Cellular Decomposition and Classification of a Hybrid System

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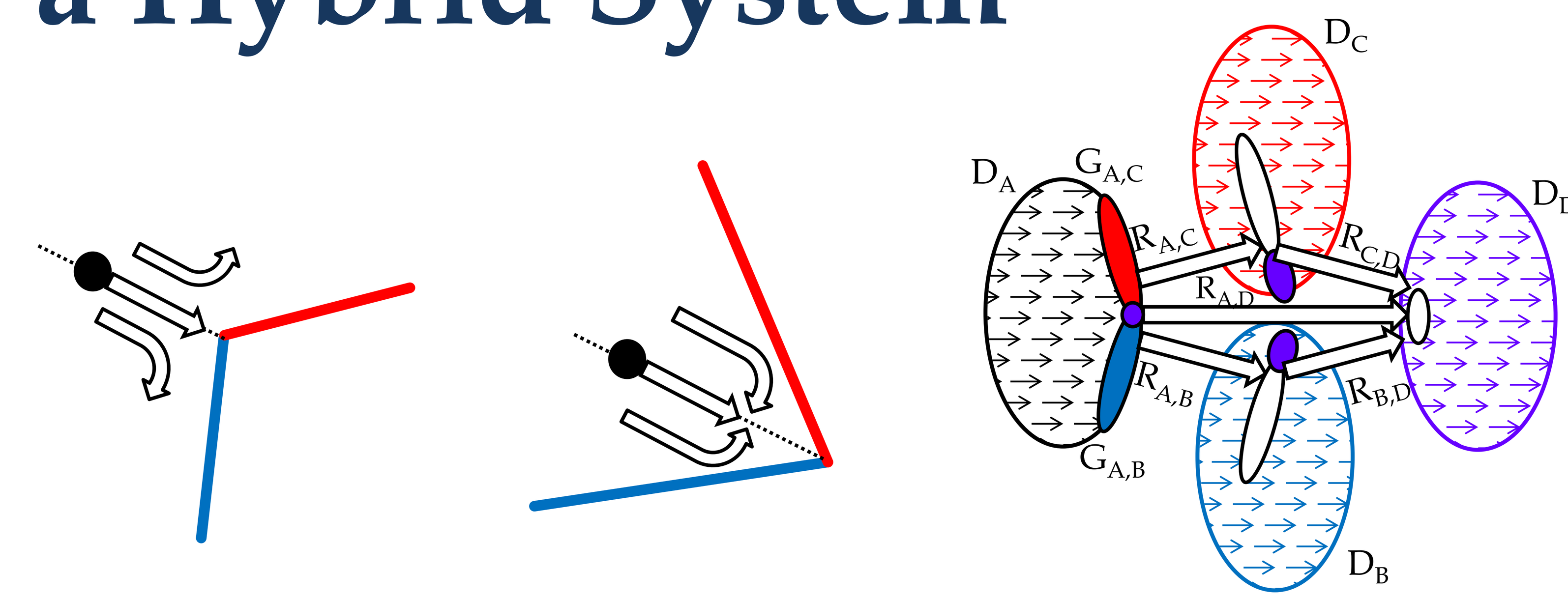
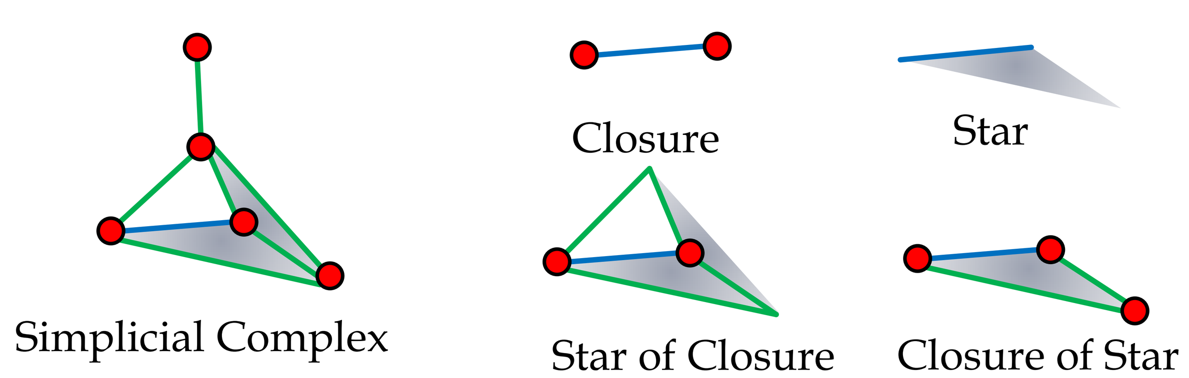


Abstract

Robots are often modeled as hybrid systems providing a consistent, formal account of the varied dynamics associated with the loss and gain of kinematic freedom as a machine impacts and breaks away from its environment [1]. This hybrid structure induces an abstract simplicial complex indexed by the active contact constraints, where each vertex in the complex is a single constraint. This complex provides a concise description of the possible edges of the hybrid system through impacts – they must lie in the closure of the star of the current cell (i.e. $(I,J) \in \Gamma \Leftrightarrow J \in \mathrm{Cl\,St}\, I$). This structure is in some sense dual to the "ground reaction complex", [2], wherein constraints reduce dimension and the equivalent adjacency property is instead the star of the closure (i.e.$(I,J) \in \Gamma \Leftrightarrow J \in \mathrm{St\,Cl}\, I$). Under either formulation, sequences of contact conditions ("letters") define smooth families of executions ("words"). Points of discontinuity lie within the boundaries between words, but in certain cases the evaluation can still be continuous over an open set including these boundaries, even though the associated words change abruptly. We present examples of these "convergent" and "divergent" word boundaries.



Cell Complex Review [3]

- •A *simplex* is the set of points defined by the convex hull of a set of *vertices*.
- •A *face* of a simplex is a simplex generated by a subset of its vertices.
- •A *simplicial complex* is a finite collection of simplices that contains the face of any simplex in the complex, as well as the intersection of any two simplices in the complex.
- •The *empty set* is a member of any simplicial complex.
- •An *abstract simplicial complex* is a finite collection of sets such that the subset of any member set is also a member.
- •The *closure* of a simplex is the smallest sub-complex containing that simplex.
- •The *star* of a simplex is the collection of all simplexes that have that simplex as a face.

Hybrid Systems Review [1]

Definition 2. *A C^r hybrid dynamical system, $r \in \mathbb{N} \cup \{\infty, \omega\}$, is a tuple $\mathcal{H} := (\mathcal{J}, \Gamma, \mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{R})$, where the constituent parts are defined as:*

1) $\mathcal{J} = \{I, J, \ldots, K\}$ *is the finite set of* discrete modes;
2) $\Gamma \subset \mathcal{J} \times \mathcal{J}$ *is the set of* discrete transitions, *forming a directed graph structure over* $\mathcal{J}$;
3) $\mathcal{D} = \coprod_{I \in \mathcal{J}} D_I$ *is the collection of* domains, *where D_I is a C^r manifold with corners that can be C^r embedded in $\mathbb{R}^{n_I}$, some $n_I \in \mathbb{N}$, to obtain a closed submanifold;*
4) $\mathcal{F} : \mathcal{D} \to T\mathcal{D}$ *is a C^r hybrid map that restricts to a vector field $F_I = \mathcal{F}|_{D_I}$ for each $I \in \mathcal{J}$;*
5) $\mathcal{G} = \coprod_{(I,J) \in \Gamma} G_{I,J}$ *is the set of* guards, *where the components $G_{I,J} \subset D_I$ are each C^r manifolds with corners;*
6) $\mathcal{R} : \mathcal{G} \to \mathcal{D}$ *is a continuous map called the* reset *that restricts as $R_{I,J} = \mathcal{R}|_{G_{I,J}} : G_{I,J} \to D_J$ for each $(I,J) \in \Gamma$.*

Definition 6. *A self-manipulation hybrid system is defined as follows,*

$$\mathcal{J} = \left\{ I \in 2^{\mathcal{K}} : \mathbf{a}_{I_n}^{-1}(0) \neq \varnothing \wedge \alpha(I_t) \subset I \right\},$$

$$\Gamma = \{(I,J) \in \mathcal{J} \times \mathcal{J} : I \cup J \in \mathcal{J}\}$$

$$D_I = \{(\dot{\mathbf{q}}, \mathbf{q}) \in (T\mathcal{Q}) : \mathbf{a}_{I_n}(\mathbf{q}) = 0, \mathbf{a}_{\mathcal{K}_n}(\mathbf{q}) \geq 0, \mathbf{A}_I(\mathbf{q})\dot{\mathbf{q}} = 0\},$$

$$F_I(\dot{\mathbf{q}}, \mathbf{q}) = \left[\overline{\mathbf{M}}^{\dagger} (\Upsilon - \overline{\mathbf{C}}\dot{\mathbf{q}} - \overline{\mathbf{N}}) - \mathbf{A}_I^{\dagger T} \dot{\mathbf{A}}_I \dot{\mathbf{q}}, \quad \dot{\mathbf{q}} \right]$$

$$G_{I,J} = \Big\{ (\dot{\mathbf{q}}, \mathbf{q}) \in D_I : \mathrm{NTD}(I,J,\mathbf{q}),\ J_n \subseteq I_n \Rightarrow \mathrm{FFA}(I,J,\mathbf{Tq}),\ J_n \nsubseteq I_n \Rightarrow \mathrm{PIV}(I,J,\mathbf{Tq}) \Big\}$$

$$R_{I,J}(\dot{\mathbf{q}}, \mathbf{q}) = [\dot{\mathbf{q}} - \Delta_{\dot{\mathbf{q}}},\ \mathbf{q}]$$

$$\mathrm{NTD}(I,J,\mathbf{q}) := \bigwedge_{k \in \mathcal{K}_n \setminus I} (k \in J) \Leftrightarrow \mathrm{TD}(k, \mathbf{q}),$$

$$\mathrm{TD}(k,\mathbf{q}) := \mathbf{a}_k(\mathbf{q}) = 0 \wedge \dot{\mathbf{a}}_k(\mathbf{q}) \preceq 0,$$

$$\mathrm{FFA}(I,J,\mathbf{Tq}) := \bigwedge_{k \in \mathcal{I}_{\mathrm{FFA}}} (k \in J) \Leftrightarrow (\mathbf{A}_k \dot{\mathbf{q}} = 0 \wedge \mathbf{U}_k(\lambda_{J \cup \{k\}}) \succeq 0)$$

$$\mathrm{PIV}(I,J,\mathbf{Tq}) := \bigwedge_{k \in \mathcal{I}_{\mathrm{IV}}} (k \in J) \Leftrightarrow (\mathbf{U}_k(\widehat{\mathbf{P}}_K) \geq 0 \vee \mathbf{U}_k(\widehat{\mathbf{P}}_K + \widetilde{\mathbf{P}}_K) \geq 0)$$

Cellular Decomposition of Mechanical Systems

Define two structures on this hybrid system:

1. The *constraint complex*, an abstract simplicial complex whose vertices are individual normal-direction constraints and whose simplices are the discrete modes.
2. The *configuration complex*, previously called the *ground reaction complex* [2], whose vertices are the maximally constrained configurations and whose simplices' dimension and adjacency are related to that of the hybrid domains through some quotient map.

The constraint complex is well defined,

Theorem *The set $\mathcal{J}_n = \{J \cap \mathcal{K}_n \,|\, J \in \mathcal{J}\}$ forms an abstract simplicial complex over the universal set $\mathcal{K}_n$.*

Proof. First note that the set $\mathcal{J}_n$ is a collection of finite subsets of $\mathcal{K}_n$. Given $I \in \mathcal{J}_n$ we have that $\exists \mathbf{q} \in \mathcal{Q} : \forall i \in I, \mathbf{a}_i(\mathbf{q}) = 0$, and so for every subset $J \subset I$ it must be the case that $\mathbf{a}_j(\mathbf{q}) = 0$ for all $j \in J$ and hence $J \in \mathcal{J}_n$. □

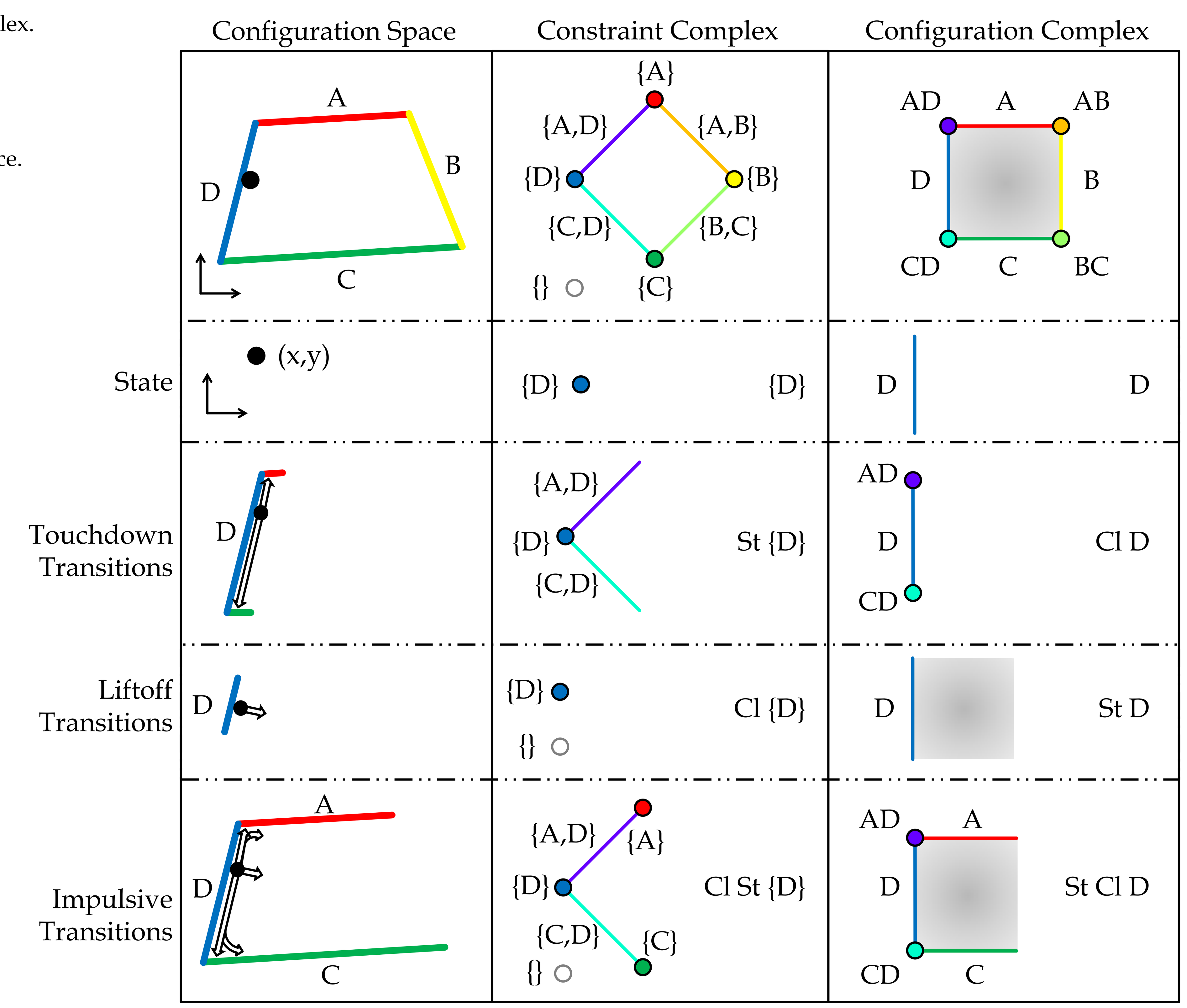
while the configuration complex is only a simplicial complex for certain systems. The set of edges has a particular structure in the constraint complex,

Theorem *Given the simplicial complex $\mathcal{J}_n$, define equivalently the set of edges $\Gamma_n = \{(I,J) \in \mathcal{J}_n \times \mathcal{J}_n \,|\, I \cup J \in \mathcal{J}_n\}$. Then for every $J \in \mathcal{J}_n$ in the closure of the star of $I \in \mathcal{J}_n$, the pair (I,J) is in Γ (i.e. $(I,J) \in \Gamma \Leftrightarrow J \in \mathrm{Cl\,St}\, I$)*

Proof. The simplex $I \cup J$, if it exists in the complex $\mathcal{J}_n$, has I as a face and therefore is in the star of I. But simplex J is also a face of $I \cup J$ and therefore is in its closure. □

When well defined, the equivalent condition on the configuration complex is ,

$$(I,J) \in \Gamma \Leftrightarrow J \in \mathrm{St\,Cl}\, I$$



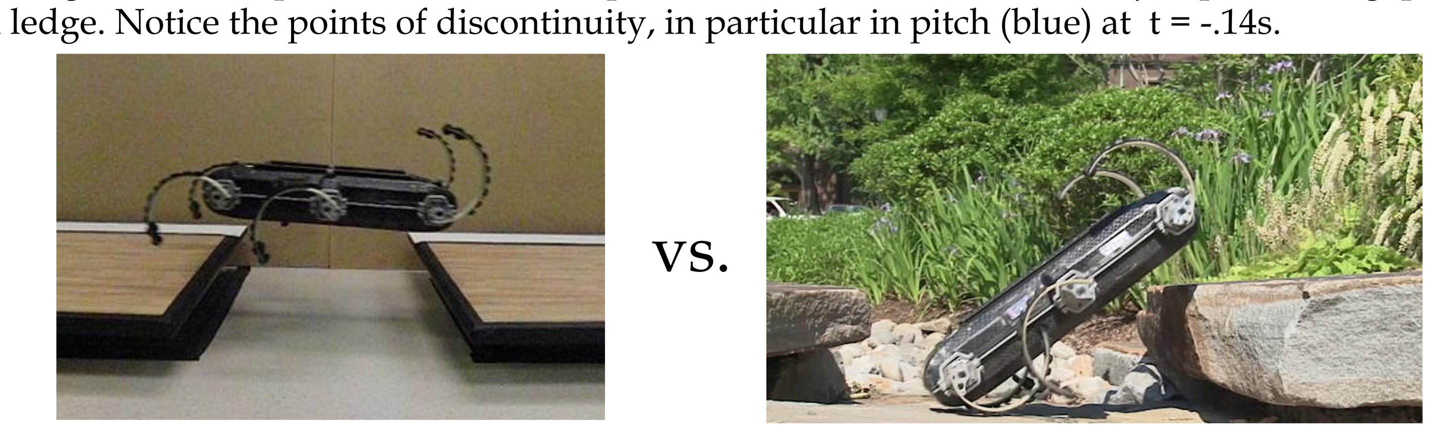
Classification of Convergence

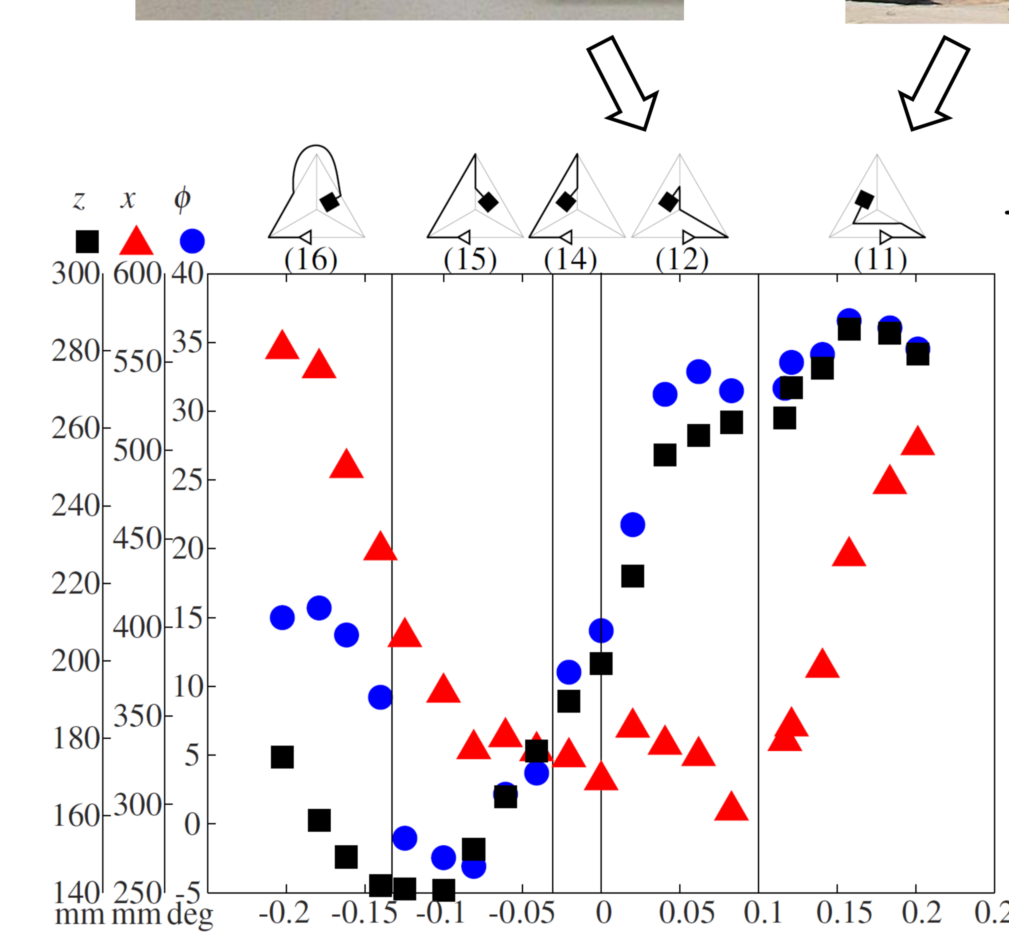
An execution of the hybrid system follows a sequence of contact modes. If we consider each to be a *letter* then the sequence of letters in an execution could be called a *word*, and the set of possible words a *vocabulary* [2]. We can now consider the *grammar* of this system, i.e. the relationship between the words.

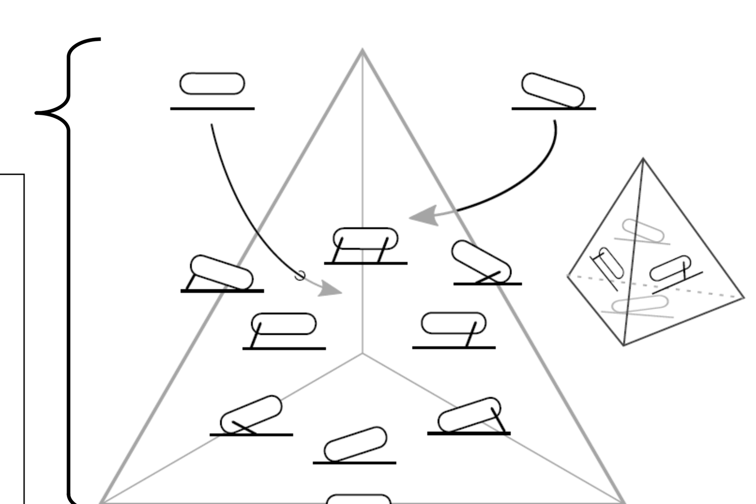
In general the boundary between words will be *divergent* points of discontinuity in the execution – any neighborhood of such points will contain points that reach formally disjoint domains (above, left). However, when the two words differ only by a letter and the guard manifolds that correspond to those letters intersect at an acute angle, the two words will *converge* (above, middle). In such cases the execution is continuous, although not smooth. In the cartoon hybrid system (above, right), there exists a neighborhood of the single point $G_{A,D}$ for which the execution follows words ABD, AD, and ACD, but the result has continuous dependence on initial condition.

Example on RHex [2]

This figure presents the apex result of executing a maximal torque forward leap on a legged robot. The data has been hand labeled with a pictogram representing the word, or path through the configuration complex. The words are quite different and can be used to jump across a gap or onto a ledge. Notice the points of discontinuity, in particular in pitch (blue) at t = -.14s.







Configuration complex for a planar, 2 leg, rigid RHex

Conclusion and Future Work

This work has only begun to explore the topological properties of these cellular decompositions and the grammatical rules for such mechanical hybrid systems. In particular, it will be important to work out the conditions under which the configuration complex is a proper cellular or simplicial complex. Furthermore, this poster presents simple examples and the scalability and computational tractability of these ideas has not been tested. We believe that this should be possible and that grammatical rules like the convergence property will aid in the design and analysis of closed loop behaviors. Finally, we believe that this work will lead to a better general understanding of dynamic transitions for manipulation and self-manipulation systems.

Acknowledgments

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[1] A. M. Johnson, S. A. Burden, and D. E. Koditschek, "A hybrid systems model for simple manipulation and self-manipulation systems," 2014, in prep.
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