Markets with Within-Type Adverse Selection*

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Abstract

We study a bilateral trade problem in which the seller owns multiple units of a good and each unit is of potentially *different* quality. A bundled purchase prohibits the seller from self-selecting only her lemons for trade, hence mitigating the effects of "within-type adverse selection," where the seller privately knows the quality of each individual unit that she owns. However, the seller still privately knows the overall quality of her endowment ("across-type adverse selection"), and a bundled purchase can exacerbate its effects. We characterize the bargaining equilibria in such a setting and discuss their properties and implementations.

Keywords: Multiple-unit lemon market; Adverse selection; Bundled purchase
JEL Classification: D21, D82, D86

1 Introduction

Adverse selection is a key contributing factor of market failure. The seminal work of Akerlof (1970) highlighted its effects in a model in which a seller owns an indivisible good of privately known quality and attempts to sell this good to an uninformed buyer. Subsequent works extended the analysis to consider a seller owning multiple units of homogenous quality who can sell any part of her endowment (i.e., a divisible good). In this paper, we study bilateral trade with asymmetric information and correlated valuations with the innovation that the

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seller not only has multiple units of goods but also that the units are possibly of *heterogenous* quality that is indistinguishable by the buyer ex ante.

Such a setup describes many wholesale trades, such as the trade of used cars between car dealers and car rental companies and the liquidation of assets after a business shutdown. There are also examples from many markets known to be plagued with asymmetric information. For instance, in the health insurance market, the purchasing decision for health insurance is typically made at the household level, and each member has a different health risk. Thus, a household is a seller of multiple units of heterogeneous risks. In the financial market, banks securitize their loans and sell them to institutional investors. The quality of these loans varies and is often the bank's private information. In patent trades, the project owner (such as a startup company) often owns multiple projects and has private information about each project's potential.

A distinctive feature in such lemon markets is the presence of "*within-type adverse selection*," where, fixing the seller's endowment realization (or type), the buyer cannot distinguish the lemons from the high-quality goods in the seller's endowment.[1] Therefore, at any per-unit price, the buyer is more likely to be supplied with only the seller's lemons. In the examples above, a household is more eager to buy insurance for its sickest member at any given premium; a loan originator is more likely to securitize and sell its riskier loans to investors; and a startup is more likely to sell the property rights to less valuable projects and keep for itself those with high potential.[2] Such adverse effects can potentially be mitigated by a bundled purchase that forces the seller to trade her entire endowment or nothing at all because it prevents the seller from self-selecting only her lemons for trade.[3] However, the *overall* quality of the seller's endowment is also her private information ("*across-type adverse selection*"). Therefore, a given bundle price is also more likely to attract a seller who owns more lemons but completely drive out a seller with more high-quality goods, which then

[1]Note that this does not mean that every unit in the seller's endowment must appear identical; rather, it only requires that there is no observable characteristics that the buyer can use to distinguish his valuation of each of the units ex ante.

[2]For example, Nguyen (2020) provides empirical evidence that households in Vietnam self-select members with the highest health risks into health insurance, and Calem et al. (2011) document that banks cherry-picked their riskier loans to securitize during the subprime mortgage crisis.

[3]Such a strategy has been adopted in many patent trades. For instance, in 2012, Microsoft purchased 800 patents and the licenses to more than 300 patents from AOL Inc. for more than \$1 billion. Similarly, when Google purchased Motorola's mobile business arm for \$12.5 billion, the sale included the transfer of all of Motorola's more than 22000 patents (see Sandhu et al. (2013) for details). Bundled purchasing also manifests as "output contracts" that are often used in trade with smaller farms to prevent farmers from keeping their better products or side-selling them to other buyers. In the entertainment industry, telecommunication companies also often purchase all of the programs produced by a studio (Slattery, 2016).

makes it even more difficult for the buyer to purchase any high-quality good.

In this paper, we study a bilateral trade model with a buyer (he) facing a seller (she) who owns multiple units of a good. Each unit can be of either high or low quality. The two players' valuations of the good are interdependent, but the quality of each unit is privately known to only the seller. We characterize the trade equilibria across various bargaining weights, including both the buyer-optimal and seller-optimal equilibria, and we discuss their properties and implementations.

To elaborate, let H and L denote a high-quality and a low-quality good, respectively. The seller's opportunity costs of the goods are c_H and c_L, respectively, with $c_H > c_L$, and the buyer's valuations are such that there are gains from a trade for both types of goods. We will use a few examples to illustrate our key results. For simplicity, we assume here that the seller has a finite number of goods (whereas the main model considers a continuum of goods) and the buyer makes a take-it-or-leave-it offer to the seller (whereas the main analysis studies the equilibrium under various bargaining positions).

Suppose first that the seller has two units and her type is either $\{H, H\}$ or $\{L, L\}$ with equal probabilities. In this case, the best that the buyer can do is either offer a per-unit price of c_L (and buy from only type $\{L, L\}$) or c_H (and buy from both types but give some rent to type $\{L, L\}$ in the process). Suppose instead that the quality distribution changes, with the seller having exactly 1 H and 1 L with a probability of one. Note that the expected quality of a given unit is unchanged from that of the previous distribution. However, the buyer will now offer a two-unit bundle price of $c_H + c_L$ for the seller's entire endowment, giving himself the first-best utility. This efficient outcome arises even though the buyer cannot distinguish the H from the L in the seller's endowment before trade.

Observe that in this example, under the first quality distribution, there is no within-type adverse selection because the quality within the seller's endowment is always homogenous; however, there is significant across-type adverse selection because the overall quality of the endowment of the two types of sellers is very different. By contrast, under the second quality distribution, there is only within-type and no across-type adverse selection, and a bundled purchase eliminates all inefficiency. In Section 4.2, we formalize this intuition that some form of a bundled purchase indeed becomes more effective when adverse selection is "shifted" (appropriately defined) from across-type to within-type.

Next, assume the seller has three units and her type is either $\bar{\theta} = \{H, H, L\}$ or $\underline{\theta} = \{H, L, L\}$. Regardless of the probabilities of types $\bar{\theta}$ and $\underline{\theta}$, there is not a full-bundle price that allows the buyer to obtain the first-best utility now. If the buyer offers a high three-unit bundle price of $2c_H + c_L$, then both types of sellers will accept the offer, and type $\underline{\theta}$ earns a rent of $c_H - c_L$. Suppose that this is suboptimal for the buyer because his expected utility is already higher from offering a lower three-unit bundle price of $c_H + 2c_L$, which allows him to buy from only type $\underline{\theta}$ but without giving her any rent.[4] Observe that the buyer can further increase his utility by complementing this bundle offer with a low per-unit price offer of c_L, which will induce type $\bar{\theta}$ to sell her one unit of L and generate an additional surplus for the buyer. However, it is not immediately clear if the buyer can improve his utility even further by offering some two-unit bundle option to also buy one H from type $\bar{\theta}$.

A key equilibrium feature that we show is that the trade outcome must occur under one of the following two forms: either the seller sells her entire endowment, or the seller sells only all of her Ls and none of her Hs. Therefore, in this example, if the buyer finds it worthwhile to offer a two-unit bundle to entice type $\bar{\theta}$ to sell an H, then he must also find it worthwhile to entice type $\bar{\theta}$ to sell her entire endowment, which is assumed not to be the case. Thus, the low three-unit bundle price (meant for $\underline{\theta}$) complemented with a low per-unit price (meant for $\bar{\theta}$) are jointly the buyer's optimal pricing strategy. Recall that in the classical Akerlof single-unit seller lemon market with binary quality, the trade equilibrium always takes one of the following two forms: either all of the goods in the market are traded, or only the Ls and none of the Hs are traded. By contrast, in our example here, an H unit is traded if it belongs to type $\underline{\theta}$ but is not traded if it belongs to type $\bar{\theta}$.

The remainder of this paper proceeds as follows. We first discuss the related literature in the next subsection. Next, we describe our setup and problem in Sections 2 and 3. Subsequently, we analyze the equilibria in Sections 4 to 6 and discuss some extensions in Section 7. Finally, we conclude in Section 8. Unless stated otherwise, all of the proofs are provided in Appendix A, and an Online Appendix provides additional details.

1.1 Related literature

Within the adverse selection literature, our paper is most closely related to works on the lemon market with a divisible good. Under this setting, Rothschild and Stiglitz (1976), Attar et al. (2011, 2014, 2017), and Ales and Maziero (2016) consider competition on the buyer's side; Stiglitz (1977) and Chade and Schlee (2012, 2020) consider the monopsony case; and Gerardi et al. (2020) consider a dynamic model. Our main point of departure is that we

[4]Let $\bar{p}$ and $\underline{p} = 1 - \bar{p}$ be the probabilities of types $\bar{\theta}$ and $\underline{\theta}$, respectively. This condition means that $\underline{p}\left[(v_H + 2v_L) - (c_H + 2c_L)\right] > \underline{p}\,(v_H + 2v_L) + \bar{p}\,(2v_H + v_L) - (2c_H + c_L)$, where v_H (v_L) is the buyer's valuation of an H (L).

consider *heterogeneous* quality within the seller's endowment, which leads to within-type adverse selection that is absent in these aforementioned papers.

Huangfu and Liu (2020) study a bargaining problem with a seller owning two goods with different but correlated quality, and Crocker and Snow (2011) study an insurance screening problem in which the insuree has multiple perils with correlated risk type. In their settings, each "unit" is a distinct good to the buyer, even though the quality of each good is the seller's private information. Our setting is substantially different from theirs, with the buyer unable to tell the difference among every unit in the seller's endowment in our model. Therefore, under a per-unit price, the seller in our model has full flexibility to choose which unit to sell, but this is not the case in Huangfu and Liu (2020) and Crocker and Snow (2011).[5]

Our paper is also related to Samuelson (1984), who considers the bargaining problem with one-sided incomplete information, with the seller having an indivisible unit of a good. Because of the linearity in the preferences in Samuelson's model, the buyer cannot benefit from offering a menu of contracts to screen the seller. Our preference specification is similar to Samuelson's; however, because our seller's endowment is divisible and contains portions with different quality, screening and signaling that lead to partial separation are possible in our model. Another notable difference is that the buyer-optimal equilibrium in Samuelson's analysis can be quite different from the seller-optimal equilibrium, whereas all of the equilibria in our model have the threshold schedule property (but for different reasons).

Because of within-type adverse selection, the buyer in our model runs down the seller's endowment of lemons using low prices for the early units. Such a trade pattern is reminiscent of that in dynamic lemon markets (e.g., Daley and Green (2012), Fuchs et al. (2016), Kim (2017), Kaya and Kim (2018) and Gerardi et al. (2020)), where the low-quality sellers are "skimmed off" over time with low prices in the early stages.[6] However, there are two substantial differences. First, in a dynamic lemon market, the buyer's cost of offering a low price for the early units is a delay in trade with the high-quality seller, whereas the associ-

[5]As a concrete example of the setup considered in these two papers, consider an individual ("seller") who owns both the risk of an early death and the risk of longevity. Although the "buyer" (insurance company) does not know the quality of each of the individual's risks, he can still distinguish the two units of risk. Therefore, if trade is organized separately, the first risk is sold by purchasing life insurance, whereas the second risk is sold by purchasing annuity. Crucially, the buyer can prevent the seller from selling her longevity risk in the market that insures against the risk of early death, and vice versa. This last feature is not possible in our setup because the buyer has no means to distinguish between any two given units in the seller's endowment, which then gives rise to within-type adverse selection. Thus, the issues that we study are quite different from those in Huangfu and Liu (2020) and Crocker and Snow (2011).

[6]The basic trade pattern is as follows: the buyer offers a low price in the early stages, and a low-quality seller randomizes between accepting and rejecting/waiting, whereas a high-quality seller always waits. Over time, the buyer becomes more confident of the quality and more willing to offer a high price.

ated cost in our model is the potential of driving a seller with many high-quality units out of the market. Second, in a dynamic lemon market, the buyer usually has no commitment to future prices, whereas the prices of each unit in our model are determined from the start.[7]

Finally, our paper is also related to the literature on commodity bundling (e.g., Adams and Yellen, 1976; McAfee et al., 1989; Fang and Norman, 2006; Armstrong, 2013; Chen and Riordan, 2013; Chen and Li, 2018), whereby a multiproduct seller finds it worthwhile to offer a discount price for a predetermined basket of distinct products. Framed in our context, product bundling benefits the buyer because the distribution of the seller's costs to supply the bundle tends to be less dispersed than the costs to supply individual goods, hence decreasing the information rent required for the seller. Our setup is different from the commodity bundling literature in two ways, which makes the logic behind the gains from bundling different. First, in our setup, the valuations of the buyer and the seller are correlated, whereas the players usually have private values in the commodity bundling literature. Second, and more substantially, the commodity bundling literature considers the bundling of *distinct* products (for example, a flight ticket bundled with hotel accommodations), whereas the bundled goods in our model are perfectly indistinguishable to the buyer ex ante. Therefore, bundling in our model also helps prevent a seller from self-selecting only her lemons to trade, whereas this is not an issue in problems considered in the commodity bundling literature.[8]

2 Model

A buyer (he) faces a seller (she) endowed with a continuum of a good with a measure of one. Each infinitesimal unit of good is of either quality L or quality H. Henceforth, an L-quality (H-quality) unit is simply referred to as L (H). The seller's endowment can consist of both Ls and Hs, and the quality of each unit is the seller's private information. A trade contract is (q, t), where $q \in [0, 1]$ is the quantity and $t \in \mathbb{R}$ is the transfer from the buyer to the seller. Because all of the units appear identical to the buyer ex ante, we assume that after accepting a contract (q, t), the choice of which q units to trade is determined by the seller.

We first describe the players' preferences, delaying the description on how the contract is determined to the next section. The buyer's valuations of L and H are v_L and v_H,

[7] This distinction on the commitment to prices is of particular relevance to Gerardi et al. (2020), who consider the case in which the seller's good is divisible and the buyer can buy small portions of it over time. They show that when there is diminishing marginal utility from consumption, the lack of commitment to future contracts harms the buyer à la Coasian dynamics.

[8] For example, the seller cannot use a low-quality flight to pass off as a high-quality hotel accommodation.

respectively, and the seller's opportunity costs of L and H are c_L and c_H, respectively. If a trade occurs under a contract (q,t) with a seller who fulfills the quantity obligation with x_L units of L and x_H units of H, where $x_L + x_H = q$, then the buyer's utility is $x_L v_L + x_H v_H - t$, and the seller's utility is $t - x_L c_L - x_H c_H$. In the absence of a trade, both of the players' outside options are zero. Let $s_L := v_L - c_L$ and $s_H := v_H - c_H$.

Assumption 1. *$c_H > c_L$, $v_H > v_L$, and $s_L, s_H > 0$.*

Because $c_H > c_L$, the seller will run down her Ls first before providing any H. Let $[x]^+ := \max\{x, 0\}$. Thus, the cost for a seller with λ units of L to supply q units is

$$C(q,\lambda) := qc_L + [q-\lambda]^+ (c_H - c_L), \tag{1}$$

and the buyer's valuation of these q units is

$$V(q,\lambda) := qv_L + [q-\lambda]^+ (v_H - v_L). \tag{2}$$

Therefore, the seller's private information is fully captured by λ, which is henceforth referred to as the *seller's type*. When the buyer trades with a type-λ seller under a contract (q,t), the utilities of the buyer and the seller are $U^B(q,t;\lambda) := V(q,\lambda) - t$ and $U^S(q,t;\lambda) := t - C(q,\lambda)$, respectively. Let F denote the distribution of λ. To minimize technical details, we assume that F is continuous and admits a continuous density f that is strictly positive over $(0,1)$. Throughout, let $E[\lambda]$, $E[v]$, and $E[c]$ denote the expectations of λ, the buyer's valuation, and the seller's cost, respectively.[9]

Remark 1. The crucial distinction between the two players in the model is that only one of them has private information. We adopt the commonly used convention that the player with (without) the private information is labeled the "seller" ("buyer"). These labels are not important because we also do not restrict the signs of v, c and t. For example, in the insurance market, the roles of the "buyer" and the "seller" are different from their literal meanings — the "buyer" is the insurance company, the "seller" is the insuree, and the "good" is the insuree's risk. In this case, the valuation (v) and the cost (c) of the good are both negative; therefore, the transfer t will also be negative, representing an insurance premium payment from the insuree to the insurance company.

[9] That is, $E[\lambda] = \int_0^1 \lambda f(\lambda)\, d\lambda$, and, for $z \in \{v, c\}$, $E[z] = E[\lambda] z_L + (1 - E[\lambda]) z_H$.

Single-Unit Seller Benchmark. To compare our results with the classical lemon market, we provide a "*single-unit seller benchmark*" that fixes the quantity and quality of the goods in the economy but lets each good be owned by a different seller. Specifically, this benchmark economy has a buyer facing a continuum of sellers of a total measure of one. Each seller owns one infinitesimal unit of good; hence, the total quantity of goods in the economy is still of a measure of one. The probability of facing a seller with an L is $E[\lambda]$. An equilibrium in this benchmark economy is a posted price. The following is a well-known result:

Lemma 1. *Say that there is mild (severe) adverse selection if $E[v] \geq (<) c_H$. There are only two possible equilibrium outcomes in the single-unit seller benchmark.*

1. *Under mild adverse selection, all of the goods in the economy are traded.*

2. *Under severe adverse selection, all of the Ls are traded, but none of the Hs are traded.*

3 Bargaining Equilibria and Incentive Compatibility

We assume that the buyer and the seller bargain over the contract and study the equilibria across various bargaining positions for the two players. Following Myerson (1979), we frame this bargaining problem with one-sided incomplete information as a direct revelation game in which a mechanism designer designs a direct menu of contracts (or mechanism) that satisfies the relevant truth-telling and participation constraints.

Let $(q(\lambda), t(\lambda))$ denote the contract for type λ. The seller's incentive-compatibility and participation constraints and the buyer's participation constraint are, respectively,

$$U^S(q(\lambda), t(\lambda); \lambda) \geq U^S(q(\lambda'), t(\lambda'); \lambda) \qquad \forall \lambda, \lambda', \tag{IC_S}$$

$$U^S(q(\lambda), t(\lambda); \lambda) \geq 0 \qquad \forall \lambda, \tag{IR_S}$$

$$\int_0^1 U^B(q(\lambda), t(\lambda); \lambda) f(\lambda) d\lambda \geq 0. \tag{IR_B}$$

Let $\sigma \in [0, 1]$ parameterize the seller's bargaining power. Our problem is as follows:

$$\begin{array}{c} \max_{q(\cdot),t(\cdot)} \int_0^1 \left[\sigma U^S(q(\lambda), t(\lambda); \lambda) + (1-\sigma) U^B(q(\lambda), t(\lambda); \lambda)\right] f(\lambda) d\lambda \\ \text{s.t.} \quad (IC_S), (IR_S) \text{ and } (IR_B) \end{array} \tag{3}$$

$\sigma = 0$ is the *buyer-optimal equilibrium*, $\sigma = 1$ is the *seller-optimal equilibrium*, and $\sigma = \frac{1}{2}$

maximizes the equally weighted sum of the two players' utility.[10]

Let $\mathcal{I}$ be the indicator function, where $\mathcal{I}(x) = 1$ if x holds and $\mathcal{I}(x) = 0$ otherwise.

Lemma 2. *The solution to program (3) must satisfy the following conditions:*

- *For all λ, $U^S(q(\lambda), t(\lambda); \lambda) = U^S(q(0), t(0); 0) + (c_H - c_L)\int_0^\lambda \mathcal{I}(q(l) > l)\, dl$.*
- *$q(\lambda)$ is nondecreasing in λ.*

Constraint (IC_S) holds whenever these two conditions are satisfied.

Because single-crossing holds only weakly here, it is possible that constraint (IC_S) holds under a nonmonotonic $q(\cdot)$. The additional aspect of Lemma 2 (relative to standard incentive-compatibility results) is establishing that a nonmonotonic quantity schedule is always strictly suboptimal, which allows us to still only consider monotonic schedules.[11]

Using Lemma 2, we express the players' utility in their virtual valuations. Let

$$\psi^S(q,\lambda) := \mathcal{I}(q > \lambda)(c_H - c_L)\frac{1-F(\lambda)}{f(\lambda)}, \tag{4}$$

$$\psi^B(q,\lambda) := qs_L + \mathcal{I}(q > \lambda)\left[(q-\lambda)(s_H - s_L) - (c_H - c_L)\frac{1-F(\lambda)}{f(\lambda)}\right], \tag{5}$$

$$\begin{aligned} VS_\sigma(q,\lambda) &:= \sigma\psi^S(q,\lambda) + (1-\sigma)\psi^B(q,\lambda) \\ &= (1-\sigma)\left(qs_L + \mathcal{I}(q > \lambda)\left[(q-\lambda)(s_H - s_L) - \frac{1-2\sigma}{1-\sigma}(c_H - c_L)\frac{1-F(\lambda)}{f(\lambda)}\right]\right). \end{aligned} \tag{6}$$

Denoting $U^S((q(0), t(0)); \lambda)$ by u^0, type-λ seller's virtual utility is $u^0 + \psi^S(q,\lambda)$, and the buyer's virtual utility with type λ is $\psi^B(q,\lambda) - u^0$. Thus, program (3) becomes

$$\begin{array}{c} \max_{q(\cdot),u^0} \quad (2\sigma - 1)u^0 + \int_0^1 VS_\sigma(q(\lambda),\lambda) f(\lambda)\, d\lambda \\ \text{s.t.} \quad q(\cdot) \text{ is nondecreasing}, \quad u^0 \geq 0, \quad \text{and} \quad (IR'_B) \end{array}, \tag{$\mathcal{P}_\sigma$}$$

[10] $\sigma = 1$ ($\sigma = 0$) also admits the interpretation of a monopolist (monopsony) facing many buyers (sellers). Note that $\sigma = \frac{1}{2}$ is different from the first-best efficiency (which is when all the goods are traded) because the players' incentive and participation constraints must still be satisfied.

[11] Intuitively, although single-crossing does not hold strictly, $\frac{\partial U^S((q,t);\lambda)}{\partial q}$ is still never strictly decreasing in λ anywhere. Thus, if $\lambda_2 > \lambda_1$ but $q(\lambda_2) < q(\lambda_1)$, then incentive-compatibility implies that type λ_2 must be indifferent between her contract and type λ_1's contract. In turn, replacing type λ_2's contract in the menu with the contract of type λ_1 will always maintain incentive-compatibility, and doing so also increases the value of the objective function because the contract of type λ_1 trades more units and hence generates more surplus. Thus, this schedule with $q(\lambda_2) < q(\lambda_1)$ cannot be optimal.

where constraint (IR'_B) is

$$\int_0^1 \psi^B (q(\lambda), \lambda) f(\lambda) d\lambda - u^0 \geq 0. \tag{IR'_B}$$

Let $\bar{q}_\sigma(\lambda) := \arg\max_q VS_\sigma(q, \lambda)$ — i.e., the pointwise optimal of $VS_\sigma(\cdot, \lambda)$. Observe that, fixing λ, $VS_\sigma(q, \lambda)$ increases linearly in q when $q \in [0, \lambda)$, has a discrete jump (that is possibly upward or downward) at $q = \lambda$, and then increases linearly again in q all the way to $q = 1$. This implies that $\bar{q}_\sigma(\lambda)$ is either λ or 1. However, $\bar{q}_\sigma(\cdot)$ might not be monotonic and might not satisfy constraint (IR'_B).[12] Therefore, the solution to program ($\mathcal{P}_\sigma$) is generally not $\bar{q}_\sigma(\cdot)$. In the next two sections, we will solve program ($\mathcal{P}_\sigma$). The following class of quantity schedules will be important for the solutions.

Definition 1. $q(\cdot)$ is a "*threshold schedule*" if there exists a threshold $\hat{\lambda} \in [0, 1]$ such that $q(\lambda) = \lambda$ for all $\lambda < \hat{\lambda}$ and $q(\lambda) = 1$ for all $\lambda \geq \hat{\lambda}$.

For brevity, a threshold schedule with a threshold of $\hat{\lambda}$ is called the "*threshold-$\hat{\lambda}$ schedule*." Note that under a threshold schedule, there is never a partial trade of the seller's endowment of a particular quality of goods — if $\lambda < \hat{\lambda}$, the seller trades all of her Ls but none of her Hs; if $\lambda \geq \hat{\lambda}$, the seller trade all of her Ls and all of her Hs.

4 Buyer-Optimal Equilibrium

In this section, we study program ($\mathcal{P}_\sigma$) under $\sigma = 0$. This case applies particularly to markets in which the buyer has monopsony power.[13] We will discuss the other "buyer-advantaged" equilibria of $\sigma \in \left(0, \frac{1}{2}\right)$ later in Section 6.

[12]Note that $VS_\sigma(1, \lambda) \geq VS_\sigma(\lambda, \lambda)$ if and only if

$$\left(\frac{1-2\sigma}{1-\sigma}\right) \frac{1 - F(\lambda)}{(1-\lambda) f(\lambda)} \leq \frac{s_H}{c_H - c_L} \tag{7}$$

When $\sigma \geq \frac{1}{2}$, equation (7) always holds; thus, $\bar{q}_\sigma(\lambda) = 1$ for all λ. However, this schedule always violates constraint (IR'_B) under severe adverse selection. By contrast, equation (7) does not always hold when $\sigma < \frac{1}{2}$. In turn, a sufficient condition for $\bar{q}_\sigma(\cdot)$ to be nondecreasing in this case is that $\frac{1-F(\lambda)}{(1-\lambda)f(\lambda)}$ is nonincreasing in λ. This is a stronger requirement than an increasing hazard rate $\frac{f(\lambda)}{1-F(\lambda)}$, which is a standard assumption for screening models (e.g., Mussa and Rosen (1978); Maskin and Riley (1984)).

[13]For example, in the insurance market, Zhu et al. (2017) and Parys (2018) have documented instances in which insurance companies providing health insurance possess significant monopolist power within the geographical locations in which they operate. To restore stability in troubled economies, governments also sometimes act as the sole purchaser of troubled companies' assets and stock, such as the Troubled Asset Relief Program (TARP) set up by the US Treasury in the wake of the 2008 financial crisis.

4.1 Buyer-optimal Menu of Contracts

When $\sigma < \frac{1}{2}$, decreasing u^0 increases the objective value of program ($\mathcal{P}_\sigma$) and also relaxes constraint (IR'_B). Therefore, the optimal u^0 is zero. Moreover, since $VS_0(q, \lambda) = \psi^B(q, \lambda)$, we can ignore constraint (IR'_B). Thus, the buyer-optimal equilibrium is the solution to the problem of choosing a nondecreasing $q(\cdot)$ to maximize $\int_0^1 \psi^B(q(\lambda), \lambda) f(\lambda) d\lambda$.

Lemma 3. *The quantity schedule of the buyer-optimal equilibrium is a threshold schedule.*

Note that Lemma 3 does not require any additional assumption on the type distribution. To explain the basic argument behind Lemma 3, first observe that when $q < \lambda$, $\psi^B(q, \lambda)$ is strictly increasing in q. Therefore, the optimal schedule must have $q(\lambda) \geq \lambda$ for all λ.

Consider the quantity schedule in the left panel of Figure 1. From equation (5), when $q > \lambda$, $\psi^B(q, \lambda)$ is also strictly increasing in q. Because $q(\lambda) > \lambda$ for $\lambda \in [\lambda_1, \lambda_2)$, the objective value from this schedule can be increased by raising the quantity for $\lambda \in [\lambda_1, \lambda_2)$ while respecting the monotonicity constraint on the quantity schedule, and this is illustrated in the schedule in the middle panel. This implies that, if $q(\lambda) > \lambda$ for some λ, the optimal schedule must feature some bunching of the form in the middle panel.[14]

Next, fixing λ_1, consider quantity schedules of the form in the middle panel and parameterized by $\lambda_2 \in [\lambda_1, 1]$. The right panel illustrates an example in which the value of λ_2 is slightly higher than that in the middle panel. Let $\xi(\lambda_2)$ denote the value of the objective function under this set of schedules parameterized by λ_2. We show that $\xi(\lambda_2)$ is strictly convex in λ_2; thus, the optimal λ_2 is either 1 or λ_1, hence making it a threshold schedule. To see why $\xi(\cdot)$ is convex, consider what happens to the objective function when λ_2 is slightly raised. The marginal gain in surplus is $[F(\lambda_2) - F(\lambda_1)] s_H$, which is proportional to the horizontal strip of the schedule. This gain increases with λ_2 — visually, the horizontal strip is longer when λ_2 is higher. The marginal increase in (virtual) information rent from raising λ_2 is $[1 - F(\lambda_2)](c_H - c_L)$, which always decreases with λ_2. Therefore, the net marginal change in the objective value from raising λ_2 increases with λ_2. Thus, $\xi(\cdot)$ is convex.

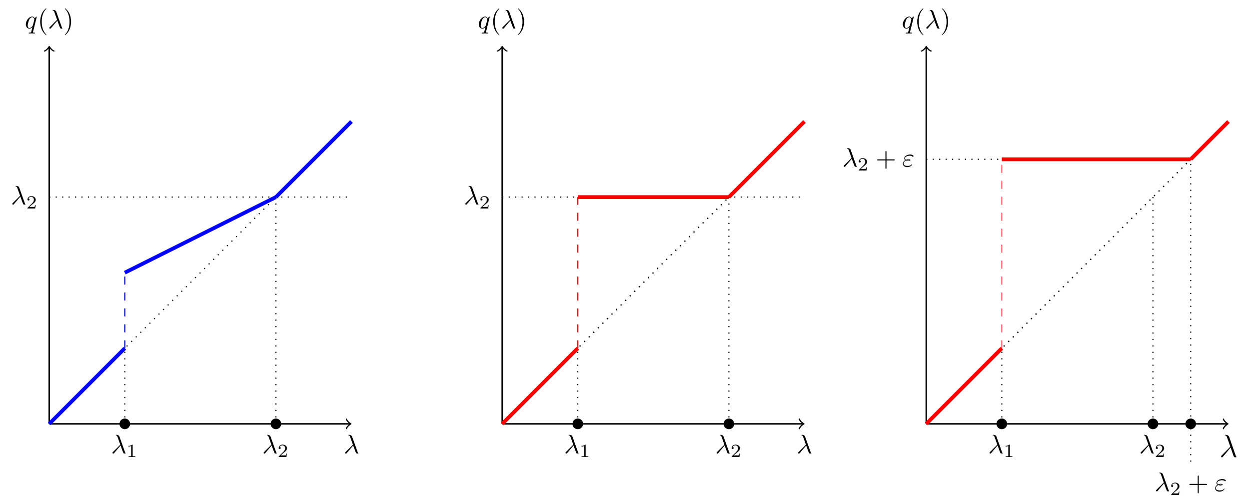
By Lemma 3, we can restrict our search for the optimal quantity schedule to threshold schedules. Under the threshold-$\hat{\lambda}$ quantity schedule, the value of the objective function is

$$\begin{aligned} \Psi^B\left(\hat{\lambda}\right) &:= \int_0^{\hat{\lambda}} \psi^B(\lambda, \lambda) f(\lambda) d\lambda + \int_{\hat{\lambda}}^1 \psi^B(1, \lambda) f(\lambda) d\lambda \\ &= \int_0^1 \lambda s_L f(\lambda) d\lambda \ + \ \int_{\hat{\lambda}}^1 \left[(1-\lambda) s_H - (c_H - c_L) \frac{1 - F(\lambda)}{f(\lambda)}\right] f(\lambda) d\lambda. \qquad (8) \end{aligned}$$

[14] The formal statement of this property is given in Lemma 5 in Appendix A.

Figure 1: On the Buyer-Optimal Equilibrium

(The diagonal dotted lines are the 45-degree lines.)



Proposition 1. *The set of buyer-optimal (i.e., $\sigma = 0$) menus of contracts is*

$$\left(q^*(\lambda), t^*(\lambda)\right) = \begin{cases} (\lambda, \lambda c_L) & , \ \forall\lambda < \lambda^B \\ \left(1, C\left(1, \lambda^B\right)\right) & , \ \forall\lambda \geq \lambda^B \end{cases},$$

where $\lambda^B \in \arg\max_{\hat{\lambda}} \Psi^B\left(\hat{\lambda}\right)$*. The buyer's equilibrium expected utility is* $\Psi^B\left(\lambda^B\right)$*. If* $\lambda^B \neq 0, 1$*, then* λ^B *must satisfy* $R\left(\lambda^B\right) = \frac{s_H}{c_H - c_L}$*, where* $R(\lambda) := \frac{1-F(\lambda)}{(1-\lambda)f(\lambda)}$*.*

- *If* $R(\lambda)$ *is strictly increasing in* λ *for all* $\lambda \in (0,1)$*, then* λ^B *is either* 0 *or* 1*.*
- *If* $R(\lambda)$ *is strictly decreasing in* λ *for all* $\lambda \in (0,1)$*, then* λ^B *is unique, and the optimal schedule is the pointwise optimal schedule,* $\bar{q}_0(\cdot)$*.*

The first-order condition of $\max_{\hat{\lambda}} \Psi^B\left(\hat{\lambda}\right)$ is $R(\lambda) = \frac{s_H}{c_H - c_L}$; therefore, this is a necessary condition for an interior optimal threshold. If $R(\lambda)$ is strictly increasing, then $\Psi^B\left(\hat{\lambda}\right)$ is strictly quasiconvex, hence leading to a corner solution. In contrast, if $R(\lambda)$ is strictly decreasing, then as noted in footnote 12, the pointwise optimum of $\psi^B(\cdot, \lambda)$ — i.e., schedule $\bar{q}_0(\cdot)$ — is nondecreasing; therefore, $\bar{q}_0(\cdot)$ is the unique solution.

Relative to the equilibria in the single-unit seller benchmark in Lemma 1, there is a richer set of equilibrium trade outcomes here. In particular, whenever $\lambda^B \in (0,1)$, there is a trade for the Hs of some but not all types of seller. This property — whether an

H is traded depends on the type of seller that owns it — is distinct from the benchmark equilibrium properties. Moreover, in the benchmark, the first-best efficiency is attained whenever $E[v] \geq c_H$ (mild adverse selection); by contrast, even if this condition holds, it is still possible that $\lambda^B > 0$, which implies that inefficiency still arises in equilibrium here.

Corollary 1. *(Implementation.) Let λ^B be the threshold of the optimal menu characterized in Proposition 1. The following are two possible ways of implementing the optimal menu:*

1. *(Mixed bundling.) The buyer offers the seller the options between selling her entire endowment at a bundle price of $C\left(1, \lambda^B\right)$ and selling à la carte at marginal price c_L.*
2. *(Marginal pricing with a switch.) The buyer offers a marginal price of c_L for the first λ^B units and a marginal price of c_H for all subsequent units.*

Despite there being a continuum of types and the optimal menu having "bunching," the optimal bundling strategy is simple, consisting of only a full bundle price complemented with a per-unit price. When the transfers are negative (e.g., in insurance), "marginal pricing with a switch" is simply a quantity discount, which is a commonly observed pricing strategy.

Viewing the optimal menu as determining when to switch the marginal price also provides further intuition on the optimality of threshold schedules. With within-type adverse selection, the seller always supplies her Ls first; hence, the buyer runs *up* the valuation of the seller's endowment. This means that if the buyer becomes confident enough in the expected quality to offer a high marginal price for the k-th (infinitesimal) unit, then he must be even more confident about the quality of the $(k+1)$-th unit, hence leading to his offering the high price to the last unit. This also means that if the seller is still in the market when the buyer switches to the high marginal price, she will remain in the market until her entire endowment is sold. This trade pattern is equivalent to a threshold quantity schedule.

4.2 Type Distribution and Buyer's Utility

Next, we consider how the buyer's expected utility changes with the distribution of λ. Let G denote another distribution of λ, and let π_F^{B*} and π_G^{B*} be the buyer's expected utility under the buyer-optimal equilibrium when the type distributions are F and G, respectively.

In the introduction, we provided a two-unit example wherein, when there is only within-type and no across-type adverse selection, a bundled purchase allows the buyer to attain the first-best utility; however, when there is only across-type and no within-type adverse selection, a bundled purchase is impotent. Our first result in this subsection considers the effect of a change in the relative severity of within-type and across-type adverse selection more generally. Specifically, we use mean-preserving contraction (MPC) as a measure of the relative severity of the two forms of adverse selection.[15] To motivate this, note that a MPC fixes the overall expected quality and moves probability weights from the extreme types to the mean type. Because the difference in the overall quality is the greatest between $\lambda = 0$ and $\lambda = 1$, across-type adverse selection is more severe when the likelihood is concentrated at the extreme types. By contrast, within-type adverse selection is more severe for the "middle" types than the extreme types — for the extreme types $\lambda = 0$ and $\lambda = 1$, the quality within the seller's endowment is homogenous. Therefore, we interpret a MPC as shifting adverse selection from across-type to within-type. Because the optimal menu is a form of bundling, we should expect a MPC to increase the buyer's equilibrium expected utility. The following result confirms this intuition.

Proposition 2. *If G is a mean-preserving contraction of F, then $\pi_G^{B*} \geq \pi_F^{B*}$. The inequality is strict if, under F or G, there is an optimal menu of contracts that has an interior threshold.*

Next, whereas the previous result fixes the overall expected quality, the next result illustrates that the buyer's expected utility is not always monotonic in the overall expected quality of the seller's goods. Say that $G >_{(\lambda_1, \lambda_2)} F$ if $G(\lambda) > F(\lambda)$ for all $\lambda \in (\lambda_1, \lambda_2)$ and $G(\lambda) = F(\lambda)$ for all $\lambda \notin (\lambda_1, \lambda_2)$.

Proposition 3. *Let λ_F^B be the threshold of some optimal menu of contracts of the buyer-optimal equilibrium under distribution F. Suppose that $\lambda_F^B > 0$.*

- *$G >_{\left(\lambda_F^B, 1\right)} F$ implies that $\pi_G^{B*} > \pi_F^{B*}$.*
- *$F >_{\left(0, \lambda_F^B\right)} G$ implies that $\pi_G^{B*} > \pi_F^{B*}$.*

$G >_{\left(\lambda_F^B, 1\right)} F$ implies that G is a shift in probability weights of F within the types that sell their entire endowment, from the high to the low λs, which increases the expected quality of a given unit of good. The first point of Proposition 3 states that this change increases the buyer's equilibrium expected utility. Note that this holds even if L generates more surplus than H (i.e., $s_L > s_H$). This is because the higher types always earn higher information rents. Thus, the distributional shift from higher to lower λs always decreases the expected information rent that the buyer gives up, and this rent effect always outweighs the negative surplus effect (if any).

[15]Following Rothschild and Stiglitz (1970), G is a mean-preserving contraction of F if $\int_0^1 \lambda dG(\lambda) = \int_0^1 \lambda dF(\lambda)$ and $\int_0^\lambda G(l)\, dl < \int_0^\lambda F(l)\, dl$ for all $\lambda \in (0, 1)$.

In the second point, $F >_{\left(0,\lambda_F^B\right)} G$ implies that G is a shift in probability weights within the types selling only their Ls, from the low to the high λs. In contrast to the previous point, such a change decreases the expected quality of the goods but still increases the buyer's equilibrium utility by increasing the trade volume of the Ls. The decrease in the quality of the goods of these types is irrelevant because their Hs are not traded.

5 Seller-Optimal Equilibrium

In this section, we consider program ($\mathcal{P}_\sigma$) for $\sigma \geq \frac{1}{2}$.

Observe that when $\sigma > \frac{1}{2}$, the objective function of program ($\mathcal{P}_\sigma$) strictly increases with u^0. Therefore, constraint (IR'_B) must bind; if not, we can increase u^0 slightly without violating constraint (IR'_B) and still increase the objective value. Substituting in the binding (IR'_B), program ($\mathcal{P}_\sigma$) becomes

$$\begin{aligned} \max_{q(\cdot)} \quad & \sigma \int_0^1 \underbrace{\left[\psi^B\left(q\left(\lambda\right),\lambda\right) + \psi^S\left(q\left(\lambda\right),\lambda\right)\right]}_{=\ V(q(\lambda),\lambda)-C(q(\lambda),\lambda)} f\left(\lambda\right) d\lambda \geq 0 \\ \text{s.t.} \quad & \int_0^1 \psi^B\left(q\left(\lambda\right),\lambda\right) f\left(\lambda\right) d\lambda \geq 0 \quad \text{and} \quad q\left(\cdot\right) \text{ is nondecreasing.} \end{aligned} \tag{9}$$

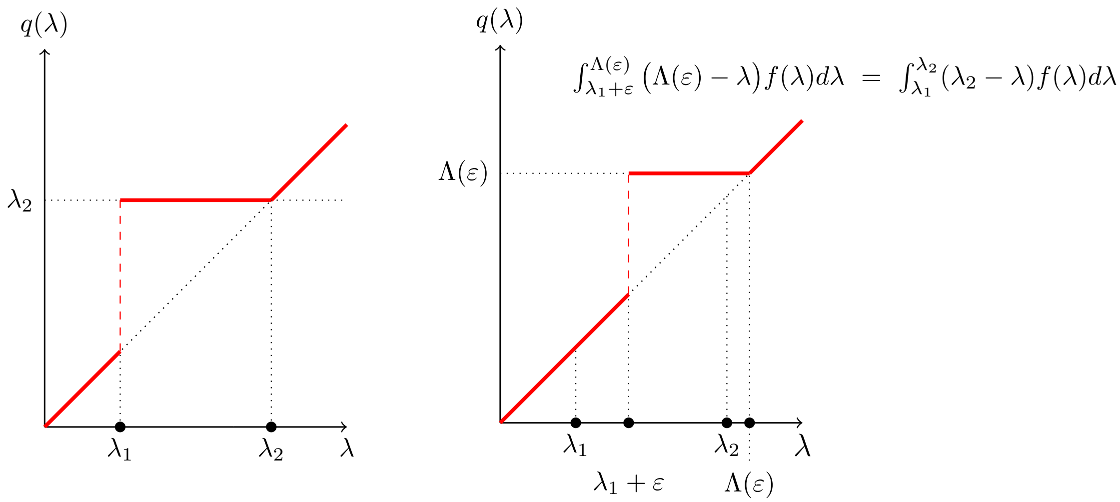
Observe that the solution to program (9) is the same across all $\sigma > \frac{1}{2}$. Additionally, program (9) will give the same solution for $q\left(\cdot\right)$ as program ($\mathcal{P}_\sigma$) when $\sigma = \frac{1}{2}$.

Corollary 2. *The optimal quantity schedule is the same across* $\sigma \in \left[\frac{1}{2}, 1\right]$. *Thus the seller-optimal equilibrium is also the equilibrium that maximizes the sum of the two players' utility.*

Lemma 4. *If* $\frac{f(\lambda)}{1-F(\lambda)}$ *is weakly increasing for* $\lambda \in (0,1)$, *the optimal quantity schedule under* $\sigma \in \left[\frac{1}{2}, 1\right]$ *must be a threshold schedule.*

As is the case for the buyer-optimal equilibrium (see Lemma 3), the quantity schedule of the seller-optimal equilibrium (and that for all $\sigma \geq \frac{1}{2}$) is also a threshold schedule. However, the mechanism driving this common property is quite different across the two equilibria, with the seller-optimal equilibrium requiring an additional assumption that the hazard rate of the type distribution is increasing. To explain the basic argument behind Lemma 4, we first note from program (9) that the seller-optimal problem is the same as maximizing the expected trade surplus, subject to the buyer's participation constraint. Because $s_L, s_H > 0$, a trade of any good increases the objective value. However, to trade the Hs, the buyer must give up information rent, which might adversely affect his participation constraint.

Figure 2: On the Seller-optimal Equilibrium



Consider a schedule involving types in $[\lambda_1, \lambda_2)$ trading some of their Hs. For the same reason as the buyer-optimal quantity schedule, if $q(\lambda) > \lambda$ for some λ, the quantity for these types must be "bunched," as illustrated in the left panel of Figure 2. Under this schedule, each $\lambda \in [\lambda_1, \lambda_2)$ sells $(\lambda_2 - \lambda)$ units of H. Let $K = s_H \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \lambda) f(\lambda) d\lambda$ denote the additional expected surplus that is generated from trading these Hs. From the expression of $\psi^B(\cdot)$, to obtain K, the expected additional (virtual) information rent incurred is $R_0 = (c_H - c_L) \int_{\lambda_1}^{\lambda_2} 1 - F(\lambda) d\lambda$. Instead, consider obtaining the same surplus K from types above $\lambda_1 + \varepsilon$. Specifically, let $\Lambda(\varepsilon)$ be the type such that $s_H \int_{\lambda_1+\varepsilon}^{\Lambda(\varepsilon)} (\Lambda(\varepsilon) - \lambda) f(\lambda) d\lambda = K$ and consider the schedule in the right panel of Figure 2. The rent required to obtain K from types in $\Big[\lambda_1 + \varepsilon, \Lambda(\varepsilon)\Big)$ is $R_\varepsilon = (c_H - c_L) \int_{\lambda_1+\varepsilon}^{\Lambda(\varepsilon)} 1 - F(\lambda) d\lambda$.

The value of $\Lambda(\varepsilon)$ depends on the density $f(\cdot)$, and the rent R_ε depends on the distribution $F(\cdot)$. We show that when $\frac{f(\lambda)}{1-F(\lambda)}$ is increasing in λ, R_ε is always smaller than R_0. Therefore, in terms of satisfying the buyer's participation constraint, it is always easier to generate additional surplus from trading Hs through the higher types. This implies that the types who sell Hs (if any) must extend to $\lambda = 1$ — i.e., a threshold schedule. Heuristically, extracting the Hs from the high types (instead of the low types) is less favored only when $f(\lambda)$ is decreasing, particular for $\lambda \geq \lambda_2$, because it means that the value of $\Lambda(\varepsilon)$ is high. An increasing hazard rate restricts that $1 - F(\lambda)$ must decrease faster relative to $f(\lambda)$, which helps nullify this upward effect of $\Lambda(\varepsilon)$ on R_ε.

Next, let $\psi^{S+B}(q, \lambda) = \psi^S(q, \lambda) + \psi^B(q, \lambda)$. Under the threshold-$\hat{\lambda}$ schedule, the value of the objective function in program (9) is

$$\begin{aligned}\Psi^{S+B}\left(\hat{\lambda}\right) =& \int_0^{\hat{\lambda}} \psi^{S+B}\left(\lambda,\lambda\right) f\left(\lambda\right) d\lambda + \int_{\hat{\lambda}}^1 \psi^{S+B}\left(1,\lambda\right) f\left(\lambda\right) d\lambda \\ =& E\left[\lambda\right] s_L + s_H \int_{\hat{\lambda}}^1 \left(1-\lambda\right) f\left(\lambda\right) d\lambda, \end{aligned} \tag{10}$$

and the buyer's participation constraint is equivalent to $\Psi^B\left(\hat{\lambda}\right) \geq 0$.

Proposition 4. *Suppose that $\sigma > \frac{1}{2}$. Assume that $\frac{f(\lambda)}{1-F(\lambda)}$ is weakly increasing for $\lambda \in (0,1)$. Let λ^S be the smallest $\hat{\lambda} \in [0,1]$ such that $\Psi^B\left(\hat{\lambda}\right) \geq 0$. The optimal menu of contracts is unique and is as follows:*

- *If there is mild adverse selection, $\lambda^S = 0$. The optimal menu is $\left(q^*\left(\lambda\right), t^*\left(\lambda\right)\right) = \left(1, E\left[v\right]\right)$ for all $\lambda \in [0,1]$.*

- *If there is severe adverse selection, $\lambda^S > 0$. The optimal menu is*

$$\left(q^*\left(\lambda\right), t^*\left(\lambda\right)\right) = \begin{cases} \left(\lambda, \lambda c_L\right) & , \ \forall \lambda < \lambda^S \\ \left(1, C\left(1, \lambda^S\right)\right) & , \ \forall \lambda \geq \lambda^S \end{cases},$$

The buyer's equilibrium expected utility is always zero.

A straightforward implication of Propositions 1 and 4 is that $\lambda^S \leq \lambda^B$, and the inequality is always strict when $\lambda^B > 0$. Note that comparing among threshold schedules, the resulting social efficiency always decreases with the threshold. Thus, unless the buyer-optimal equilibrium attains the first-best efficiency (i.e., $\lambda^B = 0$), the buyer-optimal equilibrium is *strictly* less socially efficient than any "seller-advantaged" equilibrium of $\sigma > \frac{1}{2}$.

For $\sigma = \frac{1}{2}$, the optimal menu of contracts is the same as that in Proposition 4, except that when there is mild adverse selection, there is some flexibility in the optimal transfer. In particular, under mild adverse selection, $\lambda^S = 0$; hence, there is only a full bundle price. Under $\sigma = \frac{1}{2}$, the objective function places equal weight on the two players' utility. Because the players have quasilinear utility, there is no loss in the total surplus by transferring utility from the seller to the buyer through t. Therefore, t can take any value in between c_H and $E\left[v\right]$. Moreover, when t is less than $E\left[v\right]$, the buyer's expected utility is positive, which never occurs when $\sigma > \frac{1}{2}$.[16]

[16]Such flexibility does not arise under severe adverse selection because the optimal threshold is uniquely determined as the lowest $\hat{\lambda}$ that satisfies the buyer's participation constraint, which means that the buyer's expected utility must be zero.

The menu of contracts in Proposition 4 can be sustained as a perfect Bayesian equilibrium with the seller offering a contract to the buyer. In this signaling game, the seller's strategy is a mapping from her type to a contract offer. Therefore, her equilibrium strategy is the optimal quantity and transfer schedule in Proposition 4. The off-equilibrium path beliefs can be specified as follows: given an off-equilibrium contract (q, t), if $q < \lambda^S$, the buyer places a probability of one on the seller's type being $\lambda = q$; if $q \geq \lambda^S$, the buyer's belief is the density $\frac{f(\lambda)}{1-F(\lambda^S)}$, supported on $\lambda \in \left[\lambda^S, 1\right]$.

Finally, a set of comparative static results similar to Propositions 2 and 3 also hold for the solution for $\sigma \geq \frac{1}{2}$. We relegate the details to Appendix B.1.

6 Other Buyer-advantaged Equilibria: $\sigma \in \left(0, \frac{1}{2}\right)$

The equilibria for $\sigma \in \left(0, \frac{1}{2}\right)$ are found in Appendix B.2. We discuss the main points here.

A key difference between the buyer-optimal problem ($\sigma = 0$) in Section 4 and the $\sigma \geq \frac{1}{2}$ problems in Section 5 is that constraint (IR'_B) can be ignored in the buyer-optimal problem. For certain values of $\sigma \in \left(0, \frac{1}{2}\right)$, particularly those close to 0, this property still holds. In these cases, the solution is similar to that in Proposition 1, with the appropriate change to the objective function. In particular, the solution must still be a threshold schedule. The value of the objective function of program ($\mathcal{P}_\sigma$) under the threshold-$\hat{\lambda}$ schedule is[17]

$$\Psi_\sigma\left(\hat{\lambda}\right) := \int_0^{\hat{\lambda}} VS_\sigma\left(\lambda, \lambda\right) f\left(\lambda\right) d\lambda + \int_{\hat{\lambda}}^1 VS_\sigma\left(1, \lambda\right) f\left(\lambda\right) d\lambda. \tag{11}$$

If there exists λ_σ that is both a solution to $\max_{\hat{\lambda}} \Psi_\sigma\left(\hat{\lambda}\right)$ *and* satisfies $\Psi^B\left(\lambda_\sigma\right) \geq 0$, then the threshold-$\lambda_\sigma$ schedule is a solution to program ($\mathcal{P}_\sigma$).

If such a threshold does not exist, then (IR'_B) becomes a binding constraint. However, this is still quite a different problem from $\sigma \geq \frac{1}{2}$ because (as noted at the start of Sections 4 and 5) when $\sigma < \frac{1}{2}$, the optimal u^0 must be zero, whereas this is not necessarily the case for $\sigma \geq \frac{1}{2}$. Therefore, the solution here is *not* simply the menu characterized in Proposition 4. Nevertheless, the tradeoff between trade surplus and rent provision, which leads to Lemma 4, still applies here because information rent provision has a more negative effect on constraint (IR'_B) than on the objective function.[18] Therefore, if the hazard rate of the type distribution

[17] Therefore, $\Psi_0\left(\hat{\lambda}\right) = \Psi^B\left(\hat{\lambda}\right)$ and $\Psi_{\frac{1}{2}}\left(\hat{\lambda}\right) = \frac{1}{2}\Psi^{S+B}\left(\hat{\lambda}\right)$.

[18] In the objective function $VS_\sigma(\cdot)$, the negative effect from information rent provision is weighted by $\frac{1-2\sigma}{1-\sigma}$, whereas in $\psi^B(\cdot)$ (i.e., constraint (IR'_B)), the same effect is weighted by $1 > \frac{1-2\sigma}{1-\sigma}$.

is increasing, surplus from trading Hs should still be created from the higher types as far as possible, hence implying that the optimal schedule is still a threshold schedule.

7 Extensions

Our analysis has focused on the situation with binary quality and common knowledge of the size of the seller's endowment. Without either of these assumptions, the seller's payoff-relevant information will be multidimensional, and such a problem is generally not very tractable. In this section, we discuss the progress that we can make to extend the main results from our baseline model to situations in which these restrictions are relaxed.

7.1 Nonbinary Quality

To simplify the exposition, we consider only three quality levels, but the arguments extend to any finite number of quality levels. Suppose now that there is an additional quality M. The seller's opportunity cost of M is c_M, and the buyer's valuation of M is v_M, with $c_L < c_M < c_H$ and $v_L < v_M < v_H$. Let $s_M = v_M - c_M$ and assume that $s_M > 0$.

With the additional quality level, the seller's type is two-dimensional, which makes it difficult to provide a single-crossing condition on the "cross types."[19] To make progress, we assume that the seller's type is summarized by a unidimensional $\theta \in [0,1]$ and two functions, $\tau_L, \tau_M : [0,1] \to [0,1]$, with the property that $\tau_L(\theta) \leq \tau_M(\theta)$ for all θ. Type-θ seller has $\tau_L(\theta)$ units of L, $[\tau_M(\theta) - \tau_L(\theta)]$ units of M, and $1 - \tau_M(\theta)$ units of H. Given the assumption on the costs, to fulfill any quantity obligation, the seller will first run down her endowment of Ls, followed by her endowment of Ms, before providing any Hs. Abusing notations, let the cost for type-θ seller to supply q units be

$$C(q,\theta) = qc_L + [q - \tau_L(\theta)]^+ (c_M - c_L) + [q - \tau_M(\theta)]^+ (c_H - c_M), \tag{12}$$

and the buyer's valuation of these q units be

$$V(q,\theta) = qv_L + [q - \tau_L(\theta)]^+ (v_M - v_L) + [q - \tau_M(\theta)]^+ (v_H - v_M). \tag{13}$$

Thus, under a contract (q,t), the buyer's and the seller's utility are $U^B(q,t;\theta) = V(q,\theta) - t$

[19] An example of a pair of cross types is a type θ_1 with 0.5 Ls and 0.5 Hs (and no Ms) and a type θ_2 with only Ms (and no Ls nor Hs) — type θ_1 has a lower marginal cost when $q < 0.5$ but a higher marginal cost when $q > 0.5$.

and $U^S(q,t;\theta) = t - C(q,\theta)$, respectively.

Let J denote the distribution of θ and assume that J is continuous and admits a density j that is strictly positive over $(0,1)$.

Assumption 2. *$\tau_L(\cdot)$ and $\tau_M(\cdot)$ are both differentiable, with $\tau_L'(\theta), \tau_M'(\theta) > 0$ for all $\theta \in (0,1)$. For any $\theta < \theta'$ in which $\tau_M(\theta) = \tau_L(\theta')$, it holds that $\tau_M'(\theta)/\tau_L'(\theta') \geq j(\theta)/j(\theta')$.*

This formulation of the type space restricts the kind of cross types that can coexist, but it is still flexible enough to accommodate a large range of type spaces. The monotonicity of the τ-functions allows us to define a single-crossing condition for the type space. The last property in Assumption 2 restricts the ratio of the slopes of the distribution (i.e., the density) to the ratio of the slopes of the τ-functions for certain pairs of types. This is the additional restriction required to extend the arguments used in the baseline model.[20]

Let $(q(\theta), t(\theta))$ denote the contract for type θ in a direct menu of contracts. Say that a schedule $q(\cdot)$ in our three-quality context is a *threshold schedule* if there exists θ_L and θ_M such that $0 \leq \theta_L \leq \theta_M \leq 1$ and $q(\theta) = \tau_L(\theta)$ for all $\theta < \theta_L$, $q(\theta) = \tau_M(\theta)$ for all $\theta \in [\theta_L, \theta_M)$, and $q(\theta) = 1$ for all $\theta \geq \theta_M$. As in the binary-quality case, under a threshold schedule in the three-quality case, there is never a partial trade of the seller's endowment of a particular quality — a type θ lower than θ_L trades all of her Ls but none of her Ms and Hs; a type between θ_L and θ_M trades all of her Ls and Ms but none of her Hs; and a type higher than θ_M trades her entire endowment.

Proposition 5. *Under Assumption 2, if $\{q^*(\cdot), t^*(\cdot)\}$ is a buyer-optimal equilibrium, $q^*(\cdot)$ must be a threshold schedule, and there exist θ_L^* and θ_M^*, with $0 \leq \theta_L^* \leq \theta_M^* \leq 1$, such that*

$$(q^*(\theta), t^*(\theta)) = \begin{cases} (\tau_L(\theta) \ , \ \tau_L(\theta) c_L) & , \ \textit{if } \theta < \theta_L^* \\ (\tau_M(\theta) \ , \ \tau_L(\theta_L^*) c_L + [\tau_M(\theta) - \tau_L(\theta_L^*)] c_M) & , \ \textit{if } \theta_L^* \leq \theta < \theta_M^* \\ (1 \ , \ \tau_L(\theta_L^*) c_L + [\tau_M(\theta_M^*) - \tau_L(\theta_L^*)] c_M + [1 - \tau_M(\theta_M^*)] c_H) & , \ \textit{if } \theta_M^* \leq \theta \end{cases}.$$

The basic arguments are similar to Section 4; thus, we relegate the details and the proof for Proposition 5 to Appendix B.3. Proposition 5 shows that the key property of the buyer-optimal equilibrium in our baseline model extends to the situation in which there are multiple quality levels. Similar to the second part of Corollary 1, the buyer-optimal equilibrium here can be implemented via marginal pricing with switches — the seller offers a marginal price

[20] The differentiability requirements in Assumption 2 are only to minimize the technical details and can be relaxed. A parametric example satisfying Assumption 2 is a set of $\tau_M(\cdot)$ and $\tau_L(\cdot)$ that are both affine, with $\tau_M(\cdot)$ steeper than $\tau_L(\cdot)$, and J is the uniform distribution.

of c_L for the first $\tau_L(\theta_L^*)$ units, a marginal price of c_M for the next $\tau_M(\theta_M^*) - \tau_L(\theta_L^*)$ units, and a marginal price of c_H for all subsequent units.

Extending the seller-optimal equilibrium analysis in Section 5 to the nonbinary context appears to be a more difficult problem, and we leave it to future work. However, we note that in our numerical simulations for the seller-optimal equilibrium under the type space described here, threshold schedules perform very well across various parameterizations of the τ-functions and type distributions. Moreover, fixing J to be the uniform distribution and considering a rich set of τ-functions that are affine and satisfy Assumption 2, the quantity schedule of the seller-optimal equilibrium is always a threshold schedule. We report our simulation results at the end of Appendix B.3.

7.2 Asymmetric Information on Endowment Size

Returning to the baseline model with binary quality, we consider an extension wherein the size of the seller's endowment is also her private information. Let the size of the seller's endowment be $n \in [0, 1]$. The seller now has a two-dimensional type (n, λ), with $\lambda \leq n$. Type (n, λ) has λ units of Ls and $(n - \lambda)$ units of Hs. A trade contract is still (q, t), and type (n, λ) can accept the contract only if $n \geq q$. After accepting the contract, the seller's utility is still $t - C(q, \lambda)$, and the buyer's utility is still $V(q, \lambda) - t$. Thus, the optimization problem is also still program (3) but with the type replaced by (n, λ). Given the similarity, we relegate the statement of this problem to Appendix B.4.

An important property in our baseline model is that, because of within-type adverse selection, the expected quality of the next marginal unit is always increasing because the seller is expected to supply her Ls first. This property no longer holds when there is asymmetric information on the seller's endowment size. As an example, suppose that there are only two possible types — type $\mu_1 = (1, 1)$ and type $\mu_2 = (0.5, 0)$. The buyer's expected valuation of the first 0.5 unit is strictly greater than v_L because of μ_2's Hs, but the subsequent 0.5 unit (if any) must be of quality L because it must come from μ_1, who has only Ls. Due to the lack of monotonicity in the expected valuation, the optimal menu of contracts can become very complex in general.

The problem here can be mapped to a special case of the setup with three quality levels considered in the previous subsection.[21] Therefore, Proposition 5 holds under the analogous

[21]Specifically, without loss of generality, let the binary quality considered here be H and M (instead of L). We can map a type (n, λ) — i.e., a seller who has λ units of Ms and $n - \lambda$ units of Hs — to the type (in the three-quality-level setup) who has $1 - n$ units of Ls, λ units of Ms, and $n - \lambda$ units of Hs, with the assumption that $v_L = c_L = 0$.

restriction on the type space. However, for the specific problem considered here, we are able to provide a weaker sufficient condition on the type distribution such that the trade pattern remains similar to that in the baseline model, and this condition applies to both the buyer-optimal ($\sigma = 0$) and seller-optimal ($\sigma = 1$) equilibria. Let $F(\lambda|n)$ and $f(\lambda|n)$ be the distribution and density of λ, respectively, conditional on n, and let $h(n)$ be the density of n. To simplify things, we assume that $f(\lambda|n)$ is continuous and strictly positive over $\lambda \in (0, n)$, and $h(n)$ is strictly positive over $(0, 1)$. The sufficient condition is the following:

Assumption 3. *For all* $\lambda' > \lambda$, $\frac{f(\lambda'|n\geq\lambda')}{1-F(\lambda'|n\geq\lambda')} - \frac{f(\lambda|n\geq\lambda)}{1-F(\lambda|n\geq\lambda)} \geq \gamma(\lambda', \lambda)$, *where* $\gamma(x, \lambda) := -\frac{d}{dx} \log \int_x^1 [1 - F(\lambda|n)] h(n) dn$.

Note that $\gamma(\lambda', \lambda)$ is always positive. Thus, Assumption 3 requires the conditional hazard rate to be increasing sufficiently quickly (as opposed to only increasing). An example that satisfies Assumption 3 is $F(\lambda|n)$ being the uniform distribution over $[0, n]$ and $h(n) = 2n$.[22]

Say that $q(\cdot)$ is the threshold-$\hat{\lambda}$ schedule here if $q(n, \lambda) = \lambda$ for all $\lambda < \hat{\lambda}$ and $q(n, \lambda) = n$ for all $\lambda \geq \hat{\lambda}$. As in our baseline model, under the threshold-$\hat{\lambda}$ schedule here, a type who has less than $\hat{\lambda}$ units of Ls sells all of her Ls but none of her Hs, whereas a type who has more than $\hat{\lambda}$ units of Ls sells her entire endowment, which is not necessarily of measure 1 now. Let $\hat{\Psi}^B(\hat{\lambda})$ denote the buyer's expected utility expressed in virtual valuation under the threshold-$\hat{\lambda}$ schedule:

$$\begin{aligned}\hat{\Psi}^B\left(\hat{\lambda}\right) = & \int_0^1 \int_0^n \lambda s_L f(\lambda|n) h(n) d\lambda dn \\ & + \int_{\hat{\lambda}}^1 \int_{\hat{\lambda}}^n \left[(n-\lambda) s_H - (c_H - c_L) \frac{1-F(\lambda|n)}{f(\lambda|n)}\right] f(\lambda|n) h(n) d\lambda dn\end{aligned} \tag{14}$$

The first line is the expected surplus from the Ls, and the second line is the expected surplus from the Hs from the types with $\lambda \geq \hat{\lambda}$ minus their information rent.

Proposition 6. *Suppose that Assumption 3 holds. Let* $\hat{\lambda}^B \in \arg\max_{\hat{\lambda}} \hat{\Psi}^B\left(\hat{\lambda}\right)$, *and let* $\hat{\lambda}^S$ *be the smallest* $\hat{\lambda} \in [0, 1]$ *such that* $\hat{\Psi}^B\left(\hat{\lambda}\right) \geq 0$.

- *For the buyer-optimal problem, the optimal menu of contracts is as follows:*

$$\left(q^*(n,\lambda), t^*(n,\lambda)\right) = \begin{cases} (\lambda, \lambda c_L) & , \forall \lambda < \hat{\lambda}^B \\ \left(n, \hat{\lambda}^B c_L + \left(n - \hat{\lambda}^B\right) c_H\right) & , \forall \lambda \geq \hat{\lambda}^B \end{cases},$$

[22] To be precise, this means that $f(\lambda|n) = \frac{1}{n}$ for $\lambda \in [0, n]$ and $f(\lambda|n) = 0$ for $\lambda > n$. It is readily verified that $\frac{f(\lambda|n\geq\lambda)}{1-F(\lambda|n\geq\lambda)} = \frac{2}{1-\lambda}$ and $\gamma(\lambda', \lambda) = \frac{2(\lambda'-\lambda)}{(1-\lambda')(1-\lambda)+(\lambda'-\lambda)(1-\lambda')}$; thus Assumption 3 holds.

- *For the seller-optimal problem, the optimal menu of contracts is as follows: letting* $\hat{E}[v] = \int_0^1 \int_0^n [\lambda v_L + (n-\lambda) v_H] f(\lambda|n) h(n) d\lambda dn$,
  - *if* $\hat{E}[v] \geq c_H$, *then* $(q^*(n,\lambda), t^*(n,\lambda)) = \left(n, n\hat{E}[v]\right)$;
  - *if* $\hat{E}[v] < c_H$, *then*

$$(q^*(n,\lambda), t^*(n,\lambda)) = \begin{cases} (\lambda, \lambda c_L) & , \ \forall \lambda < \hat{\lambda}^S \\ \left(n, \hat{\lambda}^S c_L + \left(n - \hat{\lambda}^S\right) c_H\right) & , \ \forall \lambda \geq \hat{\lambda}^S \end{cases}.$$

The basic argument for the optimality of threshold schedules here is the same as that for the seller-optimal problem in the baseline model, which has been explained in the previous section.[23] Thus, we relegate the details and the proof of Proposition 6 to Appendix B.4. We note that the trade pattern here is similar to that in the baseline model — there is a threshold $\hat{\lambda}$ in which each infinitesimal unit is traded at a marginal price of c_L (c_H) when the quantity traded is lower (higher) than $\hat{\lambda}$.

8 Conclusion

In this paper, we studied a model of trade with adverse selection in which a seller owns multiple units that are of potentially different quality. In our model, the seller not only privately knows the overall quality of her endowment (across-type adverse selection) but also the quality of each of her units (within-type adverse selection). We characterize the trade equilibria when the quality is binary and show that trade always occurs via a "threshold schedule," where the types higher than some threshold sell their entire endowments at a full-bundle price, whereas the types lower than the threshold sell only their lemons (and none of their Hs) at a low per-unit price. This contrasts with the equilibrium in the classical single-unit lemon market with binary quality, where either all of the goods in the market are sold or only the lemons are sold in equilibrium.

Because of within-type adverse selection, the buyer in our setup must first run down the seller's endowment of Ls before he can obtain any Hs. This feature is distinct from existing models of the lemon market and raises a few issues that could be worthwhile for further research, particularly when there are multiple buyers behaving strategically in the market.

[23] Note that in the explanation for Lemma 4, we show that we can reduce the expected rent provision while keeping the expected surplus unchanged. This is equivalent to increasing the buyer's expected utility; thus, the argument also applies to the buyer-optimal problem.

First, our analysis assumes that trade is exclusive — i.e., a seller is not allowed to sell to multiple buyers. Attar et al. (2011) have shown that when the seller's endowment is divisible but homogenous, nonexclusive competition can vastly alter the equilibrium outcome. With within-type adverse selection, nonexclusivity raises the additional problem of the buyers wanting to "free-ride" on the other buyers to run down the seller's endowment of Ls. In turn, this raises the questions of whether information on the (bilateral) trade contracts offered/accepted by a seller should be made public and whether the contracts should be allowed to be conditioned on the other contracts that the seller has entered.

Relatedly, if the seller's endowment is sold over time, information on past trade behaviors can have a very different effect under within-type adverse selection. When the quality within the seller's endowment is homogenous, information that a seller has previously sold some units at low prices signals to the market that the rest of her endowment is also of low quality. In contrast, with within-type adverse selection, knowing that the seller has already sold some units can increase the market's expected valuation of the goods left in the seller's endowment because the seller is expected to sell her lemons first. How this feature affects the equilibrium trade dynamics is then unclear. We leave the study of these issues to future research.

A Appendix: Proofs

Proof of Lemma 2

Proof. Suppose that the schedules $q(\cdot)$ and $t(\cdot)$ are jointly a solution to program (3). Since they satisfy constraint (IC_S), by the envelope theorem, $\frac{d}{d\lambda}U^S(q(\lambda), t(\lambda); \lambda) = \frac{\partial C(q(\lambda),\lambda)}{\partial\lambda} = \mathcal{I}(q(\lambda) > \lambda)(c_H - c_L)$. Therefore, the first condition in Lemma 2 is a necessary condition for constraint (IC_S). By standard arguments, if this condition holds and $q(\cdot)$ is also nondecreasing, constraint (IC_S) always holds.

Suppose that $\lambda_2 > \lambda_1$ but $q(\lambda_2) < q(\lambda_1)$. Constraint ($IC_S$) implies that

$$q(\lambda_1) \leq \lambda_1 \quad \text{or} \quad q(\lambda_2) \geq \lambda_2. \tag{15}$$

To see why, suppose, for a contradiction, that statement (15) is false. The negation of statement (15) is $q(\lambda_1) > \lambda_1$ *and* $q(\lambda_2) < \lambda_2$. Constraints (IC_S) for λ_1 and λ_2 imply that

$$\begin{aligned} & C(q(\lambda_2), \lambda_1) + C(q(\lambda_1), \lambda_2) - C(q(\lambda_1), \lambda_1) \geq C(q(\lambda_2), \lambda_2) \\ \iff \quad & [q(\lambda_2) - \lambda_1]^+ + [q(\lambda_1) - \lambda_2]^+ - [q(\lambda_1) - \lambda_1]^+ \ \geq \ [q(\lambda_2) - \lambda_2]^+ \end{aligned}$$

The RHS is zero. Since $q(\lambda_1) > \lambda_1$, at least one of $[q(\lambda_2) - \lambda_1]^+$ or $[q(\lambda_1) - \lambda_2]^+$ must be strictly positive. If $[q(\lambda_1) - \lambda_2]^+ = 0$ and only $[q(\lambda_2) - \lambda_1]^+ > 0$, then the LHS is $q(\lambda_2) - q(\lambda_1) < 0$; therefore, $[q(\lambda_1) - \lambda_2]^+$ must be strictly positive. In turn, $q(\lambda_1) > q(\lambda_2)$ implies that $q(\lambda_1) > \lambda_2$. Therefore, every term on the LHS is strictly positive. Thus, the LHS is $(q(\lambda_2) - \lambda_1) + (q(\lambda_1) - \lambda_2) - (q(\lambda_1) - \lambda_1) = q(\lambda_2) - \lambda_2$, but this is strictly negative — contradiction. Therefore, statement (15) must hold.

Suppose that $q(\lambda_1) \leq \lambda_1$. With $q(\lambda_2) < q(\lambda_1)$, it must imply that $t(\lambda_1) - t(\lambda_2) = c_L[q(\lambda_1) - q(\lambda_2)]$; if not, one of the two types has a profitable deviation from taking the other type's contract. This implies that $U^S(q(\lambda_2), t(\lambda_2); \lambda_2) = U^S(q(\lambda_1), t(\lambda_1); \lambda_2)$. In this case, replacing type λ_2's contract with $(q(\lambda_1), t(\lambda_1))$ maintains incentive-compatibility for all types because this gives type λ_2 the same utility as before, and since $(q(\lambda_1), t(\lambda_1))$ was already part of the menu, it does not affect the incentive-compatibility of the other types. With this replacement, the buyer's utility from λ_2 strictly increases by $[q(\lambda_1) - q(\lambda)]\, s_L$. Therefore, for $\sigma < 1$, the objective value strictly increases. For the case of $\sigma = 1$ (i.e., the seller-optimal problem), this replacement must imply that (IR_B) becomes slack because it increases the buyer's expected utility. In turn, $t(\cdot)$ can be increased uniformly by some small amount, which then increases the objective value (which is the seller's expected utility) without violating any of the constraints

For the other case of statement (15) that $q(\lambda_2) \geq \lambda_2$, this implies that $t(\lambda_1) - t(\lambda_2) = c_H[q(\lambda_1) - q(\lambda_2)]$. By the same argument, by replacing λ_2's contract with λ_1's contract, incentive-compatibility of all types is maintained, but the buyer's surplus from λ_2 increases by $[q(\lambda_1) - q(\lambda)]\, s_H$. We thus have a contradiction to $\lambda_2 > \lambda_1$ and $q(\lambda_2) < q(\lambda_1)$. $\square$

Properties of the Solution to Program ($\mathcal{P}_\sigma$)

We prove some important properties of the solution to program ($\mathcal{P}_\sigma$) that hold for any $\sigma \in [0,1]$. These properties will be used for the proof of Propositions 1 and 4 below.

Remark 2. It is without loss of generality to consider only schedules $q(\cdot)$ that are right-continuous — i.e., for any x, $\lim_{\lambda \downarrow x^+} q(\lambda) = q(x)$. This is because $q(\cdot)$ must be a bounded and nondecreasing function over $[0,1]$. Therefore, the set of discontinuous points is countable and has a Lebesgue measure of zero. Since F has no atom anywhere, the discontinuous points of $q(\cdot)$ do not affect the values of the objective function of program ($\mathcal{P}_\sigma$) nor constraint (IR'_B).

Lemma 5. *Suppose that $q^*(\cdot)$ is a solution to program ($\mathcal{P}_\sigma$).*

1. *It holds that $q^*(\lambda) \geq \lambda$ for all $\lambda \in [0,1]$.*

2. *Let $\mathcal{X}$ be the set of λ such that $q^*(\lambda) > \lambda$.*

   (a) *If $\mathcal{X}$ is empty, then $q^*(\cdot)$ is a threshold schedule, with $q^*(\lambda) = \lambda$ for all λ.*

   (b) *Suppose that $\mathcal{X}$ is nonempty. Let λ_1 denote the smallest λ in $\mathcal{X}$, and let λ_2 denote the smallest $\lambda \in (\lambda_1, 1]$ such that $q^*(\lambda) = \lambda$. It holds that $q^*(\lambda) = \lambda_2$ for all $\lambda \in [\lambda_1, \lambda_2]$.*

Proof. For any λ, $VS_\sigma(q, \lambda)$ and $\psi^B(q, \lambda)$ are both strictly increasing in q for any $q \in [0, \lambda)$. Thus, point 1 follows. Point 2a is a corollary of point 1.

For point 2b, we first note that if $\mathcal{X}$ is nonempty, the smallest element of $\mathcal{X}$ must exist because $q^*(\cdot)$ is right-continuous. Similarly, λ_2 must also exist because $q^*(1) = 1$ and $q^*(\cdot)$ is right-continuous. Since $q^*(\cdot)$ is nondecreasing, we know that $q^*(\lambda) \leq \lambda_2$ for all $\lambda \in [\lambda_1, \lambda_2]$. Moreover, for all $\lambda \in [\lambda_1, \lambda_2)$, $\lambda_2 > \lambda$ implies that $VS_\sigma(\lambda_2, \lambda) \geq VS_\sigma(q^*(\lambda), \lambda)$ and $\psi^B(\lambda_2, \lambda) \geq \psi^B(q^*(\lambda_2), \lambda)$, where the inequality holds strictly if $q^*(\lambda) < \lambda_2$. Therefore, $q^*(\lambda)$ must be λ_2 for all $\lambda \in [\lambda_1, \lambda_2]$. □

Corollary 3. *Suppose that $q^*(\cdot)$ is a solution to program ($\mathcal{P}_\sigma$). Let $\mathcal{X}$ and λ_2 be as defined in Lemma 5. If $\mathcal{X}$ is nonempty, $q^*(\cdot)$ is a threshold schedule if and only if $\lambda_2 = 1$*

Proof of Lemma 3

Proof. To streamline the exposition for Subsection 6, we consider the more general problem of $\max_{q(\cdot)} \int_0^1 VS_\sigma(q(\lambda), \lambda) f(\lambda) d\lambda$ for $\sigma < \frac{1}{2}$, subject to $q(\cdot)$ being nondecreasing. The buyer-optimal problem is the case of $\sigma = 0$.[24] Let $q^*(\cdot)$ be an optimal schedule. By Remark 2, we can assume that $q^*(\cdot)$ is right-continuous. Let λ_1 and λ_2 be as defined in Lemma 5. By Corollary 3, we only have to prove that $\lambda_2 = 1$.

Suppose, for a contradiction, that $\lambda_2 < 1$. There must then exist $\lambda_3 > \lambda_2$ such $q^*(\lambda) = \lambda$ for all $\lambda \in [\lambda_2, \lambda_3]$. Pick some $\varepsilon > 0$ such that $\varepsilon < \max\{\lambda_3 - \lambda_2, \lambda_2 - \lambda_1\}$. For any $x \in [\lambda_2 - \varepsilon, \lambda_2 + \varepsilon]$, define the schedule $\tilde{q}_x(\cdot)$ as follows:

$$\tilde{q}_x(\lambda) = \begin{cases} x & \text{, if } \lambda \in [\lambda_1, x) \\ \lambda & \text{, if } \lambda \in [x, \lambda_3) \\ q^*(\lambda) & \text{, if } \lambda \notin [\lambda_1, \lambda_3) \end{cases} \tag{16}$$

[24]Note that for $\sigma > 0$, the solution to this problem is not necessarily the solution to program ($\mathcal{P}_\sigma$) because we ignore constraint (IR'_B) here. This is not an issue when $\sigma = 0$ because constraint (IR'_B) holds trivially.

Note that $\tilde{q}_x(\cdot)$ is also a nondecreasing schedule. Next, let $\xi(x) := \int_0^1 VS_\sigma(\tilde{q}_x(\lambda), \lambda) f(\lambda) d\lambda$ be the objective value under schedule $\tilde{q}_x(\cdot)$. Therefore,

$$\begin{aligned}
\xi(x) = & \int_{\lambda \notin [\lambda_1, \lambda_3)} VS_\sigma(q^*(\lambda), \lambda) f(\lambda) d\lambda \\
& + \int_{\lambda_1}^{x} \underbrace{(1-\sigma)\left(x s_L + (x-\lambda)(s_H - s_L) - \frac{1-2\sigma}{1-\sigma}(c_H - c_L)\frac{1-F(\lambda)}{f(\lambda)}\right)}_{VS_\sigma(x,\lambda)} f(\lambda) d\lambda \\
& + \int_{x}^{\lambda_3} \underbrace{(1-\sigma)\lambda s_L}_{VS_\sigma(\lambda,\lambda)} f(\lambda) d\lambda
\end{aligned}$$

The first two derivatives of $\xi(x)$ are

$$\begin{aligned}
\xi'(x) = & -(1-2\sigma)(c_H - c_L)[1 - F(x)] + [F(x) - F(\lambda_1)](1-\sigma) s_H. \\
\xi''(x) = & f(x)[(1-2\sigma)(c_H - c_L) + (1-\sigma) s_H] > 0.
\end{aligned}$$

Since $\tilde{q}_{\lambda_2}(\cdot)$ is the optimal schedule $q^*(\cdot)$, $x = \lambda_2$ must be a local maximizer. However, when $\sigma < \frac{1}{2}$, $\xi''(\lambda_2)$ is strictly positive — contradiction. Therefore, λ_2 must be 1. □

Proof of Proposition 1

Proof. Since $\Psi^B\left(\hat{\lambda}\right)$ is continuous in $\hat{\lambda}$, $\max_{\hat{\lambda}\in[0,1]} \Psi^B\left(\hat{\lambda}\right)$ always has a solution. Let λ^B be a solution, and let $(q^*(\cdot), t^*(\cdot))$ be the associated schedules. $u^0 = 0$ implies that $U^S((q^*(0), t^*(0)); 0) = 0$. Using Lemma 2, $t^*(\lambda) = (c_H - c_L)\int_0^\lambda \mathcal{I}(q^*(l), l) dl + C(q^*(\lambda), \lambda)$. Therefore, for $\lambda < \lambda^B$, $t^*(\lambda) = C(\lambda, \lambda) = \lambda c_L$; for $\lambda \geq \lambda^B$, $t^*(\lambda) = (c_H - c_L)\left(\lambda - \lambda^B\right) + C(1, \lambda) = C\left(1, \lambda^B\right)$.

Next, on the optimal threshold, note that $\frac{d}{d\hat{\lambda}}\Psi^B\left(\hat{\lambda}\right) = \left(1-\hat{\lambda}\right)\left[(c_H - c_L) R\left(\hat{\lambda}\right) - s_H\right]$. Therefore, $\frac{d}{d\hat{\lambda}}\Psi^B\left(\hat{\lambda}\right) = 0$ if and only if $R\left(\hat{\lambda}\right) = \frac{s_H}{c_H - c_L}$. Suppose that $R\left(\hat{\lambda}\right)$ is strictly increasing. If $\frac{d}{d\hat{\lambda}}\Psi^B\left(\hat{\lambda}\right) = 0$, $\frac{d}{d\hat{\lambda}}\Psi^B\left(\hat{\lambda}'\right) > 0$ for all $\hat{\lambda}' > \hat{\lambda}$, which means that the optimal threshold must be 1. Therefore, the optimal threshold must be 1 or 0. If $R\left(\hat{\lambda}\right)$ is strictly decreasing and $R\left(\hat{\lambda}\right) = \frac{s_H}{c_H - c_L}$, then $\frac{d}{d\hat{\lambda}}\Psi^B\left(\hat{\lambda}'\right) > 0$ for all $\hat{\lambda}' < \hat{\lambda}$, and $\frac{d}{d\hat{\lambda}}\Psi^B\left(\hat{\lambda}\right) < 0$ for all $\hat{\lambda}' > \hat{\lambda}$. Therefore, $\Psi^B\left(\hat{\lambda}\right)$ is quasiconcave and the solution is unique. □

Proof of Proposition 2

Proof. Let $\Psi^B\left(\hat{\lambda}|F\right)$ be the expression in equation (8) under distribution F, and let $\Psi^B\left(\hat{\lambda}|G\right)$ be the associated expression under distribution G.[25] Let $S(q,\lambda) = qs_L + [q-\lambda]^+(s_H - s_L)$. Letting $q_{\hat{\lambda}}(\cdot)$ denote the threshold-$\hat{\lambda}$ schedule, by an integration by parts, we have

$$\begin{aligned}\int_0^1 S\left(q_{\hat{\lambda}}(\lambda),\lambda\right)dF(\lambda) &= S(1,1) - \int_0^1 \frac{d}{d\lambda}S\left(q_{\hat{\lambda}}(\lambda),\lambda\right)F(\lambda)\,d\lambda \\ &= s_L + \int_{\hat{\lambda}}^1 s_H F(\lambda)\,d\lambda - \int_0^1 s_L F(\lambda)\,d\lambda\end{aligned}$$

Therefore,

$$\begin{aligned}\Psi^B\left(\hat{\lambda}|F\right) &= \int_0^1 S\left(q_{\hat{\lambda}}(\lambda),\lambda\right)dF(\lambda) - (c_H - c_L)\int_{\hat{\lambda}}^1 1 - F(\lambda)\,d\lambda. \\ &= s_L - \left(1-\hat{\lambda}\right)(c_H - c_L) - \int_0^1 s_L F(\lambda)\,d\lambda + \int_{\hat{\lambda}}^1 \left[s_H + c_H - c_L\right]F(\lambda)\,d\lambda, \end{aligned} \tag{17}$$

and

$$\Psi^B\left(\hat{\lambda}|G\right) - \Psi^B\left(\hat{\lambda}|F\right) = \int_{\hat{\lambda}}^1 (s_H + c_H - c_L)\left[G(\lambda) - F(\lambda)\right]d\lambda - \int_0^1 s_L\left[G(\lambda) - F(\lambda)\right]d\lambda. \tag{18}$$

When G is a mean-preserving contraction of F, $\int_{\hat{\lambda}}^1 G(\lambda) - F(\lambda)\,d\lambda > 0$ for all $\hat{\lambda} \in (0,1)$ and $\int_0^1 G(\lambda)\,d\lambda = \int_0^1 F(\lambda)\,d\lambda$. Therefore, the second term in equation (18) is zero, whereas the first term is nonnegative, and it is strictly positive when $\hat{\lambda} \in (0,1)$. Therefore, $\Psi^B(0|G) = \Psi^B(0|F)$, $\Psi^B(1|G) = \Psi^B(1|F)$, and

$$\Psi^B\left(\hat{\lambda}|G\right) > \Psi^B\left(\hat{\lambda}|F\right)\ \forall\hat{\lambda} \in (0,1). \tag{19}$$

Let λ_F^B be an optimal threshold of the buyer-optimal problem under F — i.e., $\Psi^B\left(\lambda_F^B|F\right) = \pi_F^{B*}$. Therefore, $\pi_G^{B*} - \pi_F^{B*} \geq \Psi^B\left(\lambda_F^B|G\right) - \Psi^B\left(\lambda_F^B|F\right) \geq 0$, where the last inequality follows from equation (19), and the inequality is strict if $\lambda_F^B \in (0,1)$. □

[25] i.e., $\Psi^B\left(\hat{\lambda}|G\right) = \int_0^1 \lambda s_L dG(\lambda) + \int_{\hat{\lambda}}^1 (1-\lambda)\,s_H dG(\lambda) - \int_{\hat{\lambda}}^1 (c_H - c_L)\left[1 - G(\lambda)\right]d\lambda$.

Proof of Proposition 3

Proof. From equation (18) in the previous proof,

$$\begin{aligned}\Psi^B\left(\hat{\lambda}|G\right)-\Psi^B\left(\hat{\lambda}|F\right)&=\int_{\hat{\lambda}}^1\left(s_H-s_L+c_H-c_L\right)\left[G\left(\lambda\right)-F\left(\lambda\right)\right]d\lambda-\int_0^{\hat{\lambda}}s_L\left[G\left(\lambda\right)-F\left(\lambda\right)\right]d\lambda\\&=\int_{\hat{\lambda}}^1\left(v_H-v_L\right)\left[G\left(\lambda\right)-F\left(\lambda\right)\right]d\lambda+\int_0^{\hat{\lambda}}s_L\left[F\left(\lambda\right)-G\left(\lambda\right)\right]d\lambda. \quad (20)\end{aligned}$$

Therefore, $G>_{\left(\lambda_F^B,1\right)}F$ implies that

$$\pi_G^{B*}-\pi_F^{B*}\geq\Psi^B\left(\lambda_F^B|G\right)-\Psi^B\left(\lambda_F^B|F\right)=\int_{\lambda_F^B}^1\left(v_H-v_L\right)\left[G\left(\lambda\right)-F\left(\lambda\right)\right]d\lambda>0.$$

Next, using equation (20) again, $F>_{\left(0,\lambda_F^B\right)}G$ implies that

$$\pi_G^{B*}-\pi_F^{B*}\geq\Psi^B\left(\lambda_F^B|G\right)-\Psi^B\left(\lambda_F^B|F\right)=\int_0^{\lambda_F^B}s_L\left[F\left(\lambda\right)-G\left(\lambda\right)\right]d\lambda>0.$$

□

Proof of Lemma 4

Proof. Let $q^*\left(\cdot\right)$ be an optimal schedule. By Remark 2, we can assume that $q^*\left(\cdot\right)$ is right-continuous. Let λ_1 and λ_2 be as defined in Lemma 5. By Corollary 3, we only have to prove that $\lambda_2=1$. We know that $q^*\left(\lambda\right)=\lambda_2$ for all $\lambda\in[\lambda_1,\lambda_2]$.

Suppose, for a contradiction, that $\lambda_2<1$. Note that this rules out mild adverse selection. This is because if $E\left[v\right]\geq c_H$, then

$$\int_0^1\psi^B\left(1,\lambda\right)f\left(\lambda\right)d\lambda=E\left[v\right]-E\left[c\right]-\left(c_H-c_L\right)\underbrace{\int_0^1 1-F\left(\lambda\right)d\lambda}_{=E[\lambda]}=E\left[v\right]-c_H\geq 0. \quad (21)$$

Since the objective function is maximized by the schedule $q\left(\lambda\right)=1$ for all λ, this must be the optimal schedule. In this case, $\lambda_1=0$ and $\lambda_2=1$.

Henceforth, we consider $E\left[v\right]<c_H$. When $\lambda_2<1$, there exists $\lambda_3>\lambda_2$ such that

$q^*(\lambda) = \lambda$ for all $\lambda \in [\lambda_2, \lambda_3]$. Define the function $\phi(x)$ as follows:[26]

$$\int_x^{\phi(x)} [\phi(x) - \lambda] f(\lambda) = \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \lambda) f(\lambda) d\lambda. \tag{22}$$

Therefore, $\phi(\cdot)$ is strictly increasing and $\phi(\lambda_1) = \lambda_2$. We restrict the domain of function ϕ to be $[\lambda_1, \lambda_1 + \varepsilon]$ for some small ε such that $\phi(\lambda_1 + \varepsilon) \leq \lambda_3$. Define schedule $\hat{q}_x$ as follows:

$$\hat{q}_x(\lambda) = \begin{cases} q^*(\lambda) & , \text{if } \lambda \notin [\lambda_1, \lambda_3) \\ \lambda & , \text{if } \lambda \in [\lambda_1, x) \\ \phi(x) & , \text{if } \lambda \in [x, \phi(x)) \\ \lambda & , \text{ if } \lambda \in [\phi(x), \lambda_3) \end{cases}$$

By construction, $\int_0^1 \psi^{S+B}(\hat{q}_x(\lambda), \lambda) f(\lambda) d\lambda = \int_0^1 \psi^{S+B}(q^*(\lambda), \lambda) f(\lambda) d\lambda$ — i.e., $\hat{q}_x(\cdot)$ is a nondecreasing schedule that gives the same objective value as $q^*(\cdot)$. Let

$$\begin{aligned} D(x) &= \int_0^1 \psi^B(\hat{q}_x(\lambda), \lambda) f(\lambda) d\lambda - \int_0^1 \psi^B(q^*(\lambda), \lambda) f(\lambda) d\lambda \\ &= \underbrace{\int_0^1 \psi^{S+B}(\hat{q}_x(\lambda), \lambda) f(\lambda) d\lambda - \int_0^1 \psi^{S+B}(q^*(\lambda), \lambda) f(\lambda) d\lambda}_{=0} \\ &\quad + \left[\left(\int_{\lambda_1}^{\lambda_2} 1 - F(\lambda) d\lambda\right) - \left(\int_x^{\phi(x)} 1 - F(\lambda) d\lambda\right)\right](c_H - c_L). \end{aligned}$$

If there exists $x > \lambda_1$ such that $D(x) > 0$, this means that constraint (IR'_B) is slack under $\hat{q}_x(\cdot)$, and we can hence increase $\hat{q}_x(\lambda)$ slightly over some positive-measured subset of λ to increase the objective value while not violating (IR'_B). In turn, this contradicts the optimality of $q^*(\cdot)$, which then proves that λ_2 must be 1.

We now show that when $\frac{f(\lambda)}{1-F(\lambda)}$ is increasing, $D(x)$ is, in fact, strictly positive for all $x \in [\lambda_1, \lambda_1 + \varepsilon]$. Using the implicit function on equation (22), we have $\phi'(x) = \frac{[\phi(x)-x]f(x)}{F(k(x))-F(x)}$. Therefore,

$$\begin{aligned} D'(x) &= (1 - F(x) - [1 - F(\phi(x))]\phi'(x))(c_H - c_L) \\ &= \left(1 - F(x) - \frac{[1 - F(\phi(x))][\phi(x) - x] f(x)}{F(\phi(x)) - F(x)}\right)(c_H - c_L) \end{aligned}$$

[26] To map this to the explanation in the main text, note that $\phi(x) = \Lambda(x - \lambda_1)$.

Therefore,

$$D'(x) > 0 \iff \frac{f(x)}{1-F(x)} < \frac{F(\phi(x)) - F(x)}{[1-F(\phi(x))][\phi(x)-x]}$$

In Lemma 6 below, we show that when $\frac{f(\lambda)}{1-F(\lambda)}$ is increasing, the inequality above always holds. Since $D(\cdot)$ is continuous and $D(\lambda_1) = 0$, this implies that $D(x) > 0$ for all $x \in (\lambda_1, \lambda_1 + \varepsilon]$. As explained above, this implies that λ_2 must be 1 □

Lemma 6. *If $\frac{f(\lambda)}{1-F(\lambda)}$ is nondecreasing in λ, then for any $\lambda < \lambda'$, $\frac{f(\lambda)}{1-F(\lambda)} < \frac{F(\lambda')-F(\lambda)}{[1-F(\lambda')](\lambda'-\lambda)}$.*

Proof. Fix some $\lambda \in (0,1)$ and let $K := \frac{f(\lambda)}{1-F(\lambda)}$. For $\lambda' \in [\lambda, 1]$, let $A(\lambda') := \frac{F(\lambda')-F(\lambda)}{[1-F(\lambda')](\lambda'-\lambda)}$. Observe that $\lim_{\lambda' \downarrow \lambda} A(\lambda') = \frac{f(\lambda)}{1-F(\lambda)} = K$ and $\lim_{\lambda' \uparrow 1} A(\lambda') = \infty$. Suppose, for a contradiction, that there exist $\lambda' > \lambda$ in which $K > A(\lambda')$. Since $A(\cdot)$ is continuous, by the mean value theorem, there must exist $\hat{\lambda} \in (\lambda, 1)$ such that $A'\left(\hat{\lambda}\right) = 0$ and $A\left(\hat{\lambda}\right) < K$. By some algebra, $A'\left(\hat{\lambda}\right) = \frac{\left(\hat{\lambda}-\lambda\right) f\left(\hat{\lambda}\right)[1-F(\lambda)] - \left[F\left(\hat{\lambda}\right)-F(\lambda)\right]\left[1-F\left(\hat{\lambda}\right)\right]}{\left[1-F\left(\hat{\lambda}\right)\right]^2 \left(\hat{\lambda}-\lambda\right)^2}$. $A'\left(\hat{\lambda}\right) = 0$ implies that

$$\frac{f\left(\hat{\lambda}\right)}{1-F\left(\hat{\lambda}\right)} = \frac{F\left(\hat{\lambda}\right) - F(\lambda)}{\left(\hat{\lambda}-\lambda\right)[1-F(\lambda)]} = \underbrace{A\left(\hat{\lambda}\right)}_{<K} \underbrace{\left(\frac{1-F\left(\hat{\lambda}\right)}{1-F(\lambda)}\right)}_{<1} < K = \frac{f(\lambda)}{1-F(\lambda)}.$$

However, this contradicts the assumption that $\hat{\lambda} > \lambda$ implies $\frac{f\left(\hat{\lambda}\right)}{1-F\left(\hat{\lambda}\right)} \geq \frac{f(\lambda)}{1-F(\lambda)}$.

□

Proof of Proposition 4

Proof. Since $\Psi^{S+B}\left(\hat{\lambda}\right)$ is strictly decreasing in $\hat{\lambda}$, the solution must be λ^S.

As noted in equation (21) above, $\Psi^B(0) \geq 0$ if and only if $E[v] \geq c_H$. Therefore, if $E[v] \geq c_H$ (mild adverse selection), $\lambda^S = 0$, which implies that the optimal schedule is $q(\lambda) = 1$ for all λ. From the binding (IR'_B), $u^0 = \Psi^B(0) \implies U^S((1, t^*(0)); 0) = E[v] - c_H$. Therefore, $t^*(0) = E[v] - c_H + C(1,0) = E[v]$. Since the menu has complete pooling, $t^*(\lambda) = E[v]$ for all λ.

Next, if $E[v] < c_H$ (severe adverse selection), $\Psi^B(0) < 0$. Since $\Psi^B(1) = E[\lambda] s_L > 0$ and $\Psi^B\left(\hat{\lambda}\right)$ is continuous, there exists $\hat{\lambda} > 0$ such that $\Psi^B\left(\hat{\lambda}\right) = 0$. Therefore, λ^S always exists and is strictly positive. Given the binding (IR'_B), $u^0 = \Psi^B\left(\lambda^S\right) = 0$. Using Lemma 2, $t^*(\lambda) = (c_H - c_L)\int_0^\lambda \mathcal{I}(q^*(l), l)\, dl + C(q^*(\lambda), \lambda)$. Therefore, for $\lambda < \lambda^S$, $t^*(\lambda) = C(\lambda, \lambda) = \lambda c_L$; for $\lambda \geq \lambda^S$, $t^*(\lambda) = (c_H - c_L)\int_{\lambda^S}^\lambda 1 dl + C(1, \lambda) = C\left(1, \lambda^S\right)$. □

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Online Appendix for
"Markets with Within-Type Adverse Selection"

B Online Appendix: Additional Results

B.1 Comparative Statics for the Seller-Optimal Equilibrium

In this subsection, we provide the analogous set of comparative static results in Section 4.2 for the seller-optimal equilibrium. Let π_F^{S*} and π_G^{S*} be the seller's expected utility under the seller-optimal equilibrium when the type distributions are F and G, respectively.

Proposition 7. *If G is a mean-preserving contraction of F, then $\pi_G^{S*} \geq \pi_F^{S*}$. The inequality is strict if there is severe adverse selection under F.*

Proof. Follow the notations in the proof of Proposition 2. Let λ_F^S denote the threshold of an optimal contract under F. From equation (19), we know that $\Psi^B\left(\lambda_F^S|G\right) \geq \Psi^B\left(\lambda_F^S|F\right)$, where the inequality is strict if $\hat{\lambda} \in (0,1)$. This implies that λ_F^S is feasible under distribution G. Next, analogous to equation (17), we have

$$\Psi^{S+B}\left(\hat{\lambda}|F\right) = s_L - s_H + \int_{\hat{\lambda}}^{1} s_H F(\lambda)\, d\lambda - \int_0^1 s_L F(\lambda)\, d\lambda. \tag{23}$$

Therefore, $\Psi^{S+B}\left(\hat{\lambda}|G\right) - \Psi^{S+B}\left(\hat{\lambda}|F\right) = \int_{\hat{\lambda}}^1 s_H \left[G(\lambda) - F(\lambda)\right] d\lambda \geq 0$, where the inequality is also strict when $\hat{\lambda} \in (0,1)$. This implies that

$$\pi_G^{S*} \geq \Psi^{S+B}\left(\lambda_F^S|G\right) \geq \Psi^{S+B}\left(\lambda_F^S|F\right) = \pi_G^{S*}.$$

Moreover, the first inequality is strict when there is severe adverse selection. To see why, note that if there is severe adverse selection, $\lambda_F^S > 0$ and $\Psi^B\left(\lambda_F^S|F\right) = 0$. This implies that $\Psi^B\left(\lambda_F^S|G\right) > 0$; therefore, the optimal threshold under G, denoted by λ_G^S, must be strictly less that λ_F^S. In turn, this implies that $\Psi^{S+B}\left(\lambda_G^S|G\right) > \Psi^{S+B}\left(\lambda_F^S|G\right)$. □

Proposition 8. *Let λ_F^S be the threshold of some optimal menu of contracts of the seller-optimal equilibrium under distribution F. Suppose that there is severe adverse selection under F.*[27]

[27] If there is only mild adverse selection, then the inequality between π_G^{S*} and π_F^{S*} holds only weakly in both cases.

- *When $s_H \geq s_L$, $G >_{\left(\lambda_F^S,1\right)} F$ implies that $\pi_G^{S*} > \pi_F^{S*}$.*
- *$F >_{\left(0,\lambda_F^S\right)} G$ implies that $\pi_G^{S*} > \pi_F^{S*}$.*

Proof. In each of the two cases, $G >_{\left(\lambda_F^S,1\right)} F$ and $F >_{\left(0,\lambda_F^S\right)} G$, from equation (20), we know that $\Psi^B\left(\lambda_F^S|G\right) > \Psi^B\left(\lambda_F^S|F\right)$. Therefore, λ_F^S is always feasible under G. Since there is severe adverse selection, $\lambda_F^S > 0$ which implies that the optimal threshold under G, denoted by λ_G^S, is strictly smaller than λ_F^S. Using equation (23) above,

$$\Psi^{S+B}\left(\hat{\lambda}|G\right) - \Psi^{S+B}\left(\hat{\lambda}|F\right) = \int_{\hat{\lambda}}^{1}\left(s_H - s_L\right)\left[G\left(\lambda\right) - F\left(\lambda\right)\right]d\lambda + \int_0^{\hat{\lambda}} s_L\left[F\left(\lambda\right) - G\left(\lambda\right)\right]d\lambda$$

Therefore, if $s_H \geq s_L$, $G >_{\left(\lambda_F^S,1\right)} F$ implies that

$$\begin{aligned}\pi_G^{S*} - \pi_F^{S*} =& \Psi^{S+B}\left(\lambda_G^S|G\right) - \Psi^{S+B}\left(\lambda_F^S|F\right)\\ >& \Psi^{S+B}\left(\lambda_F^S|G\right) - \Psi^{S+B}\left(\lambda_F^S|F\right)\\ =& \int_{\lambda_F^S}^{1}\left(s_H - s_L\right)\left[G\left(\lambda\right) - F\left(\lambda\right)\right]d\lambda \ \geq \ 0.\end{aligned}$$

Next, following a similar argument, $F >_{\left(0,\lambda_F^S\right)} G$ implies that $\pi_G^{S*} - \pi_F^{S*} > \int_0^{\lambda_F^S} s_L\left[F\left(\lambda\right) - G\left(\lambda\right)\right]d\lambda \geq 0$. □

B.2 Other Buyer-advantaged Equilibria: $\sigma \in \left(0, \frac{1}{2}\right)$

In this subsection, we formalize the discussion in Section 6. Throughout this subsection, assume that $\sigma \in \left(0, \frac{1}{2}\right)$. We first prove that the solution to program ($\mathcal{P}_\sigma$) is always a threshold schedule. Subsequently, we elaborate on the associated optimal menu of contracts and its properties.

Let Q_σ^* be the set of solutions to the problem of

$$\max_{q(\cdot)} \int_0^1 VS_\sigma\left(q\left(\lambda\right), \lambda\right) f\left(\lambda\right) d\lambda \quad \text{s.t. } q\left(\cdot\right) \text{ is nondecreasing.} \tag{24}$$

Lemma 7. *If $q^*\left(\cdot\right) \in Q_\sigma^*$, $q^*\left(\cdot\right)$ must be a threshold schedule.*

This result has been shown in the proof of Lemma 3.

Corollary 4. *If $q^*\left(\cdot\right) \in Q_\sigma^*$ and $\int_0^1 \psi^B\left(q^*\left(\lambda\right), \lambda\right) f\left(\lambda\right) d\lambda \geq 0$, $q^*\left(\cdot\right)$ is a solution to program ($\mathcal{P}_\sigma$).*

The following result concerns $\sigma \in \left(0, \frac{1}{2}\right)$ in which there is no schedule in Q^*_σ that satisfies constraint (IR'_B).

Lemma 8. *Suppose that $\sigma \in \left(0, \frac{1}{2}\right)$. Assume that $\frac{f(\lambda)}{1-F(\lambda)}$ is weakly increasing for $\lambda \in (0,1)$. The solution to program ($\mathcal{P}_\sigma$) is always a threshold schedule.*

Proof. Let $q^*(\cdot)$ be an optimal schedule. By Remark 2, we can assume that $q^*(\cdot)$ is right-continuous. Let λ_1 and λ_2 be as defined in Lemma 5. By Corollary 3, we only have to prove that $\lambda_2 = 1$. Suppose, for a contradiction, that $\lambda_2 < 1$. Define λ_3, $\phi(\cdot)$, the schedule $\hat{q}_x(\cdot)$, and $D(\cdot)$ as in the proof of Proposition 4. As shown there, $D(x) := \int_0^1 \psi^B(\hat{q}_x(\lambda), \lambda) f(\lambda) d\lambda - \int_0^1 \psi^B(q^*(\lambda), \lambda) f(\lambda) d\lambda > 0$; therefore, $\hat{q}_x(\cdot)$ is feasible. Observe that

$$\int_0^1 VS_\sigma(\hat{q}_x(\lambda), \lambda) f(\lambda) d\lambda - \int_0^1 VS_\sigma(q^*(\lambda), \lambda) f(\lambda) d\lambda = \frac{1-2\sigma}{1-\sigma} D(x) > 0.$$

This contradicts the optimality of $q^*(\cdot)$. □

When the threshold-λ_σ schedule is a solution to program ($\mathcal{P}_\sigma$), the corresponding optimal menu of contracts is

$$(q^*_\sigma(\lambda), t^*_\sigma(\lambda)) = \begin{cases} (\lambda, \lambda c_L) & , \forall \lambda < \lambda_\sigma \\ (1, C(1, \lambda_\sigma)) & , \forall \lambda \geq \lambda_\sigma \end{cases}.$$

This result follows from the same argument as Proposition 1. Finally, the next result provides a tighter solution characterization under more assumptions on the type distribution. Recall that $\Psi_\sigma(\cdot)$ is defined in equation (11).

Proposition 9. *Suppose that $\frac{1-F(\lambda)}{(1-\lambda)f(\lambda)}$ is strictly decreasing for $\lambda \in (0,1)$. For any $\sigma \in \left(0, \frac{1}{2}\right)$, the solution to program ($\mathcal{P}_\sigma$) is unique. Let $\lambda_\sigma = \arg\max_\lambda \Psi_\sigma\left(\hat{\lambda}\right)$ and λ^S be as defined in Proposition 4. λ_σ is strictly decreasing in σ, and the optimal threshold is $\max\left\{\lambda^S, \lambda_\sigma\right\}$.*

Proof. Let $R(\lambda) = \frac{1-F(\lambda)}{(1-\lambda)f(\lambda)}$. $\frac{d}{d\hat{\lambda}}\Psi_\sigma\left(\hat{\lambda}\right) = \left(1-\hat{\lambda}\right)\left[\left(\frac{1-2\sigma}{1-\sigma}\right)(c_H - c_L) R\left(\hat{\lambda}\right) - s_H\right]$. If $R\left(\hat{\lambda}\right)$ is strictly decreasing and $R\left(\hat{\lambda}\right) = \left(\frac{1-2\sigma}{1-\sigma}\right)\frac{s_H}{c_H - c_L}$, then $\frac{d}{d\hat{\lambda}}\Psi_\sigma\left(\hat{\lambda}'\right) > 0$ for all $\hat{\lambda}' < \hat{\lambda}$ and $\frac{d}{d\hat{\lambda}}\Psi_\sigma < 0$ for all $\hat{\lambda}' > \hat{\lambda}$. Therefore, $\Psi_\sigma(\cdot)$ is quasiconcave and has a unique maximizer, and if it is interior, it is characterized by $R(\lambda_\sigma) = \frac{s_H}{c_H - c_L}\left(\frac{1-\sigma}{1-2\sigma}\right)$. Observe that $\frac{1-\sigma}{1-2\sigma}$ is increasing in σ; therefore, if $R(\cdot)$ is decreasing, λ_σ must be decreasing in σ. Quasiconcavity of $\Psi_\sigma(\cdot)$ then implies that the optimal threshold must be $\max\left\{\lambda^S, \lambda_\sigma\right\}$. □

B.3 Nonbinary Quality

This subsection provides the details for the extension in Section 7.1 and the proof of Proposition 5. Let C and V be as defined in equations (12) and (13). Under the first condition of Assumption 2, $C(q, \theta)$ is (weakly) submodular. Therefore, the single-crossing condition is satisfied, and we can extend the result of Lemma 2 to here. In particular, because

$$\frac{\partial C(q,\theta)}{\partial t} = \mathcal{I}(q > \tau_L(\theta))\,\tau_L'(\theta)(c_M - c_L) + \mathcal{I}(q > \tau_M(\theta))\,\tau_M'(\theta)(c_H - c_M),$$

if the menu $\{q(\cdot), t(\cdot)\}$ is a solution to the buyer-optimal problem, $q(\cdot)$ must be nondecreasing and the menu must satisfy

$$\begin{aligned} U^S(q(\theta), t(\theta); \theta) = (c_M - c_L) \int_0^\theta \tau_L'(t)\,\mathcal{I}(q(t) > \tau_L(t))\,dt \\ + (c_H - c_M) \int_0^\theta \tau_M'(t)\,\mathcal{I}(q(t) > \tau_M(t))\,dt \end{aligned} \tag{25}$$

for all θ. In turn, we can express the players' utility in their virtual valuations. Let

$$\begin{aligned} \bar{\psi}^B(q,\theta) =& q s_L + \mathcal{I}(q > \tau_L(\theta)) \left[(q - \tau_L(\theta))(s_M - s_L) - \tau_L'(\theta)(c_M - c_L)\frac{1 - J(\theta)}{j(\theta)} \right] \\ &+ \mathcal{I}(q > \tau_M(\theta)) \left[(q - \tau_M(\theta))(s_H - s_M) - \tau_M'(\theta)(c_H - c_M)\frac{1 - J(\theta)}{j(\theta)} \right], \end{aligned}$$

which is the analog of $\psi^B(\cdot)$ in equation (5). The quantity schedule in the buyer-optimal equilibrium is the solution to

$$\max_{q(\cdot)\text{ is nondecreasing}} \quad \int_0^1 \bar{\psi}^B(q(\theta), \theta)\, j(\theta)\, d\theta. \tag{26}$$

Observe that fixing any θ, $\bar{\psi}^B(q, \theta)$ is strictly increasing in q whenever $q < \tau_L(\theta)$, $q \in (\tau_L(\theta), \tau_M(\theta))$, and $q > \tau_M(\theta)$. Because $q(\cdot)$ must be nondecreasing, we have the following result:

Lemma 9. *If $q(\cdot)$ is a solution to program (26), it must satisfy the following properties:*

1. *$q(\theta) \geq \tau_L(\theta)$ for all θ.*
2. *If there exists an interval $X = \left[\underline{\theta}, \bar{\theta}\right]$ such that $\tau_L(\theta) < q(\theta) < \tau_M(\theta)$ for all $\theta \in X$, it must be the case that $q(\theta) = q(\theta')$ for all $\theta, \theta' \in X$.*

3. *If there exists an interval* $X = \left[\underline{\theta}, \bar{\theta}\right]$ *such that* $\tau_M(\theta) < q(\theta) < 1$ *for all* $\theta \in X$*, it must be the case that* $q(\theta) = q(\theta')$ *for all* $\theta, \theta' \in X$*.*

Because distribution J is atomless, it is without loss of generality to consider only quantity schedules that are right-continuous.

Lemma 10. *Under Assumption 2, if* $q^*(\cdot)$ *is a solution to program (26), then there must exist* θ_L *and* θ_M*, with* $0 \leq \theta_L \leq \theta_M \leq 1$*, such that* $q^*(\theta) = \tau_L(\theta)$ *for all* $\theta < \theta_L$*,* $q^*(\theta) = \tau_M(\theta)$ *for all* $\theta \in [\theta_L, \theta_M)$*, and* $q^*(\theta) = 1$ *for all* $\theta \geq \theta_M$*.*

Proof. By Lemma 9, we know that $q^*(\theta) \geq \tau_L(\theta)\ \forall\theta$. Let $\tilde{\Theta}$ be the set of θ such that $q^*(\theta) \neq \tau_L(\theta), \tau_M(\theta), 1$. Since $q^*(\cdot)$ must be nondecreasing and both $\tau_L(\cdot)$ and $\tau_M(\cdot)$ are increasing, Lemma 10 holds if $\tilde{\Theta}$ has a zero Lebesgue measure. Suppose, for a contradiction, that $\tilde{\Theta}$ has a strictly positive Lebesgue measure. Let $\tilde{\theta}$ be the supremum of the set $\tilde{\Theta}$.

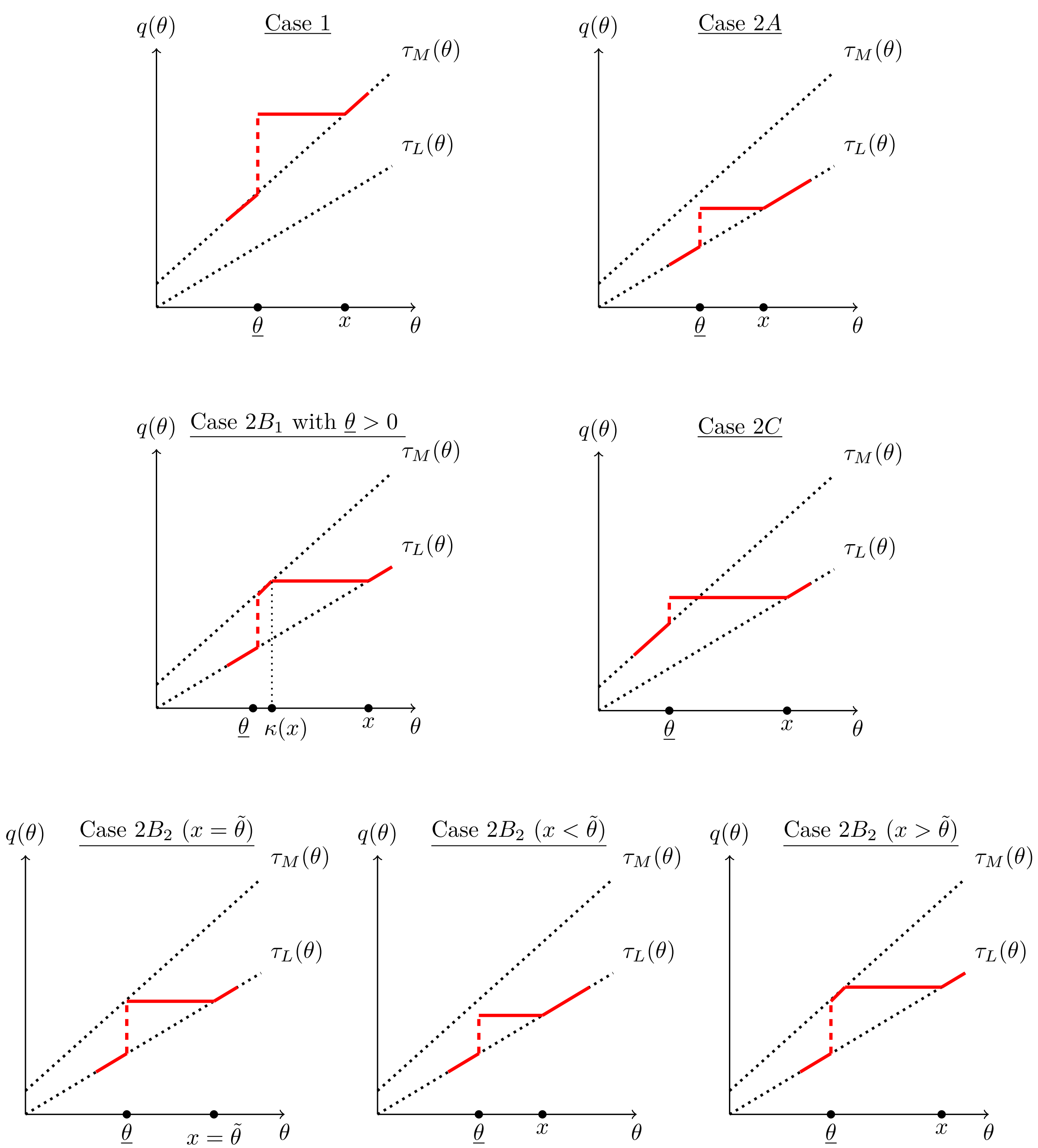
Since $\tilde{\Theta}$ has a positive Lebesgue measure, there exists some interval $\left(\tilde{\theta} - \delta, \tilde{\theta}\right)$ such that $q^*(\theta) \in (\tau_L(\theta), \tau_M(\theta))$ for all $\theta \in \left(\tilde{\theta} - \delta, \tilde{\theta}\right)$ or $q^*(\theta) \in (\tau_M(\theta), 1)$ for all $\theta \in \left(\tilde{\theta} - \delta, \tilde{\theta}\right)$. Observe that if $\tilde{\theta} = 1$ and/or $q^*\left(\tilde{\theta}\right) = 1$, then in the former case, we can increase the objective value by increasing $q^*(\theta)$ to $\tau_M(\theta)$ for all $\theta \in \left(\tilde{\theta} - \delta, \tilde{\theta}\right)$; and in the latter case, we can increase the objective value by increasing $q^*(\theta)$ to 1 for all $\theta \in \left(\tilde{\theta} - \delta, \tilde{\theta}\right)$. Therefore, $\tilde{\theta}$ must be less than 1 and $q^*\left(\tilde{\theta}\right)$ cannot be 1. By definition, $q^*(\theta) \in \{\tau_L(\theta), \tau_M(\theta), 1\}$ for all $\theta > \tilde{\theta}$. Since $q^*(\cdot)$ is right-continuous, this implies that $q^*\left(\tilde{\theta}\right)$ must be either $\tau_M\left(\tilde{\theta}\right)$ (Case 1), or $\tau_L\left(\tilde{\theta}\right)$ (Case 2).

By Lemma 9 (points 2 and 3), there exists some interval $\left(\tilde{\theta} - \delta', \tilde{\theta}\right)$ such that $q^*(\theta)$ is some $\tilde{q}$ for all θ in this interval. Define $\underline{\theta}$ to be the smallest θ such that $q^*(\underline{\theta}) = \tilde{q}$. This implies that $q^*(\theta) = \tilde{q}\ \forall\theta \in [\underline{\theta}, \tilde{\theta})$ and $q^*(\theta) < \tilde{q}$ for all $\theta < \underline{\theta}$ if $\underline{\theta} > 0$.

(Case 1.) Suppose first that $q^*\left(\tilde{\theta}\right) = \tau_M\left(\tilde{\theta}\right)$. Since $q^*(\cdot)$ is right-continuous, there exists $\eta_1 > 0$ such that $q^*(\theta) = \tau_M(\theta)$ for all $\theta \in \left[\tilde{\theta}, \tilde{\theta} + \eta_1\right]$. Choose an $\varepsilon > 0$ satisfying $\varepsilon < \eta_1$ and $\tilde{\theta} - \varepsilon > \underline{\theta}$, and consider the set of schedules $\hat{q}^1_x(\cdot)$ parameterized by $x \in \left(\tilde{\theta} - \varepsilon, \tilde{\theta} + \varepsilon\right)$, where

$$\hat{q}^1_x(\theta) = \begin{cases} q^*(\theta) & , \text{ if } \theta \notin [\underline{\theta}, \tilde{\theta} + \varepsilon) \\ \tau_M(x) & , \text{ if } \theta \in [\underline{\theta}, x] \\ \tau_M(\theta) & , \text{ if } \theta \in [x, \tilde{\theta} + \varepsilon) \end{cases}$$

Figure 3: Proof of Lemma 10: the set of $\hat{q}_x(\cdot)$ for the various cases.



Note: For Case 1 and Case 2C, the illustrated schedules consist of $q(\theta) = \tau_M(\theta)$ for some small region of $\theta < \underline{\theta}$, but these quantities could have been $\tau_L(\theta)$ instead.

Let $\xi_1(x) = \int_0^1 \bar{\psi}^B(\hat{q}_x^1(\theta), \theta) j(\theta) d\theta$. The first two derivatives are

$$\begin{aligned}
\xi_1'(x) =& \tau_M'(x)([J(x) - J(\underline{\theta})] s_H - [1 - J(x)](c_H - c_M)) \\
\xi_1''(x) =& (\tau_M''(x) / \tau_M'(x)) \xi_1'(x) + \tau_M'(x) j(x)(s_H + c_H - c_M)
\end{aligned}$$

Observe that $\hat{q}_x^1(\cdot)$ is nondecreasing $\forall x$ and $\hat{q}_{\tilde{\theta}}^1(\cdot) = q^*(\cdot)$. This implies that $\xi_1\left(\tilde{\theta}\right)$ must be a local maximum. However, if $\xi_1'\left(\tilde{\theta}\right) = 0$, $\xi_1''\left(\tilde{\theta}\right)$ is strictly positive — contradiction. Therefore, $q^*\left(\tilde{\theta}\right)$ cannot be $\tau_M\left(\tilde{\theta}\right)$.

(Case 2.) Next, suppose that $q^*\left(\tilde{\theta}\right) = \tau_L\left(\tilde{\theta}\right)$. Similar to Case 1, there must exist $\eta_2 > 0$ such that $q^*(\theta) = \tau_L(\theta)$ for all $\theta \in \left[\tilde{\theta}, \tilde{\theta} + \eta_2\right]$. We break Case 2 into three separate cases: $\tau_L\left(\tilde{\theta}\right) < \tau_M(\underline{\theta})$ (Case $2A$); $\tau_L\left(\tilde{\theta}\right) = \tau_M(\underline{\theta})$ (Case $2B$); and $\tau_L\left(\tilde{\theta}\right) > \tau_M(\underline{\theta})$ (Case $2C$).

Consider Case $2A$ first — i.e., $q^*\left(\tilde{\theta}\right) = \tau_L\left(\tilde{\theta}\right) < \tau_M(\underline{\theta})$. This implies that $q^*(\underline{\theta}) < \tau_M(\underline{\theta})$. For some $\varepsilon > 0$ such that $\varepsilon < \eta_2$, $\tau_L\left(\tilde{\theta} + \varepsilon\right) < \tau_M(\underline{\theta})$ and $\tilde{\theta} - \varepsilon > \underline{\theta}$, consider the set of schedules $\hat{q}_x^{2A}(\cdot)$ parameterized by $x \in \left(\tilde{\theta} - \varepsilon, \tilde{\theta} + \varepsilon\right)$, where

$$\hat{q}_x^{2A}(\theta) = \begin{cases} q^*(\theta) & , \text{ if } \theta \notin [\underline{\theta}, \tilde{\theta} + \varepsilon) \\ \tau_L(x) & , \text{ if } \theta \in [\underline{\theta}, x] \\ \tau_L(\theta) & , \text{ if } \theta \in [x, \tilde{\theta} + \varepsilon) \end{cases}$$

Let $\xi_{2A}(x) = \int_0^1 \bar{\psi}^B\left(\hat{q}_x^{2A}(\theta), \theta\right) j(\theta) d\theta$. The first two derivatives are

$$\begin{aligned}
\xi_{2A}'(x) =& \tau_L'(x)([J(x) - J(\underline{\theta})] s_M - [1 - J(x)](c_M - c_L)) \\
\xi_{2A}''(x) =& (\tau_L''(x) / \tau_L'(x)) \xi_{2A}'(x) + \tau_L'(x) j(x)(s_M + c_M - c_L)
\end{aligned}$$

Similar to Case 1, $\hat{q}_x^{2A}(\cdot)$ is nondecreasing $\forall x$ and $\hat{q}_{\tilde{\theta}}^{2A}(\cdot) = q^*(\cdot)$. Therefore, $\xi_{2A}\left(\tilde{\theta}\right)$ must be a local maximum, but this is contradicted by $\xi_{2A}'\left(\tilde{\theta}\right) = 0 \implies \xi_{2A}'\left(\tilde{\theta}\right) > 0$. Therefore, Case $2A$ cannot hold.

Next, we consider Case $2B$ — i.e., $q^*\left(\tilde{\theta}\right) = \tau_L\left(\tilde{\theta}\right) = \tau_M(\underline{\theta})$. This implies that $q^*(\underline{\theta}) = \tau_M(\underline{\theta})$. Observe that if $\underline{\theta} > 0$, Lemma 9 (points 2 and 3) will imply that there is some $\gamma_1 > 0$ such that $q^*(\theta) = \tau_M(\theta)$ for all $\theta \in (\underline{\theta} - \gamma_1, \underline{\theta})$ or there is some $\gamma_2 > 0$ such that $q^*(\theta) = \tau_L(\theta)$ for all $\theta \in (\underline{\theta} - \gamma_2, \underline{\theta})$. Thus, we break Case $2B$ to the following two cases:

- Case $2B_1$: $\underline{\theta} > 0$ and $q^*(\theta) = \tau_M(\theta)$ for all $\theta \in (\underline{\theta} - \gamma_1, \underline{\theta})$.

- Case $2B_2$: $\underline{\theta} = 0$, or $\underline{\theta} > 0$ and $q^*(\theta) = \tau_L(\theta)$ for all $\theta \in (\underline{\theta} - \gamma_2, \underline{\theta})$.

Consider Case $2B_1$ first. Let

$$\kappa(\theta) = \begin{cases} \tau_M^{-1}(\tau_L(\theta)) & \text{if } \tau_M(0) > \tau_L(\theta) \\ 0 & \text{if } \tau_M(0) \le \tau_L(\theta) \end{cases}$$

For some $\varepsilon > 0$ such that $\varepsilon < \max\{\eta_2, \gamma_1\}$, consider the set of schedules $\hat{q}_x^{2B_1}(\cdot)$ parameterized by $x \in \left(\tilde{\theta} - \varepsilon, \tilde{\theta} + \varepsilon\right)$, where

$$\hat{q}_x^{2B_1}(\theta) = \begin{cases} q^*(\theta) & , \text{ if } \theta \notin [\underline{\theta} - \varepsilon, \tilde{\theta} + \varepsilon) \\ \tau_M(\theta) & , \text{ if } \theta \in [\underline{\theta} - \varepsilon, \kappa(x)] \\ \tau_L(x) & , \text{ if } \theta \in [\kappa(x), x] \\ \tau_L(\theta) & , \text{ if } \theta \in [x, \tilde{\theta} + \varepsilon) \end{cases}$$

Let $\xi_{2B_1}(x) = \int_0^1 \bar{\psi}^B\left(\hat{q}_x^{2B_1}(\theta), \theta\right) j(\theta)\, d\theta$. Note that for any x such that $\kappa(x) > 0$, $\kappa'(x) = \tau_L'(x) / \tau_M'(\kappa(x))$. Using this and with some algebra, the first two derivatives of ξ_{2B_1} are

$$\begin{aligned} \xi_{2B_1}'(x) =& \tau_L'(x)\left([J(x) - J(\kappa(x))]\, s_M - [1 - J(x)]\,(c_M - c_L)\right) \\ \xi_{2B_1}''(x) =& \left(\tau_L''(x)/\tau_L'(x)\right)\xi_{2B_1}'(x) + \tau_L'(x)\left(j(x)(c_M - c_L) + s_M\left[j(x) - j(\kappa(x))\frac{\tau_L'(x)}{\tau_M'(\kappa(x))}\right]\right) \end{aligned}$$

Observe that $\hat{q}_x^{2B_1}(\cdot)$ is also nondecreasing $\forall x$ and $\hat{q}_{\tilde{\theta}}^{2B_1}(\cdot) = q^*(\cdot)$. Therefore, $\xi_{2B_1}\left(\tilde{\theta}\right)$ must be a local maximum. Under Assumption 2, the term in the square bracket of $\xi_{2B_1}''(x)$ is positive. Therefore, if $\xi_{2B_1}'\left(\tilde{\theta}\right) = 0$, then $\xi_{2B_1}''\left(\tilde{\theta}\right)$ must be strictly positive, but this contradicts $\xi_{2B_1}\left(\tilde{\theta}\right)$ being a local maximum. Therefore, we cannot have Case $2B_1$.

Next, suppose that we have Case $2B_2$. For some $\varepsilon > 0$ such that $\varepsilon < \eta_2$, consider the set of schedules $\hat{q}_x^{2B_2}(\cdot)$ parameterized by $x \in \left(\tilde{\theta} - \varepsilon, \tilde{\theta} + \varepsilon\right)$, where

$$\hat{q}_x^{2B_2}(\theta) = \begin{cases} q^*(\theta) & , \text{ if } \theta \notin [\underline{\theta}, \tilde{\theta} + \varepsilon) \\ \min\{\tau_L(x), \tau_M(\theta)\} & , \text{ if } \theta \in [\underline{\theta}, x] \\ \tau_L(\theta) & , \text{ if } \theta \in [x, \tilde{\theta} + \varepsilon) \end{cases}$$

Let $\xi_{2B_2}(x) = \int_0^1 \bar{\psi}^B\left(\hat{q}_x^{2B_2}(\theta), \theta\right) j(\theta)\, d\theta$. Note that if $x < \tilde{\theta}$, $\tau_L(x) < \tau_M(\theta)$ for all $\theta \in [\underline{\theta}, x]$. If $x > \tilde{\theta}$, then $\tau_M(\theta) < \tau_L(x)$ for all $\theta \in [\underline{\theta}, \kappa(x))$, and $\tau_L(x) < \tau_M(\theta)$ for all $\theta \in (\kappa(x), x]$.

Therefore,

$$\xi'_{2B_2}(x) = \begin{cases} \tau'_L(x)\left(\left[J(x) - J(\underline{\theta})\right]s_M - \left[1 - J(x)\right](c_M - c_L)\right) & , \text{ if } x < \tilde{\theta} \\ \tau'_L(x)\left(\left[J(x) - J(\kappa(x))\right]s_M - \left[1 - J(x)\right](c_M - c_L)\right) & , \text{ if } x > \tilde{\theta} \end{cases}$$

Since $\kappa\left(\tilde{\theta}\right) = \underline{\theta}$, $\lim_{x\uparrow\tilde{\theta}} \xi'_{2B_2}(x) = \lim_{x\downarrow\tilde{\theta}} \xi'_{2B_2}(x)$. This implies that $\xi'_{2B_2}\left(\tilde{\theta}\right)$ exists. Observe that $\hat{q}_x^{2B_2}(\cdot)$ is also nondecreasing $\forall x$ and $\hat{q}_{\tilde{\theta}}^{2B_2}(\cdot) = q^*(\cdot)$. Therefore, $\xi_{2B_2}\left(\tilde{\theta}\right)$ must be a local maximum, which implies that $\xi'_{2B_2}\left(\tilde{\theta}\right) = 0$. Observe that for $x < \tilde{\theta}$, $\xi'_{2B_2}(x) = \xi'_{2A}(x)$, and for $x > \tilde{\theta}$, $\xi'_{2B_2}(x) = \xi'_{2B_1}(x)$. From the arguments above for Case $2A$, $\xi'_{2A}\left(\tilde{\theta}\right) = 0$ implies $\xi'_{2A}(x) < 0$ for $x < \tilde{\theta}$; from the arguments above for Case $2B_1$, $\xi'_{2B_1}\left(\tilde{\theta}\right) = 0$ implies $\xi'_{2B_1}(x) > 0$ for $x > \tilde{\theta}$. Therefore, $\xi'_{2B_2}\left(\tilde{\theta}\right) = 0$ implies that $\xi'_{2B_2}(x) < (>)\, 0$ for $x < (>)\, \tilde{\theta}$, and this contradicts $\xi_{2B_2}\left(\tilde{\theta}\right)$ being a local maximum. Therefore, we cannot have Case $2B_2$ as well. Thus, we have eliminated the possibility of Case $2B$.

Finally, we consider Case $2C$ — i.e., — i.e., $q^*\left(\tilde{\theta}\right) = \tau_L\left(\tilde{\theta}\right) > \tau_M(\underline{\theta})$. This implies that $q^*(\underline{\theta}) > \tau_M(\underline{\theta})$. For some $\varepsilon > 0$ such that $\varepsilon < \eta_2$ (defined above for Case 2) and $\tilde{\theta} - \varepsilon > \underline{\theta}$, consider the set of schedules $\hat{q}_x^{2C}(\cdot)$ parameterized by $x \in \left(\tilde{\theta} - \varepsilon, \tilde{\theta} + \varepsilon\right)$, where

$$\hat{q}_x^{2C}(\theta) = \begin{cases} q^*(\theta) & , \text{ if } \theta \notin [\underline{\theta}, \tilde{\theta} + \varepsilon) \\ \tau_L(x) & , \text{ if } \theta \in [\underline{\theta}, x] \\ \tau_L(\theta) & , \text{ if } \theta \in [x, \tilde{\theta} + \varepsilon) \end{cases}$$

Let $\xi_{2C}(x) = \int_0^1 \bar{\psi}^B\left(\hat{q}_x^{2C}(\theta), \theta\right) j(\theta)\, d\theta$. Note that if $\theta \in (\underline{\theta}, \kappa(x))$, $\tau_L(x) > \tau_M(\theta)$, whereas if $\theta \in (\kappa(x), x)$, $\tau_L(x) < \tau_M(\theta)$. Therefore, the first two derivatives of $\xi_{2C}(x)$ are

$$\begin{aligned} \xi'_{2C}(x) =& \tau'_L(x)\Big(\left[J(x) - J(\kappa(x))\right]s_M - \left[1 - J(x)\right](c_M - c_L) \\ &+ \left[J(\kappa(x)) - J(\underline{\theta})\right]s_H - \left[1 - J(\kappa(x))\right](c_H - c_M) \Big) \\ \xi''_{2C}(x) =& \left(\tau''_L(x)/\tau'_L(x)\right)\xi'_{2C}(x) \\ &+ \tau'_L(x)\Big(j(x)(s_H + c_H - c_M)\kappa'(x) + j(x)(c_M - c_L) \\ &+ s_M\left[j(x) - j(\kappa(x))\frac{\tau'_L(x)}{\tau'_M(\kappa(x))}\right] \Big) \end{aligned}$$

As before, $\hat{q}_x^{2C}(\cdot)$ is also nondecreasing $\forall x$ and $\hat{q}_{\tilde{\theta}}^{2C}(\cdot) = q^*(\cdot)$. Therefore, $\xi_{2C}\left(\tilde{\theta}\right)$ must be a local maximum. As in Case $2B_1$ above, under Assumption 2, the expression in the last line of $\xi''_{2C}(x)$ is positive. Therefore, $\xi'_{2C}\left(\tilde{\theta}\right) = 0$ implies that $\xi''_{2C}\left(\tilde{\theta}\right) > 0$, which is a contradiction because $\xi_{2C}\left(\tilde{\theta}\right)$ must be a local maximum. This thus eliminates Case $2C$.

In summary, both Case 1 (i.e., $q^*\left(\tilde{\theta}\right) = \tau_M\left(\tilde{\theta}\right)$) and Case 2 (i.e., $q^*\left(\tilde{\theta}\right) = \tau_L\left(\tilde{\theta}\right)$) cannot hold. However, as argued above, if $\tilde{\Theta}$ has a positive measure, then one of these two cases must hold. Therefore, $\tilde{\Theta}$ must be of zero measure, which proves Lemma 10. $\square$

Proof of Proposition 5.

Proof. $t(\theta) = U^S(q(\theta), t(\theta); \theta) + C(q(\theta), \theta)$, with U^S defined in equation (25). The form of the optimal $q(\cdot)$ is characterized in Lemma 10. If $\theta < \theta_L$, $t(\theta) = 0 + C(\tau_L(\theta), \theta) = \tau_L(\theta) c_L$. If $\theta \in [\theta_L, \theta_M)$,

$$\begin{aligned}
t(\theta) &= (c_M - c_L) \int_{\theta_L}^{\theta} \tau'_L(t)\, dt + C(\tau_M(\theta), \theta) \\
&= (c_M - c_L)\left[\tau_L(\theta) - \tau_L(\theta_L)\right] + \tau_M(\theta) c_L + \left[\tau_M(\theta) - \tau_L(\theta)\right](c_M - c_L) \\
&= \tau_L(\theta_L) c_L + \left[\tau_M(\theta) - \tau_L(\theta_L)\right] c_M.
\end{aligned}$$

If $\theta \geq \theta_M$,

$$\begin{aligned}
t(\theta) &= (c_M - c_L) \int_{\theta_L}^{\theta} \tau'_L(t)\, dt + (c_H - c_M) \int_{\theta_M}^{\theta} \tau'_M(t)\, dt + C(\tau_M(\theta), \theta) \\
&= c_L + \left[1 - \tau_L(\theta_L)\right](c_M - c_L) + \left[1 - \tau_M(\theta_M)\right](c_H - c_M) \\
&= \tau(\theta_L) c_L + \left[\tau_M(\theta_L) - \tau_L(\theta_L)\right] c_M + \left[1 - \tau_M(\theta_M)\right] c_H
\end{aligned}$$

$\square$

Numerical Simulation for the Seller-optimal Equilibrium

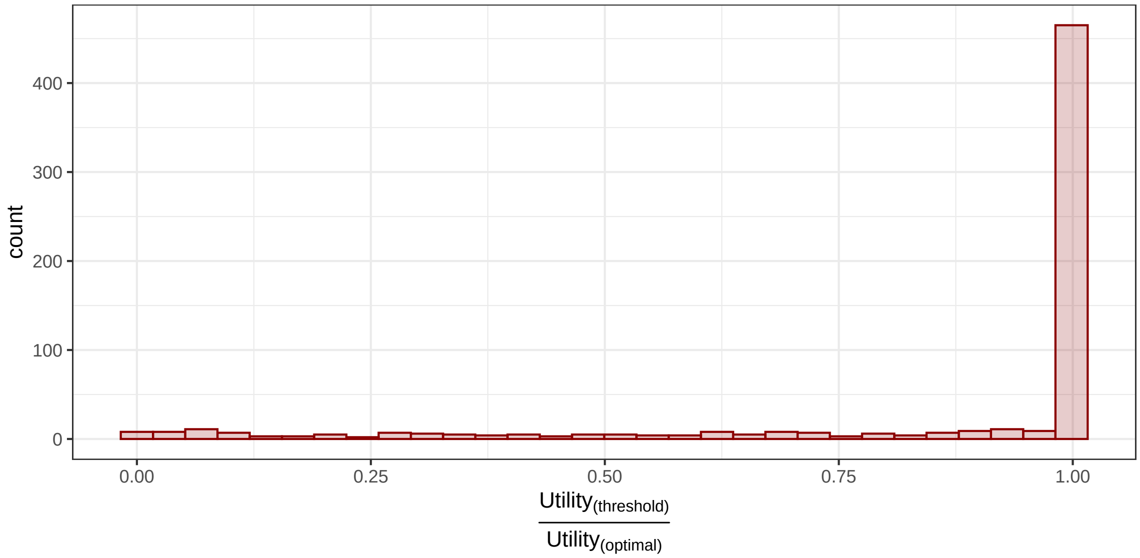
We provide two sets of numerical simulation results for the seller-optimal equilibrium in our three-quality context.

In the first set of simulations, we investigate the performance of threshold quantity schedules using the Beta distribution to parameterize J. The Beta distribution is a flexible family of continuous distributions over the interval $[0, 1]$, parameterized by two shape parameters $a_1 > 0$ and $a_2 > 0$. For example, when $a_1 = a_2 = 1$, we have the uniform distribution. We conducted 1000 simulations. For each simulation, we first randomly draw a set of nonnegative parameters $\{c_L, c_M, c_H, v_L, v_M, v_H\}$ under the restriction that $c_L < c_M < c_H$, $v_L < v_M < v_H$ and $s_L, s_M, s_H > 0$. Next, we construct J by randomly drawing a_1 and a_2 from the distribution $\log a_1, \log a_2 \sim U[-4, 4]$. Finally, we construct the τ-functions using the following procedure. We first randomly draw twenty numbers from the interval $[0, 1]$ and order them in ascending order. Let this ordered set be $\{\theta_1, \ldots, \theta_{20}\}$. Next, we draw another two sets of twenty numbers from $[0, 1]$ and order the numbers in each set in ascending order. Let these two sets be $\{x_1, \ldots, x_{20}\}$ and $\{y_1, \ldots, y_{20}\}$, and let $\bar{z}_i = \max\{x_i, y_i\}$ and $\underline{z}_i = \min\{x_i, y_i\}$. We then construct two nondecreasing piecewise linear functions $\tilde{\tau}_L, \tilde{\tau}_M : [0, 1] \to [0, 1]$ using these twenty values by setting $\tilde{\tau}_L(\theta_i) = \underline{z}_i$ and $\tilde{\tau}_M(\theta_i) = \bar{z}_i$ for $i = 1, \ldots, 20$. Thus, $\tilde{\tau}_L$ and $\tilde{\tau}_M$ have the property that $\tilde{\tau}_L(\theta) \leq \tilde{\tau}_M(\theta)$ for all θ. Finally, we apply a fifth-order spline to each of $\tilde{\tau}_L$ and $\tilde{\tau}_M$ to smooth them out, hence obtaining two nondecreasing and differentiable functions τ_L and τ_M.

For each simulation, after drawing the valuation and cost parameters, the J distribution, and the τ-functions, we compute the seller's utility under the seller-optimal equilibrium (call it $\text{Utility}_{(\text{optimal})}$) and the corresponding seller's utility with the additional constraint that the quantity schedule must be a threshold schedule (call it $\text{Utility}_{(\text{threshold})}$). Note that, although $\tilde{\tau}_M(\cdot)$ is pointwise weakly higher than $\tilde{\tau}_L(\cdot)$, this property might not be preserved between $\tau_M(\cdot)$ and $\tau_L(\cdot)$. When this arises, we discard the draw. We also discard the draws that have mild adverse selection — i.e., $E[v] \geq c_H$ — because the optimal quantity schedule under mild adverse selection is trivially a threshold schedule in which all types of sellers sell their entire endowments. Accordingly, out of 1000 simulations, we kept 637. Figure 4 plots the histogram of the ratio $\frac{\text{Utility}_{(\text{threshold})}}{\text{Utility}_{(\text{optimal})}}$ for these simulations. Of these 637 simulations, the ratio is 1 for 458 times (i.e., 72%), implying that there indeed exists a seller-optimal equilibrium with a threshold quantity schedule in these cases. Moreover, for 80% of the simulations, the optimal threshold schedule achieves at least 80% of the seller's utility under the optimal schedule (i.e., $\frac{\text{Utility}_{(\text{threshold})}}{\text{Utility}_{(\text{optimal})}} \geq 0.8$), implying that even in cases in which the seller-optimal quantity schedule is never a threshold schedule, the seller-optimal threshold quantity schedule still performs quite well.

Next, we investigate the possibility of Assumption 2 as a sufficient condition for the optimality of threshold schedules in the seller-optimal equilibrium. Given the flexibility of how the τ-functions and distribution J are drawn in the previous set of simulations, the number of draws in which part 2 of Assumption 2 holds tends to be small. However, in each of the instances in which Assumption 2 holds, the optimal schedule is indeed a threshold schedule.

Figure 4: Numerical Simulation for the Seller-Optimal Equilibrium



Instead, we conducted a second set of numerical simulations using the uniform distribution for J and parameterizing the τ-functions by affine functions that satisfy Assumption 2. Because a line is parameterized by only its vertical intercept and slope, for each simulation, we first draw two sets of two nonnegative numbers $\{x_1, x_2\}$ and $\{y_1, y_2\}$. τ_M is the affine function with vertical intercept $\max\{x_1, x_2\}$ and slope $\max\{y_1, y_2\}$, and τ_L is the affine function with vertical intercept $\min\{x_1, x_2\}$ and slope $\min\{y_1, y_2\}$. It is readily observed that Assumption 2 always holds. As in the previous set of simulations, we then compute $\text{Utility}_{(\text{optimal})}$ and $\text{Utility}_{(\text{threshold})}$. We conducted 1000 such simulations. In *all* of these simulations, we have $\text{Utility}_{(\text{threshold})} = \text{Utility}_{(\text{optimal})}$.

B.4 Asymmetric Information on Endowment Size

This subsection provides details for the extension in Section 7.2 and the proof of Proposition 6. Note that whenever we write a type (n, λ), it is taken for granted that $\lambda \leq n$. We will adopt the same notations (with some abuse) as the baseline model but modify them accordingly to account for the additional asymmetric information on n.

For $q \leq n$, let the type-(n, λ) seller's and the buyer's utility from contract (q, t) be $U^S(q, t; n, \lambda) = t - C(q, \lambda)$ and $U^B(q, t; n, \lambda) = t - V(q, \lambda)$, respectively. To shorten the notation, let $k(n,\lambda) = (q(n,\lambda), t(n,\lambda))$. Our problem is

$$\max_{\kappa(\cdot)} \int_0^1 \int_0^n \left[\sigma U^S(k(n,\lambda);n,\lambda) + (1-\sigma) U^B(k(n,\lambda);n,\lambda)\right] f(\lambda|n) h(n) \, d\lambda dn \quad \text{s.t.} \quad (IC_S^e), (IR_S^e) \text{ and } (IR_B^e) \tag{$\mathcal{P}_\sigma^e$}$$

where

$$U^S(k(n,\lambda);n,\lambda) \geq U^S(k(n',\lambda');n,\lambda) \;\; \forall (n,\lambda),(n',\lambda'), \tag{IC_S^e}$$

$$U^S(k(n,\lambda);n,\lambda) \geq 0 \qquad \forall (n,\lambda) \tag{IR_S^e}$$

$$\int_0^1 \int_0^n U^B(k(n,\lambda);n,\lambda) f(\lambda|n) h(n) \, d\lambda dn \geq 0 \tag{IR_B^e}$$

Because the type is two-dimensional, the type space does not have a complete order, which means that defining a monotonicity notion for the quantity schedule is not straightforward. The following is the appropriate monotonicity notion:

Definition 2. $q(\cdot)$ is "*monotonic*" if

- for any two types (n',λ') and (n,λ) in which $\lambda' > \lambda$, either $q(n',\lambda') \geq q(n,\lambda)$ or $q(n,\lambda) > n' = q(n',\lambda')$.
- for any two types $\left(n',\hat{\lambda}\right)$ and $\left(n,\hat{\lambda}\right)$ in which $n' > n$, either $q\left(n',\hat{\lambda}\right) = q\left(n,\hat{\lambda}\right)$ or $q\left(n',\hat{\lambda}\right) > n = q\left(n,\hat{\lambda}\right)$.

In words, when $\lambda' > \lambda$, the type with λ' (or more Ls) must trade weakly more than the type with λ whenever the endowment of λ' permits. Therefore, if the lower λ trades more than the higher λ', it must imply that λ' trades her entire endowment (i.e., her endowment constraint binds). Next, if two types have the same λ, then they must trade the same quantity whenever their endowments permit. Therefore, if $q\left(n',\hat{\lambda}\right) > q\left(n,\hat{\lambda}\right)$, it must imply that type $\left(n,\hat{\lambda}\right)$ trades her entire endowment.

Lemma 11. *The solution to program ($\mathcal{P}_\sigma^e$) must satisfy*

$$U^S(k(n,\lambda);n,\lambda) = U^S(k(n,0);n,0) + (c_H - c_L)\int_0^\lambda \mathcal{I}(q(n,l) > l)\, dl \;\; \forall (n,\lambda), \tag{27}$$

with $q(\cdot)$ being monotonic according to Definition 2.

Lemma 11 is the analog of Lemma 2 in the baseline model. The arguments are the same; thus, we omit the proof. Next, we substitute in equation (27) to express the payoffs in virtual terms. Let $S(q,\lambda) = V(q,\lambda) - C(q,\lambda)$ and

$$\psi^B(q;n,\lambda) = qs_L + \mathcal{I}(q>\lambda)\left[(q-\lambda)(s_H - s_L) - (c_H - c_L)\frac{1-F(\lambda|n)}{f(\lambda|n)}\right]$$

Let $u^0 = \int_0^1 \int_0^n U^S(k(n,0);n,0) f(\lambda|n) h(n) d\lambda dn$. By the same argument as for the baseline model, in the buyer-optimal problem, u^0 is optimally set to zero — i.e., $U^S(k(n,0);n,0) = 0$ for all n. In the seller-optimal problem, constraint (IR_B^e) must bind. Thus, the buyer-optimal problem is

$$\max_{q(\cdot)\text{ is nondecreasing}} \int_0^1 \int_0^n \psi^B(q(n,\lambda);n,\lambda) f(\lambda|n) h(n) d\lambda dn \quad .$$

The seller-optimal problem is

$$\begin{aligned} \max_{q(\cdot)\text{ is nondecreasing}} \quad & \int_0^1 \int_0^n S(q(n,\lambda),\lambda) f(\lambda|n) h(n) d\lambda dn. \\ \text{s.t.} \quad & \int_0^1 \int_0^n \psi^B(q(n,\lambda);n,\lambda) f(\lambda|n) h(n) d\lambda dn \geq 0 \end{aligned}$$

Without loss of generality, we assume that $q(\cdot)$ is always right-continuous in λ.[28]

Lemma 12. *For both the buyer-optimal and seller-optimal problems, if $q^*(\cdot)$ is a solution, $q^*(\cdot)$ must satisfy the following two conditions:*

1. *$q^*(n,\lambda) \geq \lambda$ for all (n,λ).*
2. *Let $\mathcal{X}$ be the set of λ such that $q^*(1,\lambda) > \lambda$.*
   (a) *If $\mathcal{X}$ is empty, then $q^*(n,\lambda) = \lambda$ for all (n,λ).*
   (b) *Suppose that $\mathcal{X}$ is nonempty. Let λ_1 denote the smallest λ in $\mathcal{X}$, and let λ_2 denote the smallest $\lambda \in (\lambda_1, 1]$ such that $q^*(1,\lambda) = \lambda$. It holds that for all $\lambda \in [\lambda_1,\lambda_2]$, $q^*(n,\lambda) = \min\{n,\lambda_2\}$.*

Proof. $S(q,\lambda)$ is strictly increasing in q. When $q < \lambda$, $\psi^B(q,\lambda)$ is also strictly increasing in q. This explains point 1. Next, monotonicity of $q^*(\cdot)$ implies that for all λ, $q^*(1,\lambda) \geq q^*(n,\lambda)$. Point 2a hence follows. Finally, for Point 2b, since $\psi^B(q,\lambda)$ is also strictly increasing in q when $q > \lambda$, it must be the case the $q^*(1,\lambda) = \lambda_2$ for all $\lambda \in [\lambda_1,\lambda_2]$. Point 2b then follows from the monotonicity of $q^*(\cdot)$. □

[28] i.e., for any x and n, $\lim_{\lambda\downarrow x^+} q(n,\lambda) = q(n,x)$.

Lemma 12 is the analog of Lemma 5. Point 2b states that unless every type sells only their Ls, the optimal quantity schedule must feature some bunching, similar to the middle panel of Figure 1 for the baseline model. The difference is that because of the endowment constraint for some types, such bunching might not always be possible. When this happens, the endowment constraint for these types must bind.

Lemma 13. *Suppose that Assumption 3 holds. For both the buyer-optimal and the seller-optimal problems, if $q^*(\cdot)$ is a solution, then there exists $\hat{\lambda}$ such that for all $\lambda < \hat{\lambda}$, $q(n,\lambda) = \lambda$, and for all $\lambda \geq \hat{\lambda}$, $q^*(n,\lambda) = n$.*

Proof. Let $q^*(\cdot)$ be an optimal schedule. Let λ_1 and λ_2 be as defined in Lemma 12. The lemma is proved by showing that $\lambda_2 = 1$.

Consider the seller-optimal problem first. Suppose, for a contradiction, that $\lambda_2 < 1$. There must then exist $\lambda_3 > \lambda_2$ such that $q^*(1,\lambda) = \lambda$ for all $\lambda \in [\lambda_2, \lambda_3]$. Since $q^*(\cdot)$ is monotonic, this also implies that for any $n < 1$ and $\lambda \in [\lambda_2, \lambda_3]$, $q^*(n,\lambda) = \lambda$. Observe that

$$\begin{aligned}
&\int_0^1 \int_{\lambda_1}^{\lambda_3} S\left(q^*(n,\lambda), n\right) f(\lambda|n)\, h(n)\, d\lambda dn \\
&= \int_{\lambda_3}^{n} \int_{\lambda_1}^{\lambda_3} S\left(q^*(n,\lambda), n\right) f(\lambda|n)\, h(n)\, d\lambda dn \;+\; \int_{\lambda_1}^{\lambda_3} \int_{\lambda_1}^{n} S\left(q^*(n,\lambda), n\right) f(\lambda|n)\, h(n)\, d\lambda dn \\
&= \int_0^1 \int_{\lambda_1}^{\lambda_3} \lambda s_L f(\lambda|n)\, h(n)\, d\lambda dn \\
&\quad + \int_{\lambda_2}^{1} \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \lambda)\, s_H f(\lambda|n)\, h(n)\, d\lambda dn \;+\; \int_{\lambda_1}^{\lambda_2} \int_{\lambda_1}^{n} (n-\lambda)\, s_H f(\lambda|n)\, h(n)\, d\lambda dn
\end{aligned}$$

For some small $\varepsilon > 0$ and $x \in [\lambda_1, \lambda_1 + \varepsilon]$, let $\phi(x)$ be such that

$$\begin{aligned}
&\int_{\phi(x)}^{1} \int_{x}^{\phi(x)} (\phi(x) - \lambda)\, s_H f(\lambda|n)\, h(n)\, d\lambda dn \;+\; \int_{x}^{\phi(x)} \int_{x}^{n} (n-\lambda)\, s_H f(\lambda|n)\, h(n)\, d\lambda dn \\
&= \int_{\lambda_2}^{1} \int_{\lambda_1}^{\lambda_2} (\lambda_2 - \lambda)\, s_H f(\lambda|n)\, h(n)\, d\lambda dn \;+\; \int_{\lambda_1}^{\lambda_2} \int_{\lambda_1}^{n} (n-\lambda)\, s_H f(\lambda|n)\, h(n)\, d\lambda dn \qquad (28)
\end{aligned}$$

We restrict ε to be small enough such that $\phi(\lambda_1 + \varepsilon) < \lambda_3$.

Define schedule $\hat{q}_x$ as follows:

$$\hat{q}_x(n,\lambda) = \begin{cases} q^*(n,\lambda) & \text{, if } \lambda \notin [\lambda_1, \lambda_3) \\ \lambda & \text{, if } \lambda \in [\lambda_1, x) \\ \min\{n, \phi(x)\} & \text{, if } \lambda \in [x, \phi(x)) \\ \lambda & \text{, if } \lambda \in [\phi(x), \lambda_3) \end{cases}$$

By construction, $\int_0^1 \int_0^n S\left(q^*\left(n,\lambda\right),\lambda\right) f\left(\lambda|n\right) h\left(n\right) d\lambda dn = \int_0^1 \int_0^n S\left(\hat{q}_x\left(n,\lambda\right),\lambda\right) f\left(\lambda|n\right) h\left(n\right) d\lambda dn$. Let

$$\mathcal{R}\left(q;n,\lambda\right) = \mathcal{I}\left(q>\lambda\right)\left(c_H - c_L\right)\left[1-F\left(\lambda|n\right)\right].$$

Therefore, $\psi^B\left(q;n,\lambda\right) f\left(\lambda|n\right) = S\left(q,\lambda\right) f\left(\lambda|n\right) - \mathcal{R}\left(q;n,\lambda\right)$. The difference in the buyer's expected utility between $\hat{q}_x\left(\cdot\right)$ and $q^*\left(\cdot\right)$ is

$$\begin{aligned}
D\left(x\right) &= \int_0^1 \int_0^n \left[\psi^B\left(\hat{q}_x\left(n,\lambda\right);n,\lambda\right) - \psi^B\left(q^*\left(n,\lambda\right);n,\lambda\right)\right] f\left(\lambda|n\right) h\left(n\right) d\lambda dn \\
&= \int_0^1 \int_0^n \left[\mathcal{R}\left(q^*\left(n,\lambda\right);n,\lambda\right) - \mathcal{R}\left(\hat{q}_x\left(n,\lambda\right);n,\lambda\right)\right] h\left(n\right) d\lambda dn \\
&= \int_{\lambda_3}^1 \int_{\lambda_1}^{\lambda_3} \left[\mathcal{R}\left(q^*\left(n,\lambda\right);n,\lambda\right) - \mathcal{R}\left(\hat{q}_x\left(n,\lambda\right);n,\lambda\right)\right] h\left(n\right) d\lambda dn \\
&\quad + \int_{\lambda_1}^{\lambda_3} \int_{\lambda_1}^{n} \left[\mathcal{R}\left(q^*\left(n,\lambda\right);n,\lambda\right) - \mathcal{R}\left(\hat{q}_x\left(n,\lambda\right);n,\lambda\right)\right] h\left(n\right) d\lambda dn \\
&= \left(c_H - c_L\right)\left[\int_{\lambda_2}^1 \left(\int_{\lambda_1}^{\lambda_2} 1-F\left(\lambda|n\right) d\lambda\right) h\left(n\right) dn + \int_{\lambda_1}^{\lambda_2} \left(\int_{\lambda_1}^{n} 1-F\left(\lambda|n\right) d\lambda\right) h\left(n\right) dn\right] \\
&\quad - \left(c_H - c_L\right)\left[\int_{\phi(x)}^1 \left(\int_{x}^{\phi(x)} 1-F\left(\lambda|n\right) d\lambda\right) h\left(n\right) dn + \int_{x}^{\phi(x)} \left(\int_{x}^{n} 1-F\left(\lambda|n\right) d\lambda\right) h\left(n\right) dn\right].
\end{aligned}$$

Differentiating $D\left(x\right)$ with respect to x, we have

$$D'\left(x\right) = \left[\int_x^1 \left[1-F\left(x|n\right)\right] h\left(n\right) dn - \left(\int_{\phi(x)}^1 \left[1-F\left(\phi\left(x\right)|n\right)\right] h\left(n\right) dn\right) \phi'\left(x\right)\right]\left(c_H - c_L\right)$$

From equation (28), we have

$$\begin{aligned}
\phi'\left(x\right) =& \frac{\int_{\phi(x)}^1 \left[\phi\left(x\right) - x\right] f\left(x|n\right) h\left(n\right) dn + \int_x^{\phi(x)} \left(n-x\right) f\left(x|n\right) h\left(n\right) dn}{\int_{\phi(x)}^1 \left[F\left(\phi\left(x\right)|n\right) - F\left(x|n\right)\right] h\left(n\right) dn} \\
>& \frac{\left[\phi\left(x\right) - x\right] \int_x^1 f\left(x|n\right) h\left(n\right) dn}{\int_{\phi(x)}^1 \left[F\left(\phi\left(x\right)|n\right) - F\left(x|n\right)\right] h\left(n\right) dn}
\end{aligned}$$

Therefore, we have $D'\left(x\right) > 0$ if

$$\begin{aligned}
& \frac{\left[\phi\left(x\right) - x\right] \int_x^1 f\left(x|n\right) h\left(n\right) dn}{\int_{\phi(x)}^1 \left[F\left(\phi\left(x\right)|n\right) - F\left(x|n\right)\right] h\left(n\right) dn} \le \frac{\int_x^1 \left[1-F\left(x|n\right)\right] h\left(n\right) dn}{\left(\int_{\phi(x)}^1 \left[1-F\left(\phi\left(x\right)|n\right)\right] h\left(n\right) dn\right)} \\
\iff & \frac{\int_x^1 f\left(x|n\right) h\left(n\right) dn}{\int_x^1 \left[1-F\left(x|n\right)\right] h\left(n\right) dn} \le \frac{\int_{\phi(x)}^1 \left[F\left(\phi\left(x\right)|n\right) - F\left(x|n\right)\right] h\left(n\right) dn}{\left[\phi\left(x\right) - x\right]\left(\int_{\phi(x)}^1 \left[1-F\left(\phi\left(x\right)|n\right)\right] h\left(n\right) dn\right)}
\end{aligned}$$

Fix some λ, let $LHS = \frac{\int_{\lambda}^{1} f(\lambda|n)h(n)dn}{\int_{\lambda}^{1}[1-F(\lambda|n)]h(n)dn}$, and let $RHS(\lambda') = \frac{\int_{\lambda'}^{1}[F(\lambda'|n)-F(\lambda|n)]h(n)dn}{(\lambda'-\lambda)\int_{\lambda'}^{1}[1-F(\lambda'|n)]h(n)dn}$. By L'Hôpital's rule, $\lim_{\lambda'\downarrow\lambda} RHS(\lambda') = LHS$ and $\lim_{\lambda'\uparrow 1} RHS(\lambda') = \infty$. Suppose, for a contradiction, that there exists $\lambda' \in (\lambda, 1)$ such that $LHS > RHS(\lambda')$. This must imply that there exists $\hat{\lambda} \in (\lambda, 1)$ such that $LHS > RHS\left(\hat{\lambda}\right)$ and $\frac{d}{d\lambda'} RHS(\lambda')\Big|_{\lambda'=\hat{\lambda}} = 0$. By some algebra, $\frac{d}{d\lambda'} RHS(\lambda')\Big|_{\lambda'=\hat{\lambda}} = 0$ implies that

$$\begin{aligned}
\frac{\int_{\hat{\lambda}}^{1} f\left(\hat{\lambda}|n\right) h(n)\, dn}{\int_{\hat{\lambda}}^{1}\left[1-F\left(\hat{\lambda}|n\right)\right] h(n)\, dn} &= RHS\left(\hat{\lambda}\right) \frac{\int_{\hat{\lambda}}^{1}\left[1-F\left(\hat{\lambda}|n\right)\right] h(n)\, dn}{\int_{\hat{\lambda}}^{1}\left[1-F(\lambda|n)\right] h(n)\, dn} + \frac{\left[1-F\left(\lambda|\hat{\lambda}\right)\right] h\left(\hat{\lambda}\right)}{\int_{\hat{\lambda}}^{1}\left[1-F(\lambda|n)\right] h(n)\, dn} \\
&< LHS + \frac{\left[1-F\left(\lambda|\hat{\lambda}\right)\right] h\left(\hat{\lambda}\right)}{\int_{\hat{\lambda}}^{1}\left[1-F(\lambda|n)\right] h(n)\, dn} \\
&= \frac{\int_{\lambda}^{1} f(\lambda|n) h(n)\, dn}{\int_{\lambda}^{1}\left[1-F(\lambda|n)\right] h(n)\, dn} - \frac{d}{d\hat{\lambda}} \log\left(\int_{\hat{\lambda}}^{1}\left[1-F(\lambda|n)\right] h(n)\, dn\right),
\end{aligned}$$

where the inequality in the second line follows from $\lambda < \hat{\lambda}$ and $LHS > RHS\left(\hat{\lambda}\right)$. However, this contradicts Assumption 3.[29] Therefore, it holds that $LHS \leq RHS(\lambda')$, which implies that $D'(x) > 0$. Since $D(\lambda_1) = 0$, there exists $x > \lambda_1$ such that $D(x) > 0$, thus implying that

$$\int_0^1 \int_0^n \psi^B\left(\hat{q}_x(n,\lambda); n, \lambda\right) f(\lambda|n) h(n)\, d\lambda dn > \int_0^1 \int_0^n \psi^B\left(q^*(n,\lambda); n, \lambda\right) f(\lambda|n) h(n)\, d\lambda dn \tag{29}$$

and $\int_0^1 \int_0^n S\left(\hat{q}_x(n,\lambda), \lambda\right) f(\lambda|n) h(n)\, d\lambda dn = \int_0^1 \int_0^n S\left(q^*(n,\lambda), \lambda\right) f(\lambda|n) h(n)\, d\lambda dn$. This contradicts the optimality of $q^*(\cdot)$ for the seller-optimal problem. Therefore, λ_2 must be 1 in the seller-optimal problem. For the buyer-optimal problem, the result also follows from the same argument because of the relationship in equation (29). □

Proof of Proposition 6.

Proof. Given Lemma 13, the optimal quantity schedules for the buyer-optimal problem and the seller-optimal problem are the threshold-$\hat{\lambda}^B$ schedule and the threshold-$\hat{\lambda}^S$ schedule, respectively. In the buyer-optimal problem, $U^S(k(n,0); n, 0) = 0$. Thus, $t(n,\lambda) = (c_H - c_L)\int_0^{\lambda} \mathcal{I}\left(q^*(n,l) > l\right) dl + C\left(q^*(n,\lambda), \lambda\right)$. Therefore, if $\lambda < \hat{\lambda}^B$, $t(n,\lambda) = C(\lambda,\lambda) =$

[29] Note that $f(\lambda|n \geq \lambda) = \frac{\int_{\lambda}^{1} f(\lambda|n)h(n)dn}{1-H(\lambda)}$ and $1 - F(\lambda|n \geq \lambda) = \frac{\int_{\lambda}^{1} h(n)dn - \int_{\lambda}^{1} F(\lambda|n)h(n)dn}{1-H(\lambda)} = \frac{\int_{\lambda}^{1}[1-F(\lambda|n)]h(n)dn}{1-H(\lambda)}$. Therefore, $\frac{f(\lambda|n\geq\lambda)}{1-F(\lambda|n\geq\lambda)} = \frac{\int_{\lambda}^{1} f(\lambda|n)h(n)dn}{\int_{\lambda}^{1}[1-F(\lambda|n)]h(n)dn}$.

λc_L; if $\lambda \geq \hat{\lambda}^B$,

$$t(n,\lambda) = (c_H - c_L)\left(\lambda - \hat{\lambda}^B\right) + C(n,\lambda).$$
$$= \hat{\lambda}^B c_L + \left(n - \hat{\lambda}^B\right) c_H$$

In the seller-optimal problem, if $\hat{E}[v] \geq c_H$, $\hat{\lambda}^S$ must be 0 — i.e., $q^*(n,\lambda) = n$ for all (n,λ). The transfer is then pinned down by the binding constraint (IR_B^e), which means that the buyer's expected utility is zero. Next, if $\hat{E}[v] < c_H$, then $\hat{\lambda}^S > 0$ and hence $\hat{\Psi}^B\left(\hat{\lambda}^S\right) = 0$. Given the binding ($IR_B^e$), $U^S(k(n,0);n,0)$ is zero for all n. The transfers are then derived as in the buyer-optimal problem above, with $\hat{\lambda}^B$ replaced by $\hat{\lambda}^S$. □

B.5 Decreasing Marginal Utility

To clearly illustrate the key mechanisms in our baseline model, we have assumed that, conditional on the quality, the buyer's utility of each marginal unit is constant. Therefore, the presence of within-type adverse selection unambiguously causes the buyer's valuation of the next marginal unit to increase with the quantity traded. In this subsection, we allow the buyer's marginal utility from consumption to decrease with the quantity consumed. Clearly, with sufficiently diminishing marginal utility, a full bundle can never be optimal. Our goal here is to provide (joint) sufficient conditions on the type distribution and the rate of diminishing marginal utility such that the optimal quantity schedule is still always a threshold schedule. Therefore, whenever there is a bundle option, it is still always the full bundle option.

Suppose that instead of $V(q,\lambda)$ in equation (2), the buyer's valuation of buying q units from type λ is now $\hat{V}(q,\lambda) := av(q,\lambda)\,u(q)$, where

$$av(q,\lambda) := \frac{qv_L + [q-\lambda]^+(v_H - v_L)}{q} = v_L + \left[1 - \frac{\lambda}{q}\right]^+ (v_H - v_L),$$

and $u(q)$ is a strictly increasing and strictly concave function, with $u(0) = 0$. $av(\cdot)$ is the "average valuation" of the q units supplied by type-λ. Fixing the value of av, $u(\cdot)$ captures diminishing marginal utility from consuming more units. Note that because of within-type adverse selection, the average quality provided by the seller increases with the volume of trade — i.e., $av(q,\lambda)$ is increasing in λ. Therefore, $\frac{\partial \hat{V}(q,\lambda)}{\partial q}$ might not be decreasing everywhere. The rest of the model is unchanged.

We consider the buyer-optimal problem ($\sigma = 0$). Substituting in the envelope theorem condition in Lemma 2, the problem is

$$\max_{q(\cdot)} \int_0^1 \hat{\psi}^B (q(\lambda), \lambda) f(\lambda) d\lambda \quad \text{s.t.} \quad q(\cdot) \text{ is nondecreasing,} \tag{30}$$

where

$$\hat{\psi}^B (q, \lambda) := q \hat{s}_L (q) + \mathcal{I}(q > \lambda) \left[(q - \lambda) [\hat{s}_H (q) - \hat{s}_L (q)] - (c_H - c_L) \frac{1 - F(\lambda)}{f(\lambda)} \right],$$

and $\hat{s}_z (q) := \frac{u(q)}{q} v_z - c_z$ for $z \in \{L, H\}$. We can interpret $\hat{s}_z (q)$ as the "*average surplus*" from a trade of quality-z good, and this average quality depends on the total quantity traded.

Assumption 4. *For all $q \in (0,1)$, $\frac{d^2}{dq^2} \left(\frac{u(q)}{q} \right) \geq 0$, $u'(q) > \max \left\{ \frac{c_L}{v_L}, \frac{c_H}{v_H} \right\}$, and $\frac{qu'(q)}{u(q)} \geq 1 - \left(\frac{v_H - v_L}{v_H} \right) \frac{qf(q)}{F(q)}$.*

Observe that $s'_z (q)$ is always negative because $u(q)$ is concave.[30] This implies that the average surplus falls with the quantity traded, and the first part of Assumption 4 states that the rate of this decrease is decreasing. Next, let $\hat{S}(q, \lambda) = \hat{V}(q, \lambda) - C(q, \lambda) = q\hat{s}_H (q) + [q - \lambda]^+ (\hat{s}_H (q) - \hat{s}_L (q))$. Therefore, $\hat{S}(q, \lambda)$ is the social surplus generated from trading q units with type λ. The second assumption on the value of $u'(q)$ ensures that $S(q, \lambda)$ is always strictly increasing in q, which means that a trade is still always efficient. Finally, for the last part of Assumption 4, recall that in our baseline model, due to within-type adverse selection, the seller's valuation of the marginal unit increases with the quantity traded. With decreasing marginal valuation, this effect is dampened, and the last assumption restricts how quickly the marginal utility can diminish with respect to the rate of increase in the quality.

An example that satisfies Assumption 4 is the power function $u(q) = q^\beta$, with $\max \left\{ \frac{c_L}{v_L}, \frac{c_H}{v_H}, \frac{v_L}{v_H} \right\} < \beta < 1$, together with F being the uniform distribution.[31]

Proposition 10. *Under Assumption 4, the solution to program (30) is a threshold schedule.*

[30] This is readily observed from noting that $\frac{u(q)}{q}$ is the slope of the line that joins the origin to $u(q)$; therefore, $\frac{u(q)}{q}$ must be decreasing.

[31] It is straightforward that $\beta < 1$ implies that $\frac{d^2}{dq^2} \left(\frac{u(q)}{q} \right) \geq 0$. The second part of Assumption 4 holds if and only if $\beta > \max \left\{ \frac{c_L}{v_L}, \frac{c_H}{v_H} \right\}$. Finally, under the power function, $\frac{qu'(q)}{u(q)} = \beta$, and under a uniform distribution, $\frac{xf(x)}{F(x)} = 1$; therefore, the last part of Assumption 4 holds if and only if $\beta \geq \frac{v_L}{v_H}$.

Proof. Let $\hat{u}(q) = \frac{u(q)}{q}$. Therefore, $\hat{s}_z(q) := \hat{u}(q)\, v_z - c_z$, and

$$\begin{aligned}\frac{\partial \hat{\psi}^B(q,\lambda)}{\partial q} &= \hat{s}_L(q) + \mathcal{I}(q>\lambda)\left[\hat{s}_H(q) - \hat{s}_L(q)\right] + q\hat{s}'_L(q) + [q-\lambda]^+\left(\hat{s}'_H(q) - \hat{s}'_L(q)\right)\\ &= \hat{u}(q)\Big(s_L + \mathcal{I}(q>\lambda)\left(s_H - s_L\right)\Big) + \hat{u}'(q)\Big(qv_L + [q-\lambda]^+\left(v_H - v_L\right)\Big)\end{aligned}$$

Observe that $\hat{u}(q) + q\hat{u}'(q) = u'(q)$. Therefore, when $q < \lambda$,

$$\frac{\partial \hat{\psi}^B(q,\lambda)}{\partial q} \;=\; \hat{u}(q)\, v_L - c_L + \hat{u}'(q)\, qv_L \;=\; u'(q)\, v_L - c_L \;>0.$$

When $q > \lambda$,

$$\begin{aligned}\frac{\partial \hat{\psi}^B(q,\lambda)}{\partial q} =& \hat{u}(q)\, v_H - c_H + \hat{u}'(q)\Big(\lambda v_L + (q-\lambda)\, v_H\Big)\\ &= \underbrace{\left[\hat{u}(q) + q\hat{u}'(q)\right]}_{=u'(q)} v_H - c_H \;-\underbrace{\hat{u}'(q)}_{<0}\lambda\left(v_H - v_L\right) \;>\; 0.\end{aligned}$$

This implies that $\psi^B(q,\lambda)$ is strictly increasing in q when $q < \lambda$ and $q > \lambda$. Therefore, Lemma 5 holds. Let $q^*(\cdot)$ be an optimal schedule, and let λ_1 and λ_2 be as defined in Lemma 5. To show that $q^*(\cdot)$ is a threshold schedule, we only have to show that $\lambda_2 = 1$. Define $\tilde{q}_x(\cdot)$ as in equation (16) and $\hat{\xi}(x) = \int_0^1 \hat{\psi}^B(\tilde{q}_x(\lambda), \lambda)\, f(\lambda)\, d\lambda$. By the same argument as the proof of Proposition 1, if $\hat{\xi}''(x) > 0$, this must imply that $\lambda_2 = 1$. We show this now.

$$\begin{aligned}\hat{\xi}(x) =& \int_{\lambda \notin [\lambda_1,\lambda_3)} \hat{\psi}^B(q^*(\lambda), \lambda)\, f(\lambda)\, d\lambda\\ &+ \int_{\lambda_1}^{x}\left(\lambda \hat{s}_L(x) + (x-\lambda)\,\hat{s}_H(x) - (c_H - c_L)\frac{1-F(\lambda)}{f(\lambda)}\right) f(\lambda)\, d\lambda\\ &+ \int_{x}^{\lambda_3} \lambda \hat{s}_L(\lambda)\, f(\lambda)\, d\lambda\end{aligned}$$

Therefore,

$$\begin{aligned}\hat{\xi}'(x) &= -(c_H - c_L)[1 - F(x)] + \int_{\lambda_1}^{x} [\lambda \hat{s}_L'(x) + (x - \lambda)\hat{s}_H'(x) + \hat{s}_H(x)] f(\lambda) d\lambda \\ &= -(c_H - c_L)[1 - F(x)] + [F(x) - F(\lambda_1)]\hat{s}_H(x) + \int_{\lambda_1}^{x} [\lambda \hat{s}_L'(x) + (x - \lambda)\hat{s}_H'(x)] f(\lambda) d\lambda\end{aligned}$$

$$\begin{aligned}\hat{\xi}''(x) =& f(x)[(c_H - c_L) + \hat{s}_H(x) + x\hat{s}_L'(x)] + \hat{s}_H'(x)[F(x) - F(\lambda_1)] \\ &+ \int_{\lambda_1}^{x} [\lambda \hat{s}_L''(x) + (x - \lambda)\hat{s}_H''(x)] f(\lambda) d\lambda\end{aligned}$$

By Assumption 4, $\hat{s}_z''(x) = \frac{d}{dq^2}^2 \left(\frac{u(q)}{q}\right) v_z \geq 0$; therefore, the second line of $\xi''(x)$ is positive. Since $\hat{s}_H'(x) < 0$, the first line is weakly larger than

$$\begin{aligned}& f(x)[(c_H - c_L) + \hat{s}_H(x) + x\hat{s}_L'(x)] + \hat{s}_H'(x) F(x) \\ =\ & f(x)[\hat{u}(x) v_H - c_L + x\hat{u}'(x) v_L] + \hat{u}'(x) v_H F(x) \\ \propto\ & [\hat{u}(x) - x\hat{u}'(x)] v_L - c_L \ + \ \hat{u}(x)(v_H - v_L) + \hat{u}'(x) v_H \frac{F(x)}{f(x)} \\ =\ & [u'(x) v_L - c_L] \ + \ \left[\hat{u}(x)(v_H - v_L) + \hat{u}'(x) v_H \frac{F(x)}{f(x)}\right]\end{aligned}$$

In the last line, the term in the first square bracket is positive. As for the second term, note that $\hat{u}'(x) = \frac{u'(x)}{x} - \frac{u(x)}{x^2}$; therefore, the term becomes

$$\begin{aligned}& \frac{u(x)}{x}(v_H - v_L) + \left(u'(x) - \frac{u(x)}{x}\right) v_H \frac{F(x)}{xf(x)} \\ =\ & u'(x) v_H \frac{F(x)}{xf(x)} \ + \ \frac{u(x)}{x}\left[v_H - v_L - v_H \frac{F(x)}{xf(x)}\right] \\ \propto\ & \frac{xu'(x)}{u(x)} + \left[\left(\frac{v_H - v_L}{v_H}\right)\frac{xf(x)}{F(x)} - 1\right] \quad \geq 0,\end{aligned}$$

where the last inequality is from the last part of Assumption 4. Therefore, $\hat{\xi}''(x) > 0$. We have thus shown that $q^*(\cdot)$ must be a threshold schedule. □

B.6 Stochastic Contracts

In the main text, we consider only deterministic contracts. Strausz (2006) showed that in a canonical screening model, the use of stochastic contracts is always suboptimal if the optimal menu of deterministic contracts does not exhibit any contract pooling. This sufficient condition is violated in our problem when the optimal threshold is greater than zero. Moreover, in the setup considered in Strausz (2006), there is not an equivalence of constraint (IR_B). In this section, we consider stochastic contracts and provide a sufficient condition under which the use of stochastic contracts is suboptimal in our problem.

We first define a stochastic contract in our model. Because the utility functions of both the seller and the buyer are linear in the transfers, it suffices to allow for stochasticity only in the quantity. Let α denote a CDF for q over $[0,1]$. A *stochastic contract* is a double (α, t), where t is the transfer from the buyer to the seller, and α is the CDF of the quantity that the seller must supply to the buyer. Let $\alpha^{\Delta}(q) = \alpha(q) - \sup_{x<q} \alpha(q)$. In words, $\alpha^{\Delta}(q)$ is the probability of quantity q, and if α does not have any mass at q, then $\alpha^{\Delta}(q) = 0$. Thus, a contract (α, t) is deterministic if there exists q such that $\alpha^{\Delta}(q) = 1$.

Let $\tilde{C}(\alpha, \lambda) = \int_0^1 C(q, \lambda)\, d\alpha(q)$ and $\tilde{V}(\alpha, \lambda) = \int_0^1 V(q, \lambda)\, d\alpha(q)$. Therefore, when the buyer trades with a type-λ seller under a stochastic contract (α, t), the expected utilities of the buyer and the seller are $\tilde{U}^B(\alpha, t; \lambda) = \tilde{V}(\alpha, \lambda) - t$ and $\tilde{U}^S(\alpha, t; \lambda) = t - \tilde{C}(\alpha, \lambda)$, respectively. Henceforth, we drop the "expected" quantifier without confusion.

A direct menu of stochastic contracts is denoted by $\{\alpha_\lambda, t_\lambda\}_\lambda$, where $(\alpha_\lambda, t_\lambda)$ is the contract meant for type λ. (Because of the CDF here, the type is denoted by a subscript instead.) Consider program (3) with stochastic contracts — i.e.,

$$\begin{array}{c} \max_{\{\alpha_\lambda, t_\lambda\}_\lambda} \int_0^1 \left[\sigma \tilde{U}^S(\alpha_\lambda, t_\lambda; \lambda) + (1-\sigma)\tilde{U}^B(\alpha_\lambda, t_\lambda; \lambda)\right] f(\lambda)\, d\lambda \\ \text{s.t. constraints } (\tilde{IC}_S),\ (\tilde{IR}_S) \text{ and } (\tilde{IR}_B) \end{array}, \tag{31}$$

where

$$\tilde{U}^S(\alpha_\lambda, t_\lambda; \lambda) \geq \tilde{U}^S(\alpha_{\lambda'}, t_{\lambda'}; \lambda) \qquad \forall \lambda, \lambda', \tag{$\tilde{IC}_S$}$$

$$\tilde{U}^S(\alpha_\lambda, t_\lambda; \lambda) \geq 0 \qquad \forall \lambda \tag{$\tilde{IR}_S$}$$

$$\int_0^1 \tilde{U}^B(\alpha_\lambda, t_\lambda; \lambda) f(\lambda)\, d\lambda \geq 0 \tag{$\tilde{IR}_B$}$$

Assumption 5. *For all $\lambda \in (0,1)$, $\frac{1-F(\lambda)}{(1-\lambda)f(\lambda)}$ is decreasing.*

Proposition 11. *Under Assumption 5, program (31) always has a solution that consists of only deterministic contracts.*

Proof. By the envelope theorem, if a menu $\{\alpha_\lambda, t_\lambda\}_\lambda$ satisfies constraint ($\tilde{IC}_S$), then for all

λ,[32]

$$\tilde{U}^S(\alpha_\lambda, t_\lambda; \lambda) = \tilde{U}^S(\alpha_0, t_0; 0) + (c_H - c_L)\int_0^\lambda \mathcal{I}(q > \lambda)\, d\alpha_l(q)\, dl. \tag{32}$$

Substituting in equation (32), the objective function of program (31) is

$$\zeta(\{\alpha_\lambda\}_\lambda; u^o) := (2\sigma - 1)\, u^0 + \int_0^1 VS_\sigma(q, \lambda)\, d\alpha_\lambda(q)\, f(\lambda)\, d\lambda,$$

and constraint ($\tilde{IR}_B$) becomes

$$\int_0^1 \int_0^1 \psi^B(q, \lambda)\, d\alpha_\lambda(q)\, f(\lambda)\, d\lambda \geq u^0 \tag{$\tilde{IR}'_B$}$$

Consider the following program:

$$\max_{\{\alpha_\lambda\}_\lambda; u^o \geq 0} \zeta(\{\alpha_\lambda\}_\lambda; u^o) \quad \text{s.t.} \quad (\tilde{IR}'_B) \tag{$\tilde{\mathcal{P}}_\sigma$}$$

The value of program ($\tilde{\mathcal{P}}_\sigma$) is weakly higher than the value of program (31) because the constraint of program ($\tilde{\mathcal{P}}_\sigma$) is more relaxed — in particular, it only has to satisfy a necessary (but not always sufficient) condition for constraint ($\tilde{IC}_S$). We will show that program ($\tilde{\mathcal{P}}_\sigma$) always has a deterministic solution that also satisfies constraints ($\tilde{IC}_S$) and ($\tilde{IR}_B$).[33] Therefore, this deterministic solution is also the solution to the original program (31). Note that when a deterministic solution satisfies constraint (IC_S) in the main text, then it must also satisfy constraint ($\tilde{IC}_S$) here. To shorten things, we will consider only $\sigma = 0$ and $\sigma = 1$. As explained in the main text, the other cases are either identical or similar to one of these two cases.

Consider $\sigma = 0$ first. As noted in Section 4, we can ignore constraint ($\tilde{IR}'_B$) and set $u^0 = 0$. The objective function becomes $\int_0^1 \int_0^1 \psi^B(q, \lambda)\, d\alpha_\lambda(q)\, f(\lambda)\, d\lambda$. Since $\psi(q, \lambda)$ is strictly increasing in q when $q < \lambda$, it is immediate that under the optimum, $\alpha_\lambda(q) - \alpha_\lambda^\Delta(q)$ must be zero for all λ. Recall that $\bar{q}_0(\lambda)$ is the pointwise optimal q of $\psi^B(q, \lambda)$. Clearly, the solution to program ($\tilde{\mathcal{P}}_\sigma$) must be the $\{\alpha_\lambda\}_\lambda$ that assigns probability one on quantity $\bar{q}_0(\lambda)$ for each λ — i.e., $\alpha_\lambda^{*\Delta}(\bar{q}_0(\lambda)) = 1$ for all λ. This is a menu of deterministic contracts. Moreover, under Assumption 5, there exists $\hat{\lambda}$ such that $\psi^B(1, \lambda) \geq \psi^B(\lambda, \lambda)$ if and only if $\lambda \geq \hat{\lambda}$; therefore, $\bar{q}_0(\lambda)$ is a threshold schedule, which hence also satisfies constraint ($\tilde{IC}_S$).

[32] Note that the partial derivative of $\tilde{U}^S((\alpha, t); \lambda)$ with respect to λ is $(c_H - c_L)\int_0^1 \mathcal{I}(q > \lambda)\, d\alpha(q) = (c_H - c_L)[1 - \alpha(\lambda)]$.

[33] A solution is deterministic if $\{\alpha_\lambda\}_\lambda$ is such that for all λ, there exists q such that $\alpha_\lambda^\Delta(q) = 1$. Note that constraint ($\tilde{IR}_S$) holds if and only if $u^0 \geq 0$.

This implies that $\{\alpha^*_\lambda\}_\lambda$ is a solution to program (31).[34]

Next, consider $\sigma = 1$. As noted in Section 5, constraint $(\tilde{IR}_B)$ must bind. Therefore, the objective function becomes $\int_0^1 \int_0^1 \psi^{S+B}(q,\lambda)\, d\alpha_\lambda(q)\, f(\lambda)\, d\lambda$, and constraint $(\tilde{IR}_B)$ becomes $\int_0^1 \int_0^1 \psi^B(q,\lambda)\, d\alpha_\lambda(q)\, f(\lambda)\, d\lambda \geq 0$. Let $\{\alpha^*_\lambda\}_\lambda$ be a solution to program $(\tilde{\mathcal{P}}_\sigma)$. Fixing any λ, when $q < \lambda$ and when $q > \lambda$, both $\psi^{S+B}(q,\lambda)$ and $\psi^B(q,\lambda)$ are strictly increasing in q. Therefore, it must be the case that $\alpha^{*\Delta}_\lambda(\lambda) + \alpha^{*\Delta}_\lambda(1) = 1$ for all λ. Next, under Assumption 5, there exists $\hat{\lambda}$ such that $\psi^B(1,\lambda) \geq \psi^B(\lambda,\lambda)$ if and only if $\lambda \geq \hat{\lambda}$. This implies that for $\lambda \geq \hat{\lambda}$, $\alpha^{*\Delta}_\lambda(1)$ must be 1. If $\hat{\lambda} = 0$, then $\{\alpha^*_\lambda\}_\lambda$ is a deterministic schedule that specifies every type to sell $q = 1$ with a probability of one; clearly, this satisfies constraint $(\tilde{IC}_S)$, and we are done.

Henceforth, suppose that $\hat{\lambda} > 0$. We first show that for any $\lambda < \hat{\lambda}$, either $\alpha^{*\Delta}_\lambda(\lambda) = 1$ or $\alpha^{*\Delta}_\lambda(1) = 1$. Suppose, for a contradiction, that this is not true. This implies that there exists an interval $[\lambda_1, \lambda_2]$, with $\lambda_1 < \lambda_2 < \hat{\lambda}$, such that for all $\lambda \in [\lambda_1, \lambda_2]$, both $\alpha^{*\Delta}_\lambda(\lambda)$ and $\alpha^{*\Delta}_\lambda(1)$ are strictly positive. Let $\tilde{\psi}^{S+B}(\alpha,\lambda) := \int_0^1 \psi^{S+B}(q,\lambda)\, d\alpha(q)$ and $\tilde{\psi}^B(\alpha,\lambda) := \int_0^1 \psi^B(q,\lambda)\, d\alpha(q)$. Note that for $\lambda \in [\lambda_1, \lambda_3]$,

$$
\begin{aligned}
\tilde{\psi}^{S+B}(\alpha^*_\lambda,\lambda) =& s_L + \alpha^{*\Delta}_\lambda(1)(1-\lambda)(s_H - s_L) \\
\tilde{\psi}^B(\alpha^*_\lambda,\lambda) =& s_L + \alpha^{*\Delta}_\lambda(1)\left[(1-\lambda)(s_H - s_L) - (c_H - c_L)\frac{1-F(\lambda)}{f(\lambda)}\right] \\
=& \tilde{\psi}^{S+B}(\alpha^*_\lambda,\lambda) - \alpha^{*\Delta}_\lambda(1)(c_H - c_L)\frac{1-F(\lambda)}{f(\lambda)}
\end{aligned}
$$

Next, let $\{\hat{\alpha}_\lambda\}_\lambda$ be the modification of $\{\alpha^*_\lambda\}_\lambda$ via the following: first, for all $\lambda \notin (\lambda_1, \lambda_2)$ and $\lambda = \frac{\lambda_2+\lambda_1}{2}$, let $\hat{\alpha}_\lambda = \alpha^*_\lambda$. Next, for each $\varepsilon \in \left(0, \frac{\lambda_2-\lambda_1}{2}\right)$, let

$$
\begin{aligned}
\hat{\alpha}^\Delta_{\lambda_1+\varepsilon}(\lambda_1+\varepsilon) &= \alpha^{*\Delta}_{\lambda_1+\varepsilon}(\lambda_1+\varepsilon) + \eta_{1(\varepsilon)} \\
\hat{\alpha}^\Delta_{\lambda_1+\varepsilon}(1) &= \alpha^{*\Delta}_{\lambda_1+\varepsilon}(1) - \eta_{1(\varepsilon)} \\
\hat{\alpha}^\Delta_{\lambda_2-\varepsilon}(\lambda_2-\varepsilon) &= \alpha^{*\Delta}_{\lambda_2-\varepsilon}(\lambda_2-\varepsilon) - \eta_{2(\varepsilon)} \\
\hat{\alpha}^\Delta_{\lambda_2-\varepsilon}(1) &= \alpha^{*\Delta}_{\lambda_2-\varepsilon}(1) + \eta_{2(\varepsilon)},
\end{aligned}
$$

where $\eta_{1(\varepsilon)}$ and $\eta_{2(\varepsilon)}$ are chosen to be strictly positive but small enough such that the prob-

[34]The argument here is the same as the technique used in Strausz (2006). Strausz shows that in a standard screening problem, if the solution to the relaxed problem that ignores global incentive-compatibility is indeed incentive-compatible, the use of stochastic contracts is always suboptimal. In standard screening problem, when the solution to the relaxed problem is incentive-compatible, this implies that there is no contract pooling in the solution, but this property does not hold here.

abilities are still in $[0,1]$, and they satisfy

$$
\begin{aligned}
& \tilde{\psi}^{S+B}\left(\alpha^*_{\lambda_1+\varepsilon}, \lambda_1+\varepsilon\right) f\left(\lambda_1+\varepsilon\right) + \tilde{\psi}^{S+B}\left(\alpha^*_{\lambda_2-\varepsilon}, \lambda_2-\varepsilon\right) f\left(\lambda_2-\varepsilon\right) \\
& = \tilde{\psi}^{S+B}\left(\hat{\alpha}_{\lambda_1+\varepsilon}, \lambda_1+\varepsilon\right) f\left(\lambda_1+\varepsilon\right) + \tilde{\psi}^{S+B}\left(\hat{\alpha}_{\lambda_2-\varepsilon}, \lambda_2-\varepsilon\right) f\left(\lambda_2-\varepsilon\right) \\
\iff \quad & \eta_{1(\varepsilon)} f\left(\lambda_1+\varepsilon\right)\left(1-\left(\lambda_1+\varepsilon\right)\right) = \eta_{2(\varepsilon)} f\left(\lambda_2-\varepsilon\right)\left(1-\left(\lambda_2-\varepsilon\right)\right) \\
\iff \quad & \tfrac{\eta_{1(\varepsilon)}}{\eta_{2(\varepsilon)}} = \tfrac{f(\lambda_2-\varepsilon)(1-(\lambda_2-\varepsilon))}{f(\lambda_1+\varepsilon)(1-(\lambda_1+\varepsilon))}
\end{aligned} \tag{33}
$$

Observe that

$$
\begin{aligned}
& \int_{\lambda_1}^{\lambda_2} \tilde{\psi}^{S+B}\left(\hat{\alpha}_\lambda, \lambda\right) f\left(\lambda\right) d\lambda \\
= & \int_{\lambda_1}^{\lambda_1+\frac{\lambda_2-\lambda_1}{2}} \tilde{\psi}^{S+B}\left(\hat{\alpha}_\lambda, \lambda\right) f\left(\lambda\right) d\lambda + \int_{\lambda_2-\frac{\lambda_2-\lambda_1}{2}}^{\lambda_2} \tilde{\psi}^{S+B}\left(\hat{\alpha}_\lambda, \lambda\right) f\left(\lambda\right) d\lambda \\
= & \int_0^{\frac{\lambda_2-\lambda_1}{2}} \tilde{\psi}^{S+B}\left(\hat{\alpha}_{\lambda_1+\varepsilon}, \lambda_1+\varepsilon\right) f\left(\lambda_1+\varepsilon\right) + \tilde{\psi}^{S+B}\left(\hat{\alpha}_{\lambda_2-\varepsilon}, \lambda_2-\varepsilon\right) f\left(\lambda_2-\varepsilon\right) d\varepsilon \\
= & \int_0^{\frac{\lambda_2-\lambda_1}{2}} \tilde{\psi}^{S+B}\left(\alpha^*_{\lambda_1+\varepsilon}, \lambda_1+\varepsilon\right) f\left(\lambda_1+\varepsilon\right) + \tilde{\psi}^{S+B}\left(\alpha^*_{\lambda_2-\varepsilon}, \lambda_2-\varepsilon\right) f\left(\lambda_2-\varepsilon\right) d\varepsilon \\
= & \int_{\lambda_1}^{\lambda_2} \tilde{\psi}^{S+B}\left(\alpha^*_\lambda, \lambda\right) f\left(\lambda\right) d\lambda.
\end{aligned}
$$

This implies that $\{\alpha^*_\lambda\}_\lambda$ and $\{\hat{\alpha}_\lambda\}_\lambda$ achieve the same objective value. Observe that for each $\varepsilon \in \left(0, \frac{\lambda_2-\lambda_1}{2}\right)$,

$$
\begin{aligned}
& \left[\tilde{\psi}^{B}\left(\hat{\alpha}_{\lambda_1+\varepsilon}, \lambda_1+\varepsilon\right) f\left(\lambda_1+\varepsilon\right) + \tilde{\psi}^{B}\left(\hat{\alpha}_{\lambda_2-\varepsilon}, \lambda_2-\varepsilon\right) f\left(\lambda_2-\varepsilon\right)\right] \\
& - \left[\tilde{\psi}^{B}\left(\alpha^*_{\lambda_1+\varepsilon}, \lambda_1+\varepsilon\right) f\left(\lambda_1+\varepsilon\right) + \tilde{\psi}^{B}\left(\alpha^*_{\lambda_2-\varepsilon}, \lambda_2-\varepsilon\right) f\left(\lambda_2-\varepsilon\right)\right] \\
= & \ \left(c_H - c_L\right)\Bigg[-\hat{\alpha}^\Delta_{\lambda_1+\varepsilon}(1)\left[1-F\left(\lambda_1+\varepsilon\right)\right] - \hat{\alpha}^\Delta_{\lambda_2-\varepsilon}(1)\left[1-F\left(\lambda_2-\varepsilon\right)\right] \\
& + \alpha^{*\Delta}_{\lambda_1+\varepsilon}(1)\left[1-F\left(\lambda_1+\varepsilon\right)\right] + \alpha^{*\Delta}_{\lambda_2-\varepsilon}(1)\left[1-F\left(\lambda_2-\varepsilon\right)\right]\Bigg] \\
\propto & \ \eta_{1(\varepsilon)}\left[1-F\left(\lambda_1+\varepsilon\right)\right] - \eta_{2(\varepsilon)}\left[1-F\left(\lambda_2-\varepsilon\right)\right] \\
\propto & \ \frac{\eta_{1(\varepsilon)}\left[1-F\left(\lambda_1+\varepsilon\right)\right]}{\eta_{2(\varepsilon)}\left[1-F\left(\lambda_2-\varepsilon\right)\right]} - 1 \\
= & \ \frac{\left[1-F\left(\lambda_1+\varepsilon\right)\right]/\left[f\left(\lambda_1+\varepsilon\right)\left(1-\left(\lambda_1+\varepsilon\right)\right)\right]}{\left[1-F\left(\lambda_2-\varepsilon\right)\right]/f\left(\lambda_2-\varepsilon\right)\left(1-\left(\lambda_2-\varepsilon\right)\right)} - 1 \ > \ 0
\end{aligned}
$$

where the last line follows from substituting in equation (33). The last inequality is because $\lambda_1 + \varepsilon < \lambda_2 - \varepsilon$ for all $\varepsilon \in \left(0, \frac{\lambda_2 - \lambda_1}{2}\right)$; hence, under Assumption 5, $\frac{1-F(\lambda_1+\varepsilon)}{f(\lambda_1+\varepsilon)(1-(\lambda_1+\varepsilon))} > \frac{1-F(\lambda_2-\varepsilon)}{f(\lambda_2-\varepsilon)(1-(\lambda_2-\varepsilon))}$. This implies that $\int_{\lambda_1}^{\lambda_2} \tilde{\psi}^B(\hat{\alpha}_\lambda, \lambda) f(\lambda) d\lambda > \int_{\lambda_1}^{\lambda_2} \tilde{\psi}^B(\hat{\alpha}_\lambda, \lambda) f(\lambda) d\lambda$, which means that $\int_0^1 \tilde{\psi}^B(\hat{\alpha}_\lambda, \lambda) f(\lambda) d\lambda > 0$. This is a contradiction to $\{\alpha^*_\lambda\}_\lambda$ being optimal because, as noted above, $\{\hat{\alpha}_\lambda\}_\lambda$ achieves the same objective value as $\{\alpha^*_\lambda\}_\lambda$; however, constraint ($\tilde{IR}_B$) is slack under $\{\hat{\alpha}_\lambda\}_\lambda$, and this implies that we can increase $\hat{\alpha}^\Delta_\lambda(1)$ slightly at the expense of $\hat{\alpha}^\Delta_\lambda(\lambda)$ for some positive measure subset of λs without violating constraint ($\tilde{IR}_B$), and this will strictly increase the objective value. Therefore, we have established that for any $\lambda < \hat{\lambda}$, either $\alpha^{*\Delta}_\lambda(\lambda) = 1$ or $\alpha^{*\Delta}_\lambda(1) = 1$.

Finally we claim that there exists $\hat{\lambda}' < \hat{\lambda}$ such that for all $\lambda < \hat{\lambda}'$, $\alpha^{*\Delta}_\lambda(\lambda) = 1$; and for all $\lambda > \hat{\lambda}'$, $\alpha^{*\Delta}_\lambda(1) = 1$. If this claim is true, then $\{\alpha^*_\lambda\}_\lambda$ is equivalent to the (deterministic) threshold-$\hat{\lambda}'$ schedule, which hence satisfies constraint ($\tilde{IC}_S$). To prove the claim, suppose that it is not true. Then there must exist $[\lambda_1, \lambda_2]$, with $\lambda_2 < \hat{\lambda}$, such that for all $\lambda \in \left(\lambda_1, \frac{\lambda_1+\lambda_2}{2}\right)$, $\alpha^{*\Delta}_\lambda(1) = 1$; and for all $\lambda \in \left(\frac{\lambda_1+\lambda_2}{2}, \lambda_2\right)$, $\alpha^{*\Delta}_\lambda(\lambda) = 1$. We can construct $\{\hat{\alpha}_\lambda\}_\lambda$ exactly as above.[35] By the same argument, we get a contradiction with the optimality of $\{\alpha^*_\lambda\}_\lambda$. □

[35] In fact, $\eta_{1(\varepsilon)}$ and $\eta_{2(\varepsilon)}$ can be any value in $(0, 1]$ now.