsilon} =& \int_0^1 \left[S\left(\hat{q}^{2A}_{\varepsilon}(\theta), \theta\right) - S(q^*(\theta), \theta)\right] f(\theta)\, d\theta \\ =& s_H \left(\int_{\theta_1+\varepsilon}^{\theta_2+\eta^{2A}(\varepsilon)} \left[\tau_M\left(\theta_2 + \eta^{2A}(\varepsilon)\right) - \tau_M(\theta)\right] f(\theta)\, d\theta - \int_{\theta_1}^{\theta_2} [\tau_M(\theta_2) - \tau_M(\theta)] f(\theta)\, d\theta\right). \end{aligned}$$

By equation (35) and Lemma 8 below, we have $\delta^{2A}_{\varepsilon} > 0$. Equation (35) implies that $\int_0^1 \psi^S\left(\hat{q}^{2A}_{\varepsilon}(\theta), \theta\right) d\theta = \int_0^1 \psi^S(q^*(\theta), \theta)\, d\theta$. This implies that $\int_0^1 \psi^B\left(\hat{q}^{2A}_{\varepsilon}(\theta), \theta\right) d\theta - \int_0^1 \psi^B(q^*(\theta), \theta)\, d\theta = \delta^{2A}_{\varepsilon} > 0$. Thus, $\hat{q}^{2A}_{\varepsilon}$ is feasible, but $\hat{q}^{2A}_{\varepsilon}$ achieves a higher objective value than q^*, which is a contradiction. This rules out Case 2A.

Consider Case 2B next. Under Case 2B, there must exist θ_3 such that $q^*(\theta) = \tau_L(\theta)$ for all $\theta \in [\theta_2, \theta_3]$. Let $\tilde{\theta}$ be the type such that $\tau_M\left(\tilde{\theta}\right) = q^*$. Note that $\theta_1 < \tilde{\theta} < \theta_2$. For some small $\varepsilon > 0$, define $\gamma_1(\varepsilon)$ and $\gamma_2(\varepsilon)$ implicitly by

$$\int_{\theta_1+\varepsilon}^{\tilde{\theta}+\gamma_1(\varepsilon)} \tau'_M(\theta)[1 - F(\theta)]\, d\theta = \int_{\theta_1}^{\tilde{\theta}} \tau'_M(\theta)[1 - F(\theta)]\, d\theta \tag{36}$$

$$\int_{\tilde{\theta}+\gamma_1(\varepsilon)}^{\theta_2+\gamma_2(\varepsilon)} \tau'_L(\theta)[1 - F(\theta)]\, d\theta = \int_{\tilde{\theta}}^{\theta_2} \tau'_L(\theta)[1 - F(\theta)]\, d\theta \tag{37}$$

Pick the ε to be small enough such that $\theta_2 + \gamma_2(\varepsilon) < \theta_3$. Let $\tilde{\theta}'$ be the type such that $\tau_M\left(\tilde{\theta}'\right) = \tau_L(\theta_2 + \gamma_2(\varepsilon))$. Let $\hat{q}^{2B}_{\varepsilon}$ be the schedule in which

$$\hat{q}^{2B}_{\varepsilon}(\theta) = \begin{cases} q^*(\theta) & \text{if } \theta \notin [\theta_1, \theta_3) \\ \tau_M(\theta) & \text{if } \theta \in [\theta_1, \theta_1 + \varepsilon) \\ \tau_M\left(\tilde{\theta} + \gamma_1(\varepsilon)\right) & \text{if } \theta \in [\theta_1 + \varepsilon, \tilde{\theta} + \gamma_1(\varepsilon)) \\ \tau_M(\theta) & \text{if } \theta \in [\tilde{\theta} + \gamma_1(\varepsilon), \tilde{\theta}') \\ \tau_L(\theta_2 + \gamma_2(\varepsilon)) & \text{if } \theta \in [\tilde{\theta}', \theta_2 + \gamma_2(\varepsilon)) \\ \tau_L(\theta) & \text{if } \theta \in [\theta_2 + \gamma_2(\varepsilon), \theta_3) \end{cases}$$

Observe that equations (36) and (37) imply that $\int_0^1 \psi^S\left(\hat{q}_\varepsilon^{2B}(\theta), \theta\right) d\theta = \int_0^1 \psi^S\left(q^*(\theta), \theta\right) d\theta$. Moreover,

$$\begin{aligned}
\delta_\varepsilon^{2B} =& \int_0^1 \left[S\left(\hat{q}_\varepsilon^{2B}(\theta), \theta\right) - S\left(q^*(\theta), \theta\right)\right] f(\theta)\, d\theta \\
=& s_H \left(\int_{\theta_1+\varepsilon}^{\tilde{\theta}+\gamma_1(\varepsilon)} \left[\tau_M\left(\tilde{\theta}+\gamma_1(\varepsilon)\right) - \tau_M(\theta)\right] f(\theta)\, d\theta - \int_{\theta_1}^{\tilde{\theta}} \left[\tau_M\left(\tilde{\theta}\right) - \tau_M(\theta)\right] f(\theta)\, d\theta\right) \qquad (38) \\
&+ s_M \Big(\int_{\tilde{\theta}+\gamma_1(\varepsilon)}^{\theta_2+\gamma_2(\varepsilon)} \left[\min\left\{\tau_M(\theta), \tau_L\left(\theta_2+\gamma_2(\varepsilon)\right)\right\} - \tau_L(\theta)\right] f(\theta)\, d\theta \qquad (39) \\
&\quad - \int_{\tilde{\theta}}^{\theta_2} \left[\tau_L(\theta_2) - \tau_L(\theta)\right] f(\theta)\, d\theta\Big) \qquad (40)
\end{aligned}$$

Equation (36) and Lemma 8 below jointly imply that line (38) is positive. Equation (37) and Lemma 7 below jointly imply that line (39) minus line (40) is positive. Therefore, $\delta_\varepsilon^{2B} > 0$. This implies that $\int_0^1 \psi^B\left(\hat{q}_\varepsilon^{2B}(\theta), \theta\right) d\theta - \int_0^1 \psi^B\left(q^*(\theta), \theta\right) d\theta = \delta_\varepsilon^{2B} > 0$. Thus, $\hat{q}_\varepsilon^{2B}$ is feasible, but $\hat{q}_\varepsilon^{2B}$ achieves a higher objective value than q^*, which is a contradiction. This rules out Case 2B as well.

Since both Case 1 and Case 2 are not possible, χ must be an empty set. Therefore, q^* must be a threshold schedule. We summarize the argument above in the following proposition:

Proposition 4. *Suppose that Condition 1 holds. If $q(\cdot)$ is a solution to program (33), $q(\cdot)$ must be a threshold schedule — i.e., there exist $\theta_M \leq \theta_H$ such that $q(\theta) = \tau_L(\theta)$ if $\theta < \theta_M$, $q(\theta) = \tau_M(\theta)$ if $\theta \in [\theta_M, \theta_H)$, and $q(\theta) = 1$ if $\theta \geq \theta_H$.*

We conclude this subsection with the proofs of Lemmas 7 and 8.

Lemma 7. *Let $\theta_1 < \theta_1' < \theta_2 < \theta_2'$. Under Condition 1, $\int_{\theta_1'}^{\theta_2'} \tau_L'(\theta)\left[1-F(\theta)\right] d\theta = \int_{\theta_1}^{\theta_2} \tau_L'(\theta)\left[1-F(\theta)\right] d\theta$ implies that*

$$\int_{\theta_1'}^{\theta_2'} \left[\min\left\{\tau_M(\theta), \tau_L(\theta_2')\right\} - \tau_L(\theta)\right] f(\theta)\, d\theta - \int_{\theta_1}^{\theta_2} \left[\min\left\{\tau_M(\theta), \tau_L(\theta_2)\right\} - \tau_L(\theta)\right] f(\theta)\, d\theta > 0.$$

Proof. For $x > \theta_1$, define $\phi(x)$ by $\int_x^{\phi(x)} \tau_L'(\theta)\left[1-F(\theta)\right] d\theta = \int_{\theta_1}^{\theta_2} \tau_L'(\theta)\left[1-F(\theta)\right] d\theta$. This implies that $\phi'(x) = \frac{\tau_L'(x)[1-F(x)]}{\tau_L'(\phi(x))[1-F(\phi(x))]}$. Let

$$D(x) = \int_x^{\phi(x)} \left[\min\left\{\tau_M(\theta), \tau_L(\phi(x))\right\} - \tau_L(\theta)\right] f(\theta)\, d\theta - \int_{\theta_1}^{\theta_2} \left[\min\left\{\tau_M(\theta), \tau_L(\theta_2)\right\} - \tau_L(\theta)\right] f(\theta)\, d\theta.$$

Suppose first that $\tau_M(x) > \tau_L(\phi(x))$. This implies that

$$D(x) = \int_x^{\phi(x)} [\tau_L(\phi(x)) - \tau_L(\theta)] f(\theta)\, d\theta - \int_{\theta_1}^{\theta_2} [\tau_L(\theta_2) - \tau_L(\theta)] f(\theta)\, d\theta,$$

and

$$\begin{aligned}
D'(x) &= [F(\phi(x)) - F(x)] \tau_L'(\phi(x)) \phi'(x) - [\tau_L(\phi(x)) - \tau_L(x)] f(x) \\
&= [F(\phi(x)) - F(x)] \frac{\tau_L'(x)[1 - F(x)]}{[1 - F(\phi(x))]} - [\tau_L(\phi(x)) - \tau_L(x)] f(x) \\
&\propto \frac{F(\phi(x)) - F(x)}{[1 - F(\phi(x))][\phi(x) - x]} - \left[\frac{\tau_L(\phi(x)) - \tau_L(x)}{\phi(x) - x} \times \frac{1}{\tau_L'(x)}\right] \frac{f(x)}{1 - F(x)} \\
&\geq \frac{F(\phi(x)) - F(x)}{[1 - F(\phi(x))][\phi(x) - x]} - \frac{f(x)}{1 - F(x)}
\end{aligned}$$

where the inequality follows from $\tau_L'(x) \geq \frac{\tau_L(\phi(x)) - \tau_L(x)}{\phi(x) - x}$ because τ_L is concave. Using the same argument that establishes statement (26) in the proof of Lemma 2, the last line is positive; thus, $D'(x) > 0$.

Next, suppose that $\tau_M(x) \leq \tau_L(\phi(x))$. Let $\hat{x} = \tau_M^{-1}(\tau_L(\phi(x)))$. Therefore,

$$D(x) = \int_x^{\hat{x}} [\tau_M(\theta) - \tau_L(\theta)] f(\theta)\, d\theta + \int_{\hat{x}}^{\phi(x)} [\tau_L(\phi(x)) - \tau_L(\theta)] f(\theta)\, d\theta - \int_{\theta_1}^{\theta_2} [\tau_L(\theta_2) - \tau_L(\theta)] f(\theta)\, d\theta$$

and

$$\begin{aligned}
D'(x) &= -[\tau_M(x) - \tau_L(x)] f(x) + [F(\phi(x)) - F(\hat{x})] \tau_L'(\phi(x)) \phi'(x) \\
&= [F(\phi(x)) - F(\hat{x})] \frac{\tau_L'(x)[1 - F(x)]}{[1 - F(\phi(x))]} - [\tau_M(x) - \tau_L(x)] f(x) \\
&\propto \frac{F(\phi(x)) - F(\hat{x})}{1 - F(\phi(x))} - \frac{\tau_M(x) - \tau_L(x)}{\tau_L'(x)} \left(\frac{f(x)}{1 - F(x)}\right) \\
&\geq \frac{F(\phi(x)) - F(\hat{x})}{1 - F(\phi(x))} - \frac{\tau_M(\hat{x}) - \tau_L(\hat{x})}{\tau_L'(\hat{x})} \left(\frac{f(\hat{x})}{1 - F(\hat{x})}\right) \\
&= \frac{F(\phi(x)) - F(\hat{x})}{1 - F(\phi(x))} - \frac{\tau_L(\phi(x)) - \tau_L(\hat{x})}{\tau_L'(\hat{x})} \left(\frac{f(\hat{x})}{1 - F(\hat{x})}\right) \\
&\propto \frac{F(\phi(x)) - F(\hat{x})}{[1 - F(\phi(x))][\phi(x) - \hat{x}]} - \left[\frac{\tau_L(\phi(x)) - \tau_L(\hat{x})}{\phi(x) - \hat{x}} \times \frac{1}{\tau_L'(\hat{x})}\right] \frac{f(\hat{x})}{1 - F(\hat{x})} \\
&\geq \frac{F(\phi(x)) - F(\hat{x})}{[1 - F(\phi(x))][\phi(x) - \hat{x}]} - \frac{f(\hat{x})}{1 - F(\hat{x})}
\end{aligned} \tag{41}$$

The inequality in (41) is due to the last part of Condition 1. Therefore, as before, $D'(x) > 0$.

For both cases, since $\lim_{x\downarrow\theta_1} D(x) = 0$, this implies that $D(x) > 0$ for $x > \theta_1$, which establishes the lemma. $\square$

Lemma 8. *Let $\theta_1 < \theta_1' < \theta_2 < \theta_2'$. Under Condition 1, $\int_{\theta_1'}^{\theta_2'} \tau_M'(\theta)[1 - F(\theta)]\, d\theta = \int_{\theta_1}^{\theta_2} \tau_M'(\theta)[1 - F(\theta)]\, d\theta$ implies that*

$$\int_{\theta_1'}^{\theta_2'} [\tau_M(\theta_2') - \tau_M(\theta)] f(\theta)\, d\theta - \int_{\theta_1}^{\theta_2} [\tau_M(\theta_2) - \tau_M(\theta)] f(\theta)\, d\theta > 0.$$

The proof of Lemma 8 follows the same argument as that for Lemma 7 for the case of $\tau_M(x) > \tau_L(\phi(x))$; thus, we omit it.

C.2 Asymmetric Information on Endowment Size

We consider an extension wherein the size of the seller's endowment is also her private information, as described in Section 5.2 of the main text.

Let $F(\cdot|n)$ denote the distribution of λ conditional on n and $J(\cdot)$ denote the distribution of n. Let their respective densities be $f(\cdot|n)$ and $j(\cdot)$. With a slight abuse of notations, let $(q(n,\lambda), t(n,\lambda))_{n\in[0,1],\lambda\in[0,n]}$ denote a direct mechanism, and let $U^B(n,\lambda) = V(q(n,\lambda),\lambda) - t(n,\lambda)$ and $U^S(n,\lambda) = t(n,\lambda) - C(q(n,\lambda),\lambda)$. Program $(\mathcal{P})$ in the current setup is

$$\max_{q(\cdot),t(\cdot)} \int_0^1 \int_0^n U^S(n,\lambda) f(\lambda|n) j(n)\, d\lambda dn, \qquad \text{s.t.} \qquad (IC_S^e), (IR_S^e) \text{ and } (IR_B^e) \tag{42}$$

where

$$U^S(n,\lambda) \geq t(n',\lambda') - C(q(n',\lambda'),\lambda) \qquad \forall (n,\lambda),(n',\lambda'), \tag{IC_S^e}$$

$$U^S(n,\lambda) \geq 0 \qquad \forall (n,\lambda) \tag{IR_S^e}$$

$$\int_0^1 \int_0^n U^B(n,\lambda) f(\lambda|n) j(n)\, d\lambda dn \geq b \tag{IR_B^e}$$

Because the type is two-dimensional, the type space does not have a complete order, which means that defining a monotonicity notion for the quantity schedule is not straightforward. The following is the appropriate monotonicity notion:

Definition 2. $q(\cdot)$ is "*monotonic*" if

- for any two types (n',λ') and (n,λ) in which $\lambda' > \lambda$, either $q(n',\lambda') \geq q(n,\lambda)$ or $q(n,\lambda) > n' = q(n',\lambda')$.

- for any two types (n', λ) and (n, λ) in which $n' > n$, either $q(n', \lambda) = q(n, \lambda)$ or $q(n', \lambda) > n = q(n, \lambda)$.

In words, when $\lambda' > \lambda$, the type with λ' (or more Ls) must trade weakly more than the type with λ whenever the endowment of λ' permits. Therefore, if the lower λ trades more than the higher λ', it must imply that λ' trades her entire endowment (i.e., her endowment constraint binds). Next, if two types have the same λ, then they must trade the same quantity whenever their endowments permit. Therefore, if $q(n', \lambda) > q(n, \lambda)$, it must imply that type (n, λ) trades her entire endowment.

Similar to Lemma 1, the seller's truth-telling constraint (IC_S^e) can be replaced by the following two conditions:

$$U^S(n, \lambda) = U^S(n, 0) + (c_H - c_L) \int_0^\lambda \mathcal{I}(q(n, l) > l)\, dl \quad \forall (n, \lambda), \tag{43}$$

$$q(\cdot) \text{ is monotonic according to Definition 2.} \tag{44}$$

From equation (43), for each n, constraint (IR_S^e) holds for all (n, λ) if it holds for $(n, 0)$. Additionally, substituting in equation (43), the objective function of program (42) becomes

$$\int_0^1 U^S(0, n)\, j(n)\, dn + \int_0^1 (c_H - c_L) \int_0^n \mathcal{I}(q(\lambda) > \lambda)\, [1 - F(\lambda|n)]\, j(n)\, d\lambda dn,$$

and constraint (IR_B^e) becomes

$$\int_0^1 \int_0^n \psi^B(q(n, \lambda), n, \lambda)\, j(n)\, d\lambda dn - \int_0^1 U^S(0, n)\, j(n)\, dn \geq b,$$

where (with an abuse of notation)

$$\psi^B(q, n, \lambda) = S(q, \lambda)\, f(\lambda|n) - (c_H - c_L)\, \mathcal{I}(q > \lambda)\, [1 - F(\lambda|n)]\, .$$

This implies that constraint (IR_B^e) must bind. Therefore, program (42) becomes

$$\begin{array}{c} \max\limits_{q(\cdot), u_0} \quad \int_0^1 \int_0^n S(q(n, \lambda), \lambda)\, f(\lambda|n)\, j(n)\, d\lambda dn - b \qquad \text{s.t.} \\ q(\cdot) \text{ is monotonic}, \quad u_0 \geq 0, \quad \text{and} \quad \int_0^1 \int_0^n \psi^B(q(n, \lambda), n, \lambda)\, j(n)\, d\lambda dn - u_0 = b \end{array} . \tag{45}$$

Lemma 9. *If $q(\cdot)$ is a solution to program (45), $q(\cdot)$ must satisfy the following two conditions:*

1. *$q^*(n, \lambda) \geq \lambda$ for all (n, λ).*

2. *Let $\mathcal{X}$ be the set of λ such that $q(1,\lambda) > \lambda$.*

   (a) *If $\mathcal{X}$ is empty, then $q(n,\lambda) = \lambda$ for all (n,λ).*

   (b) *Suppose that $\mathcal{X}$ is nonempty. Let λ_1 denote the smallest λ in $\mathcal{X}$, and let λ_2 denote the smallest $\lambda \in (\lambda_1, 1]$ such that $q^*(1,\lambda) = \lambda$. It holds that for all $\lambda \in [\lambda_1, \lambda_2]$, $q(n,\lambda) = \min\{n, \lambda_2\}$.*

Proof. $S(q,\lambda)$ is strictly increasing in q. When $q < \lambda$, $\psi^B(q,n,\lambda)$ is also strictly increasing in q. This explains point 1. Next, monotonicity of $q(\cdot)$ implies that for all λ, $q(1,\lambda) \geq q(n,\lambda)$. Point 2a hence follows. Finally, for Point 2b, since $\psi^B(q,n,\lambda)$ is also strictly increasing in q when $q > \lambda$, it must be the case the $q(1,\lambda) = \lambda_2$ for all $\lambda \in [\lambda_1, \lambda_2]$. Point 2b then follows from the monotonicity of $q(\cdot)$. □

Lemma 9 is the analog Lemma 5. Point 2b states that unless every type sells only their Ls, the optimal quantity schedule must feature some bunching, similar to the middle panel of Figure 1 for the baseline model. The difference is that because of the endowment constraint for some types, such bunching might not always be possible. When this happens, the endowment constraint for these types must bind.

The following is a sufficient condition for the solution quantity schedule to always be a threshold schedule:

Condition 2. For all $\lambda' > \lambda$, $\frac{f(\lambda'|n\geq\lambda')}{1-F(\lambda'|n\geq\lambda')} - \frac{f(\lambda|n\geq\lambda)}{1-F(\lambda|n\geq\lambda)} \geq \xi(\lambda',\lambda)$, where $\xi(x,\lambda) := -\frac{d}{dx}\log\int_x^1 [1-F(\lambda|n)]\, j(n)\, dn$.

Note that $\xi(\lambda',\lambda)$ is always positive. Thus, Condition 2 requires the conditional hazard rate to be increasing sufficiently quickly (as opposed to only increasing). The following is an example that satisfies Condition 2: $j(n) = 2n$ and $F(\lambda|n)$ is the uniform distribution over $[0,n]$.[27]

Proposition 5. *Under Condition 2, if $q(\cdot)$ is a solution to program (45), then $q(\cdot)$ must be a threshold schedule — i.e., there exists x such that $q(n,\lambda) = \lambda$ if $\lambda \leq x$ and $q(n,\lambda) = n$ if $\lambda > x$.*

Proof. Let $q^*(\cdot)$ be an optimal schedule. Let λ_1 and λ_2 be as defined in Lemma 9. The lemma is proved by showing that $\lambda_2 = 1$. Suppose, for a contradiction, that $\lambda_2 < 1$. There

[27]To be precise, this means that $f(\lambda|n) = \frac{1}{n}$ for $\lambda \in [0,n]$ and $f(\lambda|n) = 0$ for $\lambda > n$. It is readily verified that $\frac{f(\lambda|n\geq\lambda)}{1-F(\lambda|n\geq\lambda)} = \frac{2}{1-\lambda}$ and $\xi(\lambda',\lambda) = \frac{2(\lambda'-\lambda)}{(1-\lambda')(1-\lambda)+(\lambda'-\lambda)(1-\lambda')}$; thus Condition 2 holds.

must then exist $\lambda_3 > \lambda_2$ such that $q^*(1,\lambda) = \lambda$ for all $\lambda \in [\lambda_2, \lambda_3]$. Since $q^*(\cdot)$ is monotonic, this also implies that for any $n < 1$ and $\lambda \in [\lambda_2, \lambda_3]$, $q^*(n,\lambda) = \lambda$. Observe that

$$\begin{aligned}
&\int_0^1 \int_{\lambda_1}^{\lambda_3} S\left(q^*(n,\lambda),\lambda\right) f(\lambda|n)\, j(n)\, d\lambda dn \\
&= \int_{\lambda_3}^{n} \int_{\lambda_1}^{\lambda_3} S\left(q^*(n,\lambda),\lambda\right) f(\lambda|n)\, j(n)\, d\lambda dn \;+\; \int_{\lambda_1}^{\lambda_3} \int_{\lambda_1}^{n} S\left(q^*(n,\lambda),\lambda\right) f(\lambda|n)\, j(n)\, d\lambda dn \\
&= \int_0^1 \int_{\lambda_1}^{\lambda_3} \lambda s_L f(\lambda|n)\, j(n)\, d\lambda dn \\
&\quad + \int_{\lambda_2}^1 \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \lambda)\, s_H f(\lambda|n)\, j(n)\, d\lambda dn \;+\; \int_{\lambda_1}^{\lambda_2} \int_{\lambda_1}^{n} (n-\lambda)\, s_H f(\lambda|n)\, j(n)\, d\lambda dn
\end{aligned}$$

For some small $\varepsilon > 0$ and $x \in [\lambda_1, \lambda_1 + \varepsilon]$, let $\phi(x)$ be such that

$$\begin{aligned}
&\int_{\phi(x)}^1 \int_{x}^{\phi(x)} (\phi(x) - \lambda)\, s_H f(\lambda|n)\, j(n)\, d\lambda dn \;+\; \int_{x}^{\phi(x)} \int_{x}^{n} (n-\lambda)\, s_H f(\lambda|n)\, j(n)\, d\lambda dn \\
&= \int_{\lambda_2}^1 \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \lambda)\, s_H f(\lambda|n)\, j(n)\, d\lambda dn \;+\; \int_{\lambda_1}^{\lambda_2} \int_{\lambda_1}^{n} (n-\lambda)\, s_H f(\lambda|n)\, j(n)\, d\lambda dn
\end{aligned} \tag{46}$$

We restrict ε to be small enough such that $\phi(\lambda_1 + \varepsilon) < \lambda_3$.

Define schedule $\hat{q}_x$ as follows:

$$\hat{q}_x(n,\lambda) = \begin{cases} q^*(n,\lambda) & \text{, if } \lambda \notin [\lambda_1, \lambda_3) \\ \lambda & \text{, if } \lambda \in [\lambda_1, x) \\ \min\{n, \phi(x)\} & \text{, if } \lambda \in [x, \phi(x)) \\ \lambda & \text{, if } \lambda \in [\phi(x), \lambda_3) \end{cases}$$

By construction,

$$\int_0^1 \int_0^n S\left(\hat{q}_x(n,\lambda),\lambda\right) f(\lambda|n)\, j(n)\, d\lambda dn = \int_0^1 \int_0^n S\left(q^*(n,\lambda),\lambda\right) f(\lambda|n)\, j(n)\, d\lambda dn. \tag{47}$$

Let

$$\psi^S(q,n,\lambda) = \mathcal{I}(q > \lambda)(c_H - c_L)\left[1 - F(\lambda|n)\right].$$

Therefore, $\psi^B(q,n,\lambda) = S(q,\lambda) f(\lambda|n) - \psi^S(q,n,\lambda)$. The difference in the buyer's expected utility between $\hat{q}_x(\cdot)$ and $q^*(\cdot)$ is

$$
\begin{aligned}
D(x) &= \int_0^1 \int_0^n \left[\psi^B\left(\hat{q}_x(n,\lambda), n, \lambda\right) - \psi^B\left(q^*(n,\lambda), n, \lambda\right)\right] j(n)\, d\lambda dn \\
&= \int_0^1 \int_0^n \left[\psi^S\left(q^*(n,\lambda), n, \lambda\right) - \psi^S\left(\hat{q}_x(n,\lambda), n, \lambda\right)\right] j(n)\, d\lambda dn \\
&= \int_{\lambda_3}^1 \int_{\lambda_1}^{\lambda_3} \left[\psi^S\left(q^*(n,\lambda), n, \lambda\right) - \psi^S\left(\hat{q}_x(n,\lambda), n, \lambda\right)\right] j(n)\, d\lambda dn \\
&\quad + \int_{\lambda_1}^{\lambda_3} \int_{\lambda_1}^{n} \left[\psi^S\left(q^*(n,\lambda), n, \lambda\right) - \psi^S\left(\hat{q}_x(n,\lambda), n, \lambda\right)\right] j(n)\, d\lambda dn \\
&= (c_H - c_L)\left[\int_{\lambda_2}^1 \left(\int_{\lambda_1}^{\lambda_2} 1 - F(\lambda|n)\, d\lambda\right) h(n)\, dn + \int_{\lambda_1}^{\lambda_2} \left(\int_{\lambda_1}^{n} 1 - F(\lambda|n)\, d\lambda\right) j(n)\, dn\right] \\
&\quad - (c_H - c_L)\left[\int_{\phi(x)}^1 \left(\int_{x}^{\phi(x)} 1 - F(\lambda|n)\, d\lambda\right) j(n)\, dn + \int_{x}^{\phi(x)} \left(\int_{x}^{n} 1 - F(\lambda|n)\, d\lambda\right) j(n)\, dn\right].
\end{aligned}
$$

Differentiating $D(x)$ with respect to x, we have

$$
D'(x) = \left[\int_x^1 \left[1 - F(x|n)\right] j(n)\, dn - \left(\int_{\phi(x)}^1 \left[1 - F(\phi(x)|n)\right] j(n)\, dn\right) \phi'(x)\right] (c_H - c_L)
$$

From equation (46), we have

$$
\begin{aligned}
\phi'(x) =& \frac{\int_{\phi(x)}^1 \left[\phi(x) - x\right] f(x|n)\, j(n)\, dn + \int_x^{\phi(x)} (n - x) f(x|n)\, j(n)\, dn}{\int_{\phi(x)}^1 \left[F(\phi(x)|n) - F(x|n)\right] j(n)\, dn} \\
>& \frac{\left[\phi(x) - x\right] \int_x^1 f(x|n)\, j(n)\, dn}{\int_{\phi(x)}^1 \left[F(\phi(x)|n) - F(x|n)\right] j(n)\, dn}
\end{aligned}
$$

Therefore, we have $D'(x) > 0$ if

$$
\begin{aligned}
& \frac{\left[\phi(x) - x\right] \int_x^1 f(x|n)\, j(n)\, dn}{\int_{\phi(x)}^1 \left[F(\phi(x)|n) - F(x|n)\right] j(n)\, dn} \leq \frac{\int_x^1 \left[1 - F(x|n)\right] j(n)\, dn}{\left(\int_{\phi(x)}^1 \left[1 - F(\phi(x)|n)\right] j(n)\, dn\right)} \\
\iff & \frac{\int_x^1 f(x|n)\, j(n)\, dn}{\int_x^1 \left[1 - F(x|n)\right] j(n)\, dn} \leq \frac{\int_{\phi(x)}^1 \left[F(\phi(x)|n) - F(x|n)\right] j(n)\, dn}{\left[\phi(x) - x\right] \left(\int_{\phi(x)}^1 \left[1 - F(\phi(x)|n)\right] j(n)\, dn\right)}
\end{aligned}
$$

Fixing some λ, let $LHS = \frac{\int_\lambda^1 f(\lambda|n) j(n) dn}{\int_\lambda^1 [1-F(\lambda|n)] j(n) dn}$, and let $RHS(\lambda') = \frac{\int_{\lambda'}^1 [F(\lambda'|n) - F(\lambda|n)] j(n) dn}{(\lambda'-\lambda)\int_{\lambda'}^1 [1-F(\lambda'|n)] j(n) dn}$. By L'Hôpital's rule, $\lim_{\lambda' \downarrow \lambda} RHS(\lambda') = LHS$ and $\lim_{\lambda' \uparrow 1} RHS(\lambda') = \infty$. Suppose, for a contradiction, that there exists $\lambda' \in (\lambda, 1)$ such that $LHS > RHS(\lambda')$. This must imply that there exists $\hat{\lambda} \in (\lambda, 1)$ such that $LHS > RHS\left(\hat{\lambda}\right)$ and $\frac{d}{d\lambda'} RHS(\lambda')\Big|_{\lambda' = \hat{\lambda}} = 0$. By some algebra, $\frac{d}{d\lambda'} RHS(\lambda')\Big|_{\lambda'=\hat{\lambda}} = 0$ implies that

$$\begin{aligned}
\frac{\int_{\hat{\lambda}}^{1} f\left(\hat{\lambda}|n\right) j(n)\, dn}{\int_{\hat{\lambda}}^{1} \left[1 - F\left(\hat{\lambda}|n\right)\right] j(n)\, dn} &= RHS\left(\hat{\lambda}\right) \frac{\int_{\hat{\lambda}}^{1} \left[1 - F\left(\hat{\lambda}|n\right)\right] j(n)\, dn}{\int_{\hat{\lambda}}^{1} \left[1 - F(\lambda|n)\right] j(n)\, dn} + \frac{\left[1 - F\left(\lambda|\hat{\lambda}\right)\right] j\left(\hat{\lambda}\right)}{\int_{\hat{\lambda}}^{1} \left[1 - F(\lambda|n)\right] j(n)\, dn} \\
&< LHS + \frac{\left[1 - F\left(\lambda|\hat{\lambda}\right)\right] j\left(\hat{\lambda}\right)}{\int_{\hat{\lambda}}^{1} \left[1 - F(\lambda|n)\right] j(n)\, dn} \\
&= \frac{\int_{\lambda}^{1} f(\lambda|n)\, j(n)\, dn}{\int_{\lambda}^{1} \left[1 - F(\lambda|n)\right] j(n)\, dn} - \frac{d}{d\hat{\lambda}} \log\left(\int_{\hat{\lambda}}^{1} \left[1 - F(\lambda|n)\right] j(n)\, dn\right),
\end{aligned}$$

where the inequality in the second line follows from $\lambda < \hat{\lambda}$ and $LHS > RHS\left(\hat{\lambda}\right)$. However, this contradicts Condition 2.[28] Therefore, it holds that $LHS \leq RHS(\lambda')$, which implies that $D'(x) > 0$.

Since $D(\lambda_1) = 0$, there exists $x > \lambda_1$ such that $D(x) > 0$, thus implying that

$$\int_0^1 \int_0^n \psi^B\left(\hat{q}_x(n, \lambda); n, \lambda\right) j(n)\, d\lambda dn > \int_0^1 \int_0^n \psi^B\left(q^*(n, \lambda); n, \lambda\right) j(n)\, d\lambda dn \tag{48}$$

This implies that $\hat{q}_x(\cdot)$ is also feasible, and from equation (47), $\hat{q}_x(\cdot)$ is also optimal. However, equation (48) implies that constraint (IR_B^e) does not bind, which is a contradiction. □

C.3 Diminishing Marginal Utility

This subsection provides the details for the extension described in Subsection 5.3. Our main program is still program ($\mathcal{P}$), but with V now defined in Subsection 5.3.

The argument to transform program ($\mathcal{P}$) to program ($\tilde{\mathcal{P}}$) considers only the seller's incentives, which is unchanged here; thus, the argument still applies here. However, note that in the current setup,

$$S(q, \lambda) = \begin{cases} \nu_L(q) - qc_L & \text{if } q \leq \lambda \\ \nu_L(\lambda) - \lambda c_L + \nu_H(q - \lambda) - (q - \lambda)\, c_H & \text{if } q > \lambda \end{cases}. \tag{49}$$

Let

$$\bar{s}_H(x) := \nu_H(x) - xc_H.$$

[28]Note that $f(\lambda|n \geq \lambda) = \frac{\int_{\lambda}^{1} f(\lambda|n) j(n) dn}{1 - J(\lambda)}$ and $1 - F(\lambda|n \geq \lambda) = \frac{\int_{\lambda}^{1} j(n) dn - \int_{\lambda}^{1} F(\lambda|n) j(n) dn}{1 - J(\lambda)} = \frac{\int_{\lambda}^{1} [1 - F(\lambda|n)] j(n) dn}{1 - J(\lambda)}$. Therefore, $\frac{f(\lambda|n \geq \lambda)}{1 - F(\lambda|n \geq \lambda)} = \frac{\int_{\lambda}^{1} f(\lambda|n) j(n) dn}{\int_{\lambda}^{1} [1 - F(\lambda|n)] j(n) dn}$.

The condition required for the solution to program $(\tilde{\mathcal{P}})$ to be a threshold schedule is as follows:

Condition 3. For any $z, x \in [0,1]$ such that $z > x$,

$$\int_x^z \bar{s}'_H (z-\lambda) f(\lambda) \, d\lambda \geq \frac{\bar{s}_H (z-x)}{z-x} [F(z) - F(x)]. \tag{50}$$

Note that equation (50) can be written as

$$\int_x^z \left[\bar{s}'_H (z-\lambda) - \frac{\bar{s}_H (z-x)}{z-x} \right] f(\lambda) \, d\lambda \geq 0.$$

Since $\bar{s}_H$ is concave, by the mean value theorem, there exists $\bar{\lambda}$ such that $\bar{s}'_H (z-\lambda) > (=) [<] \frac{\bar{s}_H (z-x)}{z-x}$ if $\lambda > (=) [<] \bar{\lambda}$ — i.e., there are both positive and negative terms in the integrand. Thus, Condition 3 is a restriction on the curvature of $\bar{s}_H$ together with the distribution. The following is an example:

Lemma 10. *Condition 3 always holds if F is the uniform distribution.*

Proof. When F is the uniform distribution, $F(z) - F(x) = z - x$; thus, the right-hand side of equation (50) is $\bar{s}_H (z-x)$. Since $f(\lambda) = 1$, the left-hand side of equation (50) is $\int_x^z \bar{s}'_H (z-\lambda) \, d\lambda$. By the fundamental theorem of calculus, this is equal to $\bar{s}_H (z-x)$. □

Proposition 6. *Under Condition 3, if $q(\cdot)$ is a solution to program $(\tilde{\mathcal{P}})$ in the current setup, then $q(\cdot)$ must be a threshold schedule, defined in Definition 1.*

Proof. Since $\nu'_L (x) > c_L$ and $\nu'_H (x) > c_H$ for all x, $S(q, \lambda)$ defined in equation (49) is still always strictly increasing in q. In turn, $\psi^B (q, \lambda)$ is also increasing in q when $q < \lambda$ and when $q > \lambda$. Thus, Lemma 5 in the proof of Proposition 1 still holds. Let λ_1, λ_2, λ_3, $\eta(\varepsilon)$ and $\hat{q}_\varepsilon$ be as defined in that proof. Following the exact arguments, we only have to show that δ_ε in equation (25) is positive, where, over here,

$$\begin{aligned}
\delta_\varepsilon &= \int_{\lambda_1}^{\lambda_3} S\left[(\hat{q}_\varepsilon (\lambda), \lambda) - S(q^* (\lambda), \lambda)\right] f(\lambda) \, d\lambda \\
&= \int_{\lambda_1+\varepsilon}^{\lambda_2+\eta(\varepsilon)} \bar{s}_H (\lambda_2 + \eta(\varepsilon) - \lambda) f(\lambda) \, d\lambda - \int_{\lambda_1}^{\lambda_2} \bar{s}_H (\lambda_2 - \lambda) f(\lambda) \, d\lambda \, .
\end{aligned}$$

This is indeed the case from Lemma 11 below. □

Lemma 11. *Under Condition 3, when $f/(1-F)$ is nondecreasing, the following property holds: for any $\lambda_1 < \lambda_1' < \lambda_2 < \lambda_2'$, if $\int_{\lambda_1'}^{\lambda_2'} 1 - F(\lambda)\, d\lambda = \int_{\lambda_1}^{\lambda_2} 1 - F(\lambda)\, d\lambda$, then*

$$\int_{\lambda_1'}^{\lambda_2'} \bar{s}_H(\lambda_2' - \lambda) f(\lambda)\, d\lambda > \int_{\lambda_1}^{\lambda_2} \bar{s}_H(\lambda_2 - \lambda) f(\lambda)\, d\lambda. \tag{51}$$

Proof. Fix any $\lambda_1, \lambda_2 > 0$. For $x > \lambda_1$, define $\phi(x)$ to be such that $\int_x^{\phi(x)} 1 - F(\lambda)\, d\lambda = \int_{\lambda_1}^{\lambda_2} 1 - F(\lambda)\, d\lambda$. ϕ is strictly increasing and (by the implicit function theorem) differentiable, with $\phi'(x) = \frac{1-F(x)}{1-F(\phi(x))}$. Let $\bar{x} = \phi^{-1}(1)$ and

$$\bar{D}(x) := \int_x^{\phi(x)} \bar{s}_H(\phi(x) - \lambda) f(\lambda)\, d\lambda - \int_{\lambda_1}^{\lambda_2} \bar{s}_H(\lambda_2 - \lambda) f(\lambda)\, d\lambda$$

Our goal is to show that $\bar{D}(x) > 0$ for all $x \in (\lambda_1, \bar{x}]$. Note that $\bar{D}(x)$ is also differentiable for $x \in (\lambda_1, \bar{x})$, with

$$\begin{aligned}
\bar{D}'(x) &= \left(\int_x^{\phi(x)} \bar{s}_H'(\phi(x) - \lambda) f(\lambda)\, d\lambda \right) \frac{1 - F(x)}{1 - F(\phi(x))} - \bar{s}_H(\phi(x) - x) f(x) \\
&\propto \left[\frac{\int_x^{\phi(x)} \bar{s}_H'(\phi(x) - \lambda) f(\lambda)\, d\lambda}{\bar{s}_H(\phi(x) - x)} \right] \frac{1}{1 - F(\phi(x))} - \frac{f(x)}{1 - F(x)} \\
&\geq \left[\frac{F(\phi(x)) - F(x)}{\phi(x) - x} \right] \frac{1}{1 - F(\phi(x))} - \frac{f(x)}{1 - F(x)},
\end{aligned}$$

where the inequality holds because of Condition 3. In turn, from the property in equation (26) in Appendix A, $\bar{D}'(x) > 0$. Since $\bar{D}(x)$ is continuous for $x \in [\lambda_1, \bar{x}]$ and $\bar{D}(\lambda_1) = 0$, $\bar{D}'(x) > 0$ for all $x \in (\lambda_1, \bar{x})$ implies that $\bar{D}(x) > 0$ for all $x \in (\lambda_1, \bar{x}]$. □

C.4 Stochastic Mechanism

We consider the use of stochastic mechanism in this subsection. Because the utility functions of both the seller and the buyer are linear in the transfers, it suffice to allow for stochasticity only in the quantity. A stochastic contract is a double (α, t), where t is still the transfer from the buyer to the seller, and α is the CDF of the quantity that the seller must supply to the buyer. The following are two important notations:

$$\begin{aligned}
\bar{\alpha}(q) &:= 1 - \alpha(q) \\
\alpha^{\Delta}(q) &:= \alpha(q) - \sup_{x<q} \alpha(q)
\end{aligned}$$

$\bar{\alpha}(q)$ is the probability of having to supply more than q units under α. $\alpha^{\Delta}(q)$ denote the mass at q; thus, a deterministic contract consists of α where there is a q in which $\alpha^{\Delta}(q)=1$.

Let

$$\begin{aligned}\bar{C}(\alpha,\lambda) &= \int_0^1 C(q,\lambda)\,d\alpha(q)\\ \bar{V}(\alpha,\lambda) &= \int_0^1 V(q,\lambda)\,d\alpha(q)\\ \bar{S}(\alpha,\lambda) &= \bar{V}(\alpha,\lambda)-\bar{C}(\alpha,\lambda)=\int_0^1 S(q,\lambda)\,d\alpha(q)\end{aligned}$$

where C and V are defined in equations (1) and (2). Thus, under a stochastic contract (α,t) between the buyer and the type-λ seller, the buyer's and the seller's expected utility are $\bar{V}(\alpha,\lambda)-t$ and $t-\bar{C}(\alpha,\lambda)$, respectively.

Let $\{\alpha(\cdot|\lambda),t(\lambda)\}_{\lambda\in[0,1]}$ denote a direct stochastic mechanism. Let $\bar{U}^B(\lambda)=\bar{V}(\alpha(\cdot|\lambda),\lambda)-t(\lambda)$ and $\bar{U}^S(\lambda)=t(\lambda)-\bar{C}(\alpha(\cdot|\lambda),\lambda)$. Our main program is

$$\max_{\{\alpha(\cdot|\lambda),t(\lambda)\}_{\lambda\in[0,1]}} \int_0^1 \bar{U}^S(\lambda)f(\lambda)\,d\lambda, \qquad \text{s.t.} \qquad (\bar{IC}^S),(\bar{IR}^S) \text{ and } (\bar{IR}^B) \qquad , \tag{$\mathcal{P}_{stoch}$}$$

where

$$\bar{U}^S(\lambda)\geq t(\lambda')-\bar{C}(\alpha(\cdot|\lambda'),\lambda) \qquad \forall\lambda,\lambda', \tag{$\bar{IC}^S$}$$

$$\bar{U}^S(\lambda)\geq 0 \qquad \forall\lambda, \tag{$\bar{IR}^S$}$$

$$\int_0^1 \bar{U}^B(\lambda)f(\lambda)\,d\lambda\geq b. \tag{$\bar{IR}^B$}$$

By the envelope theorem, constraint ($\bar{IC}^S$) implies that

$$\frac{d\bar{U}^S(\lambda)}{d\lambda}=-\frac{\partial\bar{C}(\alpha(\cdot|\lambda),\lambda)}{\partial\lambda}=(c_H-c_L)\,\bar{\alpha}(\lambda|\lambda)$$

almost everywhere. Therefore,

$$\bar{U}^S(\lambda)=\bar{U}^S(0)+(c_H-c_L)\int_0^\lambda \bar{\alpha}(l|l)\,dl. \tag{$\bar{IC}^{S'}$}$$

Consider the program

$$\max_{\{\alpha(\cdot|\lambda),t(\lambda)\}_{\lambda\in[0,1]}} \int_0^1 \bar{U}^S(\lambda)f(\lambda)\,d\lambda, \qquad \text{s.t.} \qquad (\bar{IC}^{S'}),(\bar{IR}^S) \text{ and } (\bar{IR}^B) \qquad , \tag{52}$$

Program (52) is a relaxed version of program ($\mathcal{P}_{stoch}$) because it satisfies only a set of necessary conditions for constraint ($\bar{IC}^S$). Thus, the value of program (52) is weakly higher than the value of program ($\mathcal{P}_{stoch}$). Say that a mechanism $\{\alpha(\cdot|\lambda)\}_{\lambda\in[0,1]}$ is deterministic if $\{\alpha(\cdot|\lambda), t(\lambda)\}$ is a deterministic contract for all λ. We will provide a condition under which program (52) has a solution mechanism that is deterministic and satisfies all the constraints of program ($\mathcal{P}_{stoch}$); thus, program ($\mathcal{P}_{stoch}$) also has a solution mechanism that is deterministic.

Condition 4. $\frac{(1-\lambda)f(\lambda)}{1-F(\lambda)}$ is nondecreasing.

Proposition 7. *Under Condition (4), there is a solution mechanism to program ($\mathcal{P}_{stoch}$) that consists of an $\{\alpha(\cdot|\lambda)\}_{\lambda\in[0,1]}$ that takes the form of a deterministic threshold schedule — i.e., there exists a x such that $\alpha^{\Delta}(\lambda|\lambda) = 1$ for all $\lambda < x$ and $\alpha^{\Delta}(1|\lambda) = 1$ for all $\lambda \geq x$.*

We first provide some preliminary results to prove Proposition 7. First, constraint ($\bar{IC}^{S'}$) implies that $t(\lambda)$ must satisfy

$$t(\lambda) = \bar{U}^S(0) + (c_H - c_L)\int_0^{\lambda} \bar{\alpha}(l|l)\, dl + \bar{C}(\alpha(\cdot|\lambda), \lambda) \tag{53}$$

Doing integration by parts, we have

$$\int_0^1 t(\lambda) f(\lambda)\, d\lambda = \bar{U}^S(0) + (c_H - c_L)\int_0^1 \bar{\alpha}(\lambda]|\lambda)[1 - F(\lambda)]\, d\lambda + \int_0^1 \bar{C}(\alpha(\cdot|\lambda), \lambda) f(\lambda)\, d\lambda. \tag{54}$$

Therefore,

$$\begin{aligned}\int_0^1 \bar{U}^B(\lambda) f(\lambda)\, d\lambda &= \int_0^1 \bar{S}(\alpha(\cdot|\lambda), \lambda) f(\lambda)\, d\lambda - (c_H - c_L)\int_0^1 \bar{\alpha}(\lambda|\lambda)[1 - F(\lambda)]\, d\lambda - \bar{U}^S(0). \\ &= \int_0^1 \bar{\psi}^B(\alpha(\cdot|\lambda), \lambda)\, d\lambda - \bar{U}^S(0),\end{aligned}$$

where

$$\bar{\psi}^B(\alpha, \lambda) = \bar{S}(\alpha, \lambda) f(\lambda) - (c_H - c_L)\bar{\alpha}(\lambda)[1 - F(\lambda)]. \tag{55}$$

Following the same argument as the one for Lemma 4, constraint ($\bar{IR}^B$) must bind, and we can transform program (52) to the following program:

$$\max_{\{\alpha(\cdot|\lambda)\}_{\lambda\in[0,1]},\bar{U}^S(0)} \quad \int_0^1 \bar{S}\left(\alpha\left(\cdot|\lambda\right),\lambda\right) f\left(\lambda\right) d\lambda - b \qquad \text{s.t.}$$

$$\bar{U}^S(0) \geq 0\ , \quad \text{and} \quad \underbrace{\int_0^1 \bar{\psi}^B\left(\alpha\left(\cdot|\lambda\right),\lambda\right) d\lambda - \bar{U}^S(0) = b}_{(\bar{IR}^{B'})} \tag{$\tilde{\mathcal{P}}_{stoch}$}$$

Thus, our objective is to show that there exists a solution $\left\{\{\alpha\left(\cdot|\lambda\right)\}_{\lambda\in[0,1]}, \bar{U}^S(0)\right\}$ for program ($\tilde{\mathcal{P}}_{stoch}$) in which $\{\alpha\left(\cdot|\lambda\right)\}_{\lambda\in[0,1]}$ takes the form of a deterministic threshold schedule. We note the following property, which should be obvious:

Lemma 12. *For any $\left\{\{\alpha\left(\cdot|\lambda\right)\}_{\lambda\in[0,1]}, \bar{U}^S(0)\right\}$, if there exists $\{\hat{\alpha}\left(\cdot|\lambda\right)\}_{\lambda\in[0,1]}$ that has the property that $\bar{\psi}^B\left(\hat{\alpha}\left(\cdot|\lambda\right),\lambda\right) \geq \bar{\psi}^B\left(\alpha\left(\cdot|\lambda\right),\lambda\right)$ and $\bar{S}\left(\hat{\alpha}\left(\cdot|\lambda\right),\lambda\right) > \bar{S}\left(\alpha\left(\cdot|\lambda\right),\lambda\right)$ for a set of λ that has a strictly positive measure, then $\{\alpha\left(\cdot|\lambda\right)\}_{\lambda\in[0,1]}$ cannot be part of a solution to program ($\tilde{\mathcal{P}}_{stoch}$).*

Next, observe that

$$\begin{aligned}
\bar{S}\left(\alpha,\lambda\right) &= \int_{q\in[0,\lambda]} s_L q d\alpha\left(q\right) + \int_{q\in(\lambda,1]} \left[\lambda s_L + \left(q-\lambda\right) s_H\right] d\alpha\left(q\right) \\
&= \left[\int_{q\in[0,1]} \min\left\{q,\lambda\right\} d\alpha\left(q\right)\right] s_L \ + \ \left[\int_{q\in(\lambda,1]} \left(q-\lambda\right) d\alpha\left(q\right)\right] s_H
\end{aligned} \tag{56}$$

Lemma 13. *If $\left\{\{\alpha\left(\cdot|\lambda\right)\}_{\lambda\in[0,1]}, \bar{U}^S(0)\right\}$ is a solution to program ($\tilde{\mathcal{P}}_{stoch}$), then for all λ, $\alpha^{\Delta}\left(\lambda|\lambda\right) + \alpha^{\Delta}\left(1|\lambda\right) = 1$.*

Proof. Let $\{\alpha^*\left(\cdot|\lambda\right)\}_{\lambda\in[0,1]}$ be a solution to program ($\tilde{\mathcal{P}}_{stoch}$). Let $\{\hat{\alpha}\left(\cdot|\lambda\right)\}_{\lambda\in[0,1]}$ be another mechanism where $\hat{\alpha}^{\Delta}\left(\lambda|\lambda\right) = \alpha^*\left(\lambda|\lambda\right)$ and $\hat{\alpha}^{\Delta}\left(1|\lambda\right) = \bar{\alpha}^*\left(\lambda|\lambda\right)$. Given the expression in equation (56), it is immediate that $\bar{S}\left(\hat{\alpha}\left(\cdot|\lambda\right),\lambda\right) \geq \bar{S}\left(\alpha^*\left(\cdot|\lambda\right),\lambda\right)$, with the inequality holding strictly if $\alpha^*\left(\cdot|\lambda\right) \neq \hat{\alpha}\left(\cdot|\lambda\right)$. Next, since $\bar{\hat{\alpha}}\left(\lambda\lambda\right) = \bar{\alpha}\left(\lambda|\lambda\right)$, from equation (55), $\bar{\psi}^B\left(\hat{\alpha}\left(\cdot|\lambda\right),\lambda\right) - \bar{\psi}^B\left(\alpha^*\left(\cdot|\lambda\right),\lambda\right) = \bar{S}\left(\hat{\alpha}\left(\cdot|\lambda\right),\lambda\right) - \bar{S}\left(\alpha^*\left(\cdot|\lambda\right),\lambda\right)$, which is positive from above. Therefore, if $\alpha^*\left(\cdot|\lambda\right) \neq \hat{\alpha}\left(\cdot|\lambda\right)$ for a positive measure of λ, by Lemma 12, $\{\alpha^*\left(\cdot|\lambda\right)\}_{\lambda\in[0,1]}$ cannot be part of a solution to program ($\tilde{\mathcal{P}}_{stoch}$). $\square$

Henceforth, without loss of generality, we can restrict attention to only $\{\alpha\left(\cdot|\lambda\right)\}_{\lambda\in[0,1]}$ with the property that $\alpha^{\Delta}\left(\lambda|\lambda\right) + \alpha^{\Delta}\left(1|\lambda\right) = 1$.

Lemma 14. *Under Condition (4), if $\left\{\{\alpha\left(\cdot|\lambda\right)\}_{\lambda\in[0,1]}, \bar{U}^S(0)\right\}$ is a solution to program (52), then it must the case that for all λ, $\alpha^{\Delta}\left(\lambda|\lambda\right) = 1$ or $\alpha^{\Delta}\left(1|\lambda\right) = 1$.*

Proof. Suppose, for a contradiction, that the statement of the lemma does not hold — i.e., letting $\{\alpha^*(\cdot|\lambda)\}_{\lambda\in[0,1]}$ be a solution to program $(\tilde{\mathcal{P}}_{stoch})$, there exists an interval $[\lambda_1, \lambda_2]$ such that $\alpha^{*\Delta}(\lambda|\lambda), \alpha^{*\Delta}(1|\lambda) \in (0,1)$ for all λ. Let $\lambda^o = \frac{\lambda_1+\lambda_2}{2}$, and for $\lambda \in [\lambda_1, \lambda^o]$, let $\zeta(\lambda) = \lambda^o + \lambda - \lambda_1$. Therefore, $(\zeta(\lambda_1), \zeta(\lambda^o)] = (\lambda^o, \lambda_2]$ and $\zeta(\lambda) > \lambda$.

Let $k = \min_{\lambda\in[\lambda_1,\lambda^o]} \alpha^{\Delta}(\lambda|\lambda)$, and choose an $\varepsilon \in (0,k)$. For $\lambda \in (\lambda_1, \lambda^o)$, define $\eta(\lambda)$ as follows:

$$\begin{aligned} & [1-F(\lambda)]\,\varepsilon = [1-F(\zeta(\lambda))]\,\eta(\lambda) \\ \Longrightarrow \quad & \eta(\lambda) = \tfrac{[1-F(\lambda)]\varepsilon}{1-F(\zeta(\lambda))}. \end{aligned}$$

We choose ε small enough such that $\eta(\lambda) < \min_{\lambda\in[\lambda^o,\lambda_2]} 1 - \alpha^{*\Delta}(1|\lambda)$.

Consider the schedule $\{\hat{\alpha}(\cdot|\lambda)\}_{\lambda\in[0,1]}$, where $\hat{\alpha}(\cdot|\lambda) = \alpha^*(\cdot|\lambda)$ for all $\lambda \notin [\lambda_1, \lambda^o) \cup (\lambda^o, \lambda_2]$, and

$$\begin{aligned} &\text{for } \lambda \in [\lambda_1, \lambda^o), \quad \hat{\alpha}(\lambda|\lambda) = \alpha^*(\lambda|\lambda) + \varepsilon\ ;\ \hat{\alpha}(1|\lambda) = \alpha^*(1|\lambda) - \varepsilon \\ &\text{for } \lambda \in (\lambda^o, \lambda_1], \quad \hat{\alpha}(\lambda|\lambda) = \alpha^*(\lambda|\lambda) - \eta\left(\zeta^{-1}(\lambda)\right)\ ;\ \hat{\alpha}(1|\lambda) = \alpha^*(1|\lambda) + \eta\left(\zeta^{-1}(\lambda)\right) \end{aligned}$$

In words, a type $\lambda \in [\lambda_1, \lambda^o)$ is paired with a type $\zeta(\lambda) \in (\lambda^o, \lambda_2]$, where ζ is bijective. $\{\hat{\alpha}(\cdot|\lambda)\}_{\lambda\in[0,1]}$ is constructed as follows: for each $\lambda \in [\lambda_1, \lambda^o)$, there is an increase of probability ε for $\alpha^*(\lambda|\lambda)$ (and a decrease of ε for $\hat{\alpha}(\lambda|\lambda)$); and for its "paired" type $\zeta(\lambda)$, there is an increase of probability $\eta(\lambda)$ for $\alpha^*(1|\zeta(\lambda))$ (and a decrease of $\eta(\lambda)$ for $\hat{\alpha}(\zeta(\lambda)|\zeta(\lambda))$).

Note that $\bar{\alpha}^*(\lambda|\lambda) = \alpha^{*\Delta}(1|\lambda)$ and $\bar{\hat{\alpha}}(\lambda|\lambda) = \hat{\alpha}^{\Delta}(1|\lambda)$. Therefore,

$$\begin{aligned} &\int_{\lambda_1}^{\lambda_2} \bar{\hat{\alpha}}(1|\lambda)\,[1-F(\lambda)]\,d\lambda \\ =&\int_{\lambda_1}^{\lambda^o} \hat{\alpha}^{\Delta}(1|\lambda)\,[1-F(\lambda)]\,d\lambda + \int_{\lambda^o}^{\lambda_2} \hat{\alpha}^{\Delta}(1|\lambda)\,[1-F(\lambda)]\,d\lambda \\ =&\int_{\lambda_1}^{\lambda^o} \hat{\alpha}^{\Delta}(1|\lambda)\,[1-F(\lambda)]\,d\lambda + \int_{\lambda_1}^{\lambda^o} \hat{\alpha}^{\Delta}\left(1|\zeta^{-1}(\lambda)\right)[1-F(\zeta(\lambda))]\,d\lambda \\ =&\int_{\lambda_1}^{\lambda^o} \left(\alpha^{*\Delta}(1|\lambda) - \varepsilon\right)[1-F(\lambda)] + \left[\alpha^{*\Delta}(1|\zeta(\lambda)) + \eta(\lambda)\right][1-F(\zeta(\lambda))]\,d\lambda \\ =&\int_{\lambda_1}^{\lambda^o} \left[\alpha^{*\Delta}(1|\lambda) + \alpha^{*\Delta}(1|\zeta(\lambda))\right][1-F(\lambda)]\,d\lambda + \int_{\lambda_1}^{\lambda^o} [[1-F(\lambda)]\,\varepsilon + [1-F(\zeta(\lambda))]\,\eta(\lambda)]\,[1-F(\lambda)]\,d\lambda \\ =&\int_{\lambda_1}^{\lambda_2} \alpha^{*\Delta}(1|\lambda)\,[1-F(\lambda)]\,d\lambda. \end{aligned}$$

This implies that

$$\int_{\lambda_1}^{\lambda_2} \bar{\psi}^B\left(\hat{\alpha}\left(\cdot|\lambda\right),\lambda\right)d\lambda - \bar{\psi}^B\left(\alpha^*\left(\cdot|\lambda\right),\lambda\right)d\lambda = \int_{\lambda_1}^{\lambda_2}\left[\bar{S}\left(\hat{\alpha}\left(\cdot|\lambda\right)\right) - \bar{S}\left(\alpha^*\left(\cdot|\lambda\right)\right)\right]f\left(\lambda\right)d\lambda. \quad (57)$$

Next, similar to above,

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2}\left(1-\lambda\right)f\left(\lambda\right)\hat{\alpha}^{\Delta}\left(1|\lambda\right)d\lambda \\
=&\int_{\lambda_1}^{\lambda^o}\left(1-\lambda\right)f\left(\lambda\right)\hat{\alpha}^{\Delta}\left(1|\lambda\right)+\left(1-\zeta\left(\lambda\right)\right)f\left(\zeta\left(\lambda\right)\right)\hat{\alpha}^{\Delta}\left(1|\zeta\left(\lambda\right)\right)d\lambda \\
=&\int_{\lambda_1}^{\lambda^o}\left(1-\lambda\right)f\left(\lambda\right)\left[\hat{\alpha}^{\Delta}\left(1|\lambda\right)-\varepsilon\right]+\left[\left(1-\zeta\left(\lambda\right)\right)f\left(\zeta\left(\lambda\right)\right)\left[\hat{\alpha}^{\Delta}\left(1|\zeta\left(\lambda\right)\right)+\eta\left(\lambda\right)\right]\right]d\lambda \\
=&\int_{\lambda_1}^{\lambda_2}\left(1-\lambda\right)f\left(\lambda\right)\alpha^{*\Delta}\left(1|\lambda\right)d\lambda \\
&+\int_{\lambda_1}^{\lambda^o}\left(1-\zeta\left(\lambda\right)\right)f\left(\zeta\left(\lambda\right)\right)\eta\left(\lambda\right)-\left(1-\lambda\right)f\left(\lambda\right)\varepsilon\ d\lambda
\end{aligned} \quad (58)$$

Observe that

$$\begin{aligned}
\left(1-\zeta\left(\lambda\right)\right)f\left(\zeta\left(\lambda\right)\right)\eta\left(\lambda\right) &= \frac{\left(1-\zeta\left(\lambda\right)\right)f\left(\zeta\left(\lambda\right)\right)}{1-F\left(\zeta\left(\lambda\right)\right)}\left[1-F\left(\lambda\right)\right]\varepsilon \\
&> \frac{\left(1-\lambda\right)f\left(\lambda\right)}{1-F\left(\lambda\right)}\left[1-F\left(\lambda\right)\right]\varepsilon \ = \ \left(1-\lambda\right)f\left(\lambda\right)\varepsilon,
\end{aligned}$$

where the inequality is because of Condition (4). Therefore, the line in (58) is strictly positive, meaning that

$$\int_{\lambda_1}^{\lambda_2}\left(1-\lambda\right)f\left(\lambda\right)\hat{\alpha}^{\Delta}\left(1|\lambda\right)d\lambda > \int_{\lambda_1}^{\lambda_2}\left(1-\lambda\right)f\left(\lambda\right)\alpha^{*\Delta}\left(1|\lambda\right)d\lambda.$$

This implies that

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_2}\left[\bar{S}\left(\hat{\alpha}\left(\cdot|\lambda\right)\right)-\bar{S}\left(\alpha^*\left(\cdot|\lambda\right)\right)\right]f\left(\lambda\right)d\lambda \\
=&\int_{\lambda_1}^{\lambda_2}\left(1-\lambda\right)\left[\hat{\alpha}^{\Delta}\left(1|\lambda\right)-\alpha^{*\Delta}\left(1|\lambda\right)\right]s_H f\left(\lambda\right)d\lambda \ > \ 0.
\end{aligned}$$

Therefore, from equation (57), $\int_{\lambda_1}^{\lambda_2}\bar{\psi}^B\left(\hat{\alpha}\left(\cdot|\lambda\right),\lambda\right)d\lambda - \bar{\psi}^B\left(\alpha^*\left(\cdot|\lambda\right),\lambda\right)d\lambda > 0$. By Lemma 12, $\left\{\alpha^*\left(\cdot|\lambda\right)\right\}_{\lambda\in[0,1]}$ cannot be part of a solution to program ($\tilde{\mathcal{P}}_{stoch}$), which is a contradiction. $\square$

Henceforth, without loss of generality, we can restrict attention to only $\left\{\alpha\left(\cdot|\lambda\right)\right\}_{\lambda\in[0,1]}$ with the property that either $\alpha^{\Delta}\left(\lambda|\lambda\right)=1$ or $\alpha^{\Delta}\left(1|\lambda\right)=1$.

Lemma 15. *Under Condition (4), if* $\left\{\{\alpha(\cdot|\lambda)\}_{\lambda\in[0,1]}, \bar{U}^S(0)\right\}$ *is a solution to program* ($\tilde{\mathcal{P}}_{stoch}$)*, there must exist* x *such that* $\alpha^{\Delta}(\lambda|\lambda) = 1$ *for all* $\lambda < x$ *and* $\alpha^{\Delta}(1|\lambda) = 1$ *for all* $\lambda \geq x$*.*

Proof. Let $\{\alpha^*(\cdot|\lambda)\}_{\lambda\in[0,1]}$ be a solution to program ($\tilde{\mathcal{P}}_{stoch}$). Suppose, for a contradiction, that the statement of the lemma does not hold. This implies that there exist $\lambda_1 < \lambda_2 < \lambda_3 < 1$ such that $\alpha^{*\Delta}(1|\lambda) = 1$ for all $\lambda \in [\lambda_1, \lambda_2)$ but $\alpha^{*\Delta}(\lambda|\lambda) = 1$ for all $\lambda \in [\lambda_2, \lambda_3]$. For $x \in [\lambda_1, \lambda_1 + \varepsilon]$, define $\phi(x)$ by

$$\int_x^{\phi(x)} 1 - F(\lambda)\, d\lambda = \int_{\lambda_1}^{\lambda_2} 1 - F(\lambda)\, d\lambda. \tag{59}$$

Pick ε such that $\phi(\lambda_1 + \varepsilon) = \lambda_3$. Equation (59) implies that $\phi'(x) = \frac{1-F(x)}{1-F(\phi(x))}$. Let

$$\begin{aligned}
& D(x) := \int_x^{\phi(x)} (1-\lambda) f(\lambda)\, d\lambda \\
\implies\ & D'(x) = [1-\phi(x)] f(\phi(x)) \phi'(x) - (1-x) f(x) \\
& \qquad = [1-\phi(x)] f(\phi(x)) \frac{1-F(x)}{1-F(\phi(x))} - (1-x) f(x) \\
& \qquad \propto \frac{[1-\phi(x)] f(\phi(x))}{1-F(\phi(x))} - \frac{(1-x) f(x)}{1-F(x)} \quad \geq \quad 0,
\end{aligned}$$

where the inequality is due to Condition (4) and is strict if $x > \lambda_1$. Thus,

$$\int_{\lambda_1+\varepsilon}^{\lambda_3} (1-\lambda) f(\lambda)\, d\lambda = D(\lambda_1+\varepsilon) > D(\lambda_1) = \int_{\lambda_1}^{\lambda_2} (1-\lambda) f(\lambda)\, d\lambda. \tag{60}$$

Consider the schedule $\{\hat{\alpha}(\cdot|\lambda)\}_{\lambda\in[0,1]}$ where $\hat{\alpha}(\cdot|\lambda) = \alpha^*(\cdot|\lambda)$ for all $\lambda \notin [\lambda_1, \lambda_3]$, and $\hat{\alpha}^{\Delta}(\lambda|\lambda) = 1$ for all $\lambda \in [\lambda, \lambda_1 + \varepsilon)$, and $\hat{\alpha}^{\Delta}(1|\lambda) = 1$ for all $\lambda \in [\lambda_1 + \varepsilon, \lambda_3]$. Observe that

$$\begin{aligned}
& \int_{\lambda_1}^{\lambda_3} \left[\bar{S}(\hat{\alpha}(\cdot|\lambda)) - \bar{S}(\alpha^*(\cdot|\lambda))\right] f(\lambda)\, d\lambda \\
= & \left[\int_{\lambda_1+\varepsilon}^{\lambda_3} (1-\lambda) f(\lambda)\, d\lambda - \int_{\lambda_1}^{\lambda_2} (1-\lambda) f(\lambda)\, d\lambda\right] s_H > 0,
\end{aligned}$$

where the inequality is from equation (60). Additionally,

$$\begin{aligned}
&\int_{\lambda_1}^{\lambda_3} \left(\bar{\hat{\alpha}}(\lambda) - \bar{\alpha}^*(\lambda)\right)[1-F(\lambda)]\, f(\lambda)\, d\lambda \\
=&\int_{\lambda_1+\varepsilon}^{\lambda_3} \bar{\hat{\alpha}}^{\Delta}(\lambda)[1-F(\lambda)]\, d\lambda - \int_{\lambda_1}^{\lambda_2} \bar{\alpha}^{*\Delta}(\lambda)[1-F(\lambda)]\, d\lambda.
\end{aligned}$$

Thus, $\int_{\lambda_1}^{\lambda_3} \bar{\psi}^B(\hat{\alpha}(\cdot|\lambda)) - \bar{\psi}^B(\alpha^*(\cdot|\lambda))\, d\lambda = \int_{\lambda_1}^{\lambda_3}\left[\bar{S}(\hat{\alpha}(\cdot|\lambda)) - \bar{S}(\alpha^*(\cdot|\lambda))\right] f(\lambda)\, d\lambda > 0$. By Lemma 12, $\{\alpha^*(\cdot|\lambda)\}_{\lambda\in[0,1]}$ cannot be part of a solution to program ($\tilde{\mathcal{P}}_{stoch}$), which is a contradiction. □

Finally, Proposition 7 is a corollary of Lemma 15.

D Online Appendix: Additional Results

D.1 On the Constrained Pareto Frontier

This subsection studies the curvature of the Pareto frontier. From Proposition 2, if the solution is of class A1, the frontier is $\hat{S}(0) - b$, which has a constant slope of -1. If the solution is of class A2 , the frontier is $\hat{\psi}^S(\lambda^*(b))$.

Lemma 16. *Suppose that the density f is differentiable over $(0,1)$. For all $b \in \left(E[v] - c_H, \bar{b}\right)$, $\hat{\psi}^S(\lambda^*(b))$ is twice differentiable with respect to b, and $\frac{d^2}{db^2}\hat{\psi}^S(\lambda^*(b)) \leq (\geq)\, 0$ if $\frac{f(\lambda)(1-\lambda)}{1-F(\lambda)}$ is increasing (decreasing) at $\lambda = \lambda^*(b)$.*

Observe that $\frac{f(\lambda)(1-\lambda)}{1-F(\lambda)}$ is constant for the uniform distribution. This explains why the slope of the Pareto frontier of the A2 allocations under the uniform distribution is a constant.

Proof. Let $\mathcal{V}^S(b) = \hat{\psi}^S(\lambda^*(b))$. Twice differentiating $\mathcal{V}^S(b)$, we have

$$\begin{aligned}
\mathcal{V}^{S'}(b) &= -(c_H - c_L)[1-F(\lambda^*(b))]\,\lambda^{*'}(b) \\
\mathcal{V}^{S''}(b) &\propto f(\lambda^*(b))\left[\lambda^{*'}(b)\right]^2 - [1-F(\lambda^*(b))]\,\lambda^{*''}(b)
\end{aligned}$$

$$\implies \quad \mathcal{V}^{S''}(b) \leq 0 \quad \iff \quad \frac{f(\lambda^*(b))}{1-F(\lambda^*(b))} \leq \frac{\lambda^{*''}(b)}{[\lambda^{*'}(b)]^2} \tag{61}$$

Let $\hat{\psi}^B(x) = \hat{S}(x) - \hat{\psi}^S(x)$. Doing total differentiation twice on $\hat{\psi}^B(\lambda^*(b)) = b$, we have $\frac{\lambda^{*''}(b)}{\left[\lambda^{*'}(b)\right]^2} = -\frac{\hat{\psi}^{B''}(\lambda^*(b))}{\hat{\psi}^{B'}(\lambda^*(b))}$, where

$$\begin{aligned}\hat{\psi}^{B\prime}(\lambda) =& -(1-\lambda)\, s_H f(\lambda) + (c_H - c_L)\left[1 - F(\lambda)\right] \\ \hat{\psi}^{B\prime\prime}(\lambda) =& f(\lambda)\left[s_H - c_H + c_L\right] - (1-\lambda)\, s_H f'(\lambda)\end{aligned}$$

Therefore, equation (61) holds if and only if

$$\frac{f(\lambda^*(b))}{1 - F(\lambda^*(b))} \leq -\frac{\hat{\psi}^{B\prime\prime}(\lambda^*(b))}{\hat{\psi}^{B\prime}(\lambda^*(b))}$$

$\hat{\psi}^{B\prime}(\lambda^*(b))$ must be positive; if not, there exists $x < \lambda^*(b)$ in which $\hat{\psi}^B(x) = b$, which contradicts the definition of $\lambda^*(b)$. When $\hat{\psi}^{B\prime}(\lambda) = -(1-\lambda)\, s_H f(\lambda) + (c_H - c_L)\left[1 - F(\lambda)\right] \geq 0$,

$$\begin{aligned} & \tfrac{f(\lambda)}{1-F(\lambda)} \leq -\tfrac{f(\lambda)[s_H - c_H + c_L] - (1-\lambda) s_H f'(\lambda)}{-(1-\lambda) s_H f(\lambda) + (c_H - c_L)[1-F(\lambda)]} \\ \iff \quad & f(\lambda)\left[1 - F(\lambda)\right] \leq \left[1 - F(\lambda)\right](1-\lambda) f'(\lambda) + (1-\lambda)\left[f(\lambda)\right]^2 \\ \iff \quad & \tfrac{1}{1-\lambda} \leq \tfrac{f'(\lambda)}{f(\lambda)} + \tfrac{f(\lambda)}{1-F(\lambda)} \\ \iff \quad & -\tfrac{d}{d\lambda}\log(1-\lambda) \leq \tfrac{d}{d\lambda}\log f(\lambda) - \tfrac{d}{d\lambda}\log(1 - F(\lambda)) \\ \iff \quad & \tfrac{d}{d\lambda}\log\left[\tfrac{f(\lambda)(1-\lambda)}{1-F(\lambda)}\right] \geq 0 \\ \iff \quad & \left[\tfrac{1-F(\lambda)}{f(\lambda)(1-\lambda)}\right] \times \tfrac{d}{d\lambda}\left[\tfrac{f(\lambda)(1-\lambda)}{1-F(\lambda)}\right] \geq 0 \end{aligned}$$

Since $\frac{f(\lambda)(1-\lambda)}{1-F(\lambda)} > 0$, the last line is equivalent to $\frac{d}{d\lambda}\left[\frac{f(\lambda)(1-\lambda)}{1-F(\lambda)}\right] \geq 0$. □

D.2 The Monopsonist's Screening Problem

The mechanism implementing the buyer-optimal SB allocation is also a solution to the problem of a monopsonist who can offer a menu of trade contracts to screen the seller. However, although the mechanism that attains the buyer-optimal SB allocation is unique, the solution to the monopsonist's problem is not necessarily unique. This is because it is possible that the monopsonist can obtain the buyer-optimal second best utility ($\bar{b}$) while giving the seller a lower expected utility than what she gets under the buyer-optimal SB allocation.

In this section, we specifically study the monopsonist's screening problem in our model. We fully characterize the set of optimal screening mechanisms (which includes the mechanism in Proposition 2 for $b = \bar{b}$) and show that the quantity schedules of *all* the optimal screening mechanisms are still always threshold schedules. Moreover, this property holds even if $f/(1-F)$ is not monotonic.

Henceforth, assume that F still admits a density f, but $f/(1-F)$ is not necessarily increasing. The monopsonist's problem is

$$\max_{q(\cdot),t(\cdot)} \int_0^1 U^B(\lambda) f(\lambda)\, d\lambda, \qquad \text{s.t.} \qquad (IC^S) \text{ and } (IR^S).$$

Using a similar argument to Lemma 1, the problem becomes

$$\max_{\text{nondecreasing } q(\cdot)} \int_0^1 \psi^B(q(\lambda), \lambda)\, d\lambda. \tag{62}$$

Lemma 17. *If $q^*(\cdot)$ is a solution to program (62), $q^*(\cdot)$ must be a threshold schedule.*

Proof. Suppose that $q^*(\cdot)$ is a solution to program (62). It is straightforward to observe that the properties in Lemma 5 hold. Therefore, we only have to prove that $\lambda_2 = 1$. Suppose, for a contradiction, that $\lambda_2 < 1$. There must then exist $\lambda_3 > \lambda_2$ such $q^*(\lambda) = \lambda$ for all $\lambda \in [\lambda_2, \lambda_3]$. Pick some $\varepsilon > 0$ such that $\varepsilon < \max\{\lambda_3 - \lambda_2, \lambda_2 - \lambda_1\}$. For any $x \in [\lambda_2 - \varepsilon, \lambda_2 + \varepsilon]$, define the schedule $\tilde{q}_x(\cdot)$ as follows:

$$\tilde{q}_x(\lambda) = \begin{cases} x & \text{, if } \lambda \in [\lambda_1, x) \\ \lambda & \text{, if } \lambda \in [x, \lambda_3) \\ q^*(\lambda) & \text{, if } \lambda \notin [\lambda_1, \lambda_3) \end{cases} \tag{63}$$

Note that $\tilde{q}_x(\cdot)$ is also a nondecreasing schedule. Next, let $\xi(x) := \int_0^1 \psi^B(\tilde{q}_x(\lambda), \lambda) f(\lambda)\, d\lambda$ be the objective value under schedule $\tilde{q}_x(\cdot)$. Therefore,

$$\begin{aligned} \xi(x) = & \int_{\lambda \notin [\lambda_1, \lambda_3)} \psi^B(q^*(\lambda), \lambda) f(\lambda)\, d\lambda + + \int_x^{\lambda_3} \underbrace{\lambda s_L}_{\psi^B(\lambda,\lambda)} f(\lambda)\, d\lambda \\ & + \int_{\lambda_1}^{x} \underbrace{\left(x s_L + (x-\lambda)(s_H - s_L) - (c_H - c_L)\frac{1-F(\lambda)}{f(\lambda)} \right)}_{\psi^B(x,\lambda)} f(\lambda)\, d\lambda \end{aligned}$$

The first two derivatives of $\xi(x)$ are

$$\begin{aligned} \xi'(x) =& -(c_H - c_L)[1 - F(x)] + [F(x) - F(\lambda_1)] s_H. \\ \xi''(x) =& f(x)[(c_H - c_L) + s_H] > 0. \end{aligned}$$

Since $\tilde{q}_{\lambda_2}(\cdot)$ is the optimal schedule $q^*(\cdot)$, $x = \lambda_2$ must be a local maximizer. However $\xi''(\lambda_2)$ is strictly positive — contradiction. Therefore, λ_2 must be 1. □

By Lemma 17, we can restrict our search for the optimal quantity schedule to threshold schedules. Under the threshold-x quantity schedule, the value of the objective function is

$$\begin{aligned}\hat{\psi}^B(x) &:= \int_0^x \psi^B(\lambda,\lambda) f(\lambda)\, d\lambda + \int_x^1 \psi^B(1,\lambda) f(\lambda)\, d\lambda \\ &= \int_0^1 \lambda s_L f(\lambda)\, d\lambda \;+\; \int_x^1 \left[(1-\lambda) s_H - (c_H - c_L)\frac{1-F(\lambda)}{f(\lambda)}\right] f(\lambda)\, d\lambda. \qquad (64)\end{aligned}$$

Let $R(\lambda) := \frac{(1-\lambda)f(\lambda)}{1-F(\lambda)}$.

Proposition 8. *The monopsonist's optimal menus of contracts is*

$$(q^*(\lambda), t^*(\lambda)) = \begin{cases} (\lambda, \lambda c_L) & , \ \forall \lambda < \lambda^B \\ \left(1, C\left(1, \lambda^B\right)\right) & , \ \forall \lambda \geq \lambda^B \end{cases},$$

where $\lambda^B \in \arg\max_x \hat{\psi}^B(x)$. *The buyer's equilibrium expected utility is* $\hat{\psi}^B\left(\lambda^B\right)$.
If $\lambda^B \neq 0, 1$, *then* λ^B *must satisfy* $R\left(\lambda^B\right) = \frac{c_H - c_L}{s_H}$.

- *If* $R(\lambda)$ *is strictly decreasing in* λ *for all* $\lambda \in (0,1)$, *then* λ^B *is either* 0 *or* 1.
- *If* $R(\lambda)$ *is strictly increasing in* λ *for all* $\lambda \in (0,1)$, *then* λ^B *is unique, and the optimal schedule is the pointwise optimal schedule — i.e.,* $q^*(\lambda) = \arg\max_q \psi^B(q,\lambda)$ *for all* λ.

The first-order condition of $\max_x \hat{\psi}^B(x)$ is $R(\lambda) = \frac{c_H - c_L}{s_H}$; thus, this is a necessary condition for an interior optimal threshold. If $R(\lambda)$ is decreasing, $\hat{\psi}^B(x)$ is quasiconvex, hence leading to a corner solution. In contrast, if $R(\lambda)$ is increasing, the pointwise optimum of $\psi^B(\cdot,\lambda)$ is nondecreasing, which means that it is the unique optimum.