Impacts

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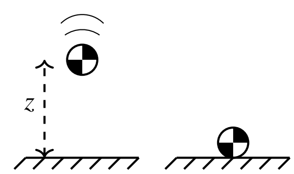
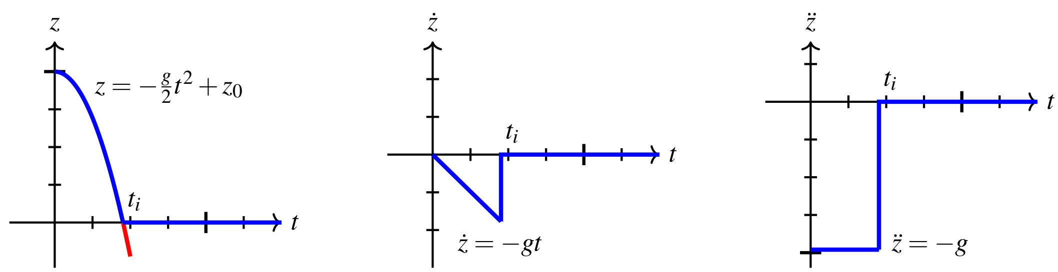
1 Plastic Impact Example

Consider a ball impacting a surface with a plastic impact assumption. Recall that plastic, or inelastic, impact means that there is no recoil, i.e. the ball does not bounce. It will fall towards the surface and then simply come to rest. While in freefall, assuming no other forces, it will fall at a constant gravitational acceleration. When it reaches the ground it will instantaneously stop (zero velocity) and, since it is now in static equilibrium, have no further velocity or acceleration. Lets look closer at this impact process.





When does impact occur? Impact happens when the ball touches the ground, so when the height z reaches zero. We can solve for the impact time t_i by integrating the dynamics before impact and then solving the equation $z(t_i) = 0$. Lets assume the ball starts at a height $z(0) = 1$ with zero velocity.

$$\ddot{z}(t) = -g \tag{1}$$

$$\dot{z}(t) = \int_{\tau=0}^{t} \ddot{z}(\tau)d\tau = -gt + c_1 = -gt \qquad (c_1 = 0 \text{ because } \dot{z}(0) = 0) \tag{2}$$

$$z(t) = \int_{\tau=0}^{t} \dot{z}(\tau)d\tau = -\frac{g}{2}t^2 + c_2 = -\frac{g}{2}t^2 + 1 \qquad (c_2 = 1 \text{ because } z(0) = 1) \tag{3}$$

$$z(t_i) = 0 \Rightarrow -\frac{g}{2}t_i^2 + 1 = 0 \Rightarrow t_i = \sqrt{\frac{2}{g}} \approx 0.4518 \qquad (\text{assuming } g = 9.8) \tag{4}$$

What happens at the time of impact (t_i)? If nothing happened, the ball would continue to fall through the ground (shown in red in the graph above). Recall from physics that the impact will induce an impulsive force, or impulse, P that stops the ball from going through the ground. We can calculate the impulse by integrating the forces on the body during the impact process. The dynamics of the system in contact are $m\ddot{z} - mg + \lambda = 0$, where λ is the contact force. In order to reason about the forces during the impact process we will call the start of the impact t_i^- and the end of impact t_i^+. Integrating, we get,

$$\int_{t_i^-}^{t_i^+} m\ddot{z}(\tau) - mg + \lambda(\tau)d\tau = m\dot{z}(t_i^+) - m\dot{z}(t_i^-) - mg(t_i^+ - t_i^-) + \int_{t_i^-}^{t_i^+} \lambda(\tau)d\tau = 0 \tag{5}$$

The impulse P is defined as the integrated contact force. For notational simplicity, we will also use $\dot{z}^-$ and $\dot{z}^+$ to represent the velocity at the start and end of the impact process. Now, let us assume that the impact process happens instantaneously, so that $t_i^- = t_i^+$. Plugging in, we get,

$$P := \int_{t_i^-}^{t_i^+} \lambda(\tau)d\tau, \qquad \dot{z}^- := \dot{z}(t_i^-), \qquad \dot{z}^+ := \dot{z}(t_i^+) \tag{6}$$

$$\lim_{t_i^+ \to t_i^-} \left(m\dot{z}^+ - m\dot{z}^- - mg(t_i^+ - t_i^-) + P \right) = m\dot{z}^+ - m\dot{z}^- + P = 0 \tag{7}$$

Finally, to solve for P we need to use one other piece of information: we know that at the end of the impact the velocity is zero, $\dot{z}^+ = 0$. With this, we can solve for the impulse $P = m\dot{z}^-$.

2 General Impact Process

The previous example is quite simple (1D configuration space, a single constraint, no applied forces, etc). However, the analysis of the impact process can be easily generalized.

For a general impact, let the configuration space be $q \in \mathcal{Q} \sim \mathbb{R}^n$. Instead of impact occurring at $z = 0$ we will define the generalized scalar distance to impact as $a(q)$, such that $a(q) > 0$ before impact and $a(q) = 0$ at impact.

$$a(q) > 0 \quad \Rightarrow \quad \text{Not in contact} \tag{8}$$
$$a(q) = 0 \quad \Rightarrow \quad \text{In contact} \tag{9}$$
$$a(q) < 0 \quad \Rightarrow \quad \text{Penetrating (not allowed)} \tag{10}$$

While any smooth function that meets these requirements will work as a constraint function, it is more useful to define $a(q)$ as a distance in the same units as other distances (e.g. meters) so that forces and impulses are in the correct units. In the single ball example, $a(q) = z$ was the height of the ball. For a single constraint, $a(q) : \mathcal{Q} \to \mathbb{R}$ maps configurations q to a scalar distance. If we consider m constraints we will stack them into a column vector so that $a(q) : \mathcal{Q} \to \mathcal{C}$ for the space of constraint distances $\mathcal{C} \sim \mathbb{R}^m$.

Similarly, if we want to talk about the relative velocity at the time of impact, we can differentiate this constraint to get $A(q)\dot{q} := \frac{d}{dt}a(q)$, where $A(q)$ is a row vector for a single constraint (and will generalize for m constraints to $A : T\mathcal{Q} \to T\mathcal{C}$ as an $n \times m$ matrix). Note that while the velocity constraint depends on the configuration q, we may write simply A instead of $A(q)$ when the configuration is clear from context. In the single ball example, $A\dot{q} = [1]\dot{z} = \dot{z}$. Based on the sign convention for $a(q)$ we similarly see that our constraint velocity,

$$A\dot{q} > 0 \quad \Rightarrow \quad \text{Separating constraint} \tag{11}$$
$$A\dot{q} = 0 \quad \Rightarrow \quad \text{Maintaining constraint} \tag{12}$$
$$A\dot{q} < 0 \quad \Rightarrow \quad \text{Approaching constraint} \tag{13}$$

With this more general definition of a constraint, we can look back at our example and work through the same derivation.

When does impact occur? Impact happens when $a(q)$ goes to zero. This means both that the distance takes a value of zero and that the velocity was approaching and not separating, as liftoff starts at a distance of zero and separates (we will look more at these velocity constraints in the complementarity section). This time can be found by solving the *impact time equation* (generalizing (4)),

$$\boxed{\text{Impact occurs at time } t_i \text{ such that } a(q(t_i)) = 0 \text{ and } A\dot{q}(t_i) < 0.} \tag{14}$$

If we have a closed form expression for $q(t)$ we can solve this equation exactly. More commonly, we will integrate the dynamics numerically to find $q(t)$ and will use a numerical root finding algorithm to solve this equation, as discussed in the collision detection chapter.

What happens at the time of impact (t_i)**?** As with the simple example, if nothing happened at time t_i when $a(q(t_i)) = 0$ and $\frac{d}{dt}a(q(t_i)) = A\dot{q}(t_i) < 0$ then by smoothness we would conclude that shortly, at a time $t_i + \varepsilon$, it must be the case that $a(q(t_i + \varepsilon)) < 0$, which is penetrating the contact surface and not allowed. As we did in the simple example, we will calculate the impulse by integrating the forces on the system during the impact process. Again we will call time at the start of impact t_i^- and the time at the end of impact t_i^+, and similarly $\dot{q}^-$ and $\dot{q}^+$ are the velocity at the start and end of impact, respectively. For our general system, the dynamics in contact are given by $M\ddot{q} + C\dot{q} + N + A^T\lambda - \Upsilon = 0$. Integrating and taking the limit as the impact duration goes to zero we get,

$$\lim_{t_i^+ \to t_i^-} \int_{t_i^-}^{t_i^+} M\ddot{q}(\tau) + C\dot{q}(\tau) + N(\tau) + A^T\lambda(\tau) - \Upsilon(\tau) d\tau = 0 \tag{15}$$

When we take this limit, only forces that can go to infinity (impulses) will remain, and so the contributions of C, N, and Υ drop out. Integrating the force term to get our impulse $P \in T^*\mathcal{Q} \sim \mathbb{R}^n$ on the system coordinates (sometimes called the "body impulse") in terms of the impulse in the contact coordinates $\hat{P} \in T^*\mathcal{C} \sim \mathbb{R}^m$ (or "contact impulse") as follows,

$$P := A^T\hat{P} \qquad \hat{P} := \lim_{t_i^+ \to t_i^-} \int_{t_i^-}^{t_i^+} \lambda(\tau) d\tau \tag{16}$$

(Note that since $A^\dagger A^T = I$, we can also write $A^\dagger P = \hat{P}$). With this notation, we can solve (15) to get (similar to (7)),

$$\boxed{M\dot{q}^+ - M\dot{q}^- + A^T\hat{P} = 0} \qquad \text{Impulse-Momentum Equation} \tag{17}$$

This is called our *impulse-momentum equation*, as it shows the effect of the impulse (P) on the momentum ($M\dot{q}$) of the system. Equivalently, it may be more intuitive to write $M\dot{q}^+ = M\dot{q}^- - A^T\hat{P}$, i.e. the outgoing momentum is equal to the incoming momentum minus the impulse loss.

This impulse-momentum equation holds for any impact law, as we have not yet used the properties of plastic or elastic impact. The particular impact assumptions introduce additional constraints.

3 Plastic Impact

Plastic, or inelastic, impact is an assumption about the net result over the infinitesimal impact process, specifically that all impacting velocities are zeroed out. With our general constraint velocity notation we can express the plastic impact law as simply,

$$\boxed{A\dot{q}^+ = 0} \qquad \text{Plastic Impact Law} \tag{18}$$

i.e. the velocity in the constrained directions after impact is zero. To determine the velocity of the rest of the system (not just in the constrained directions) we need to solve the plastic impact law and the impulse-momentum equation,

$$\text{Given } q \text{ and } \dot{q}^-, \text{ solve} \quad \begin{cases} M\dot{q}^+ - M\dot{q}^- + A^T\hat{P} = 0 \\ A\dot{q}^+ = 0 \end{cases} \qquad \text{for } \dot{q}^+ \text{ and } \hat{P}. \tag{19}$$

To do this by hand we might write out all of the equations and substitute to find an expression for each element in $\dot{q}^+$ and $\hat{P}$. Computationally, it is easier to use our block matrix solution strategy,

$$\begin{bmatrix} M & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \dot{q}^+ \\ \hat{P} \end{bmatrix} = \begin{bmatrix} M\dot{q}^- \\ 0 \end{bmatrix} \tag{20}$$

$$\begin{bmatrix} \dot{q}^+ \\ \hat{P} \end{bmatrix} = \begin{bmatrix} M & A^T \\ A & 0 \end{bmatrix}^{-1} \begin{bmatrix} M\dot{q}^- \\ 0 \end{bmatrix} = \begin{bmatrix} M^\dagger & A^{\dagger T} \\ A^\dagger & \Lambda^\dagger \end{bmatrix} \begin{bmatrix} M\dot{q}^- \\ 0 \end{bmatrix} \tag{21}$$

Looking at the top row of the solution to find the velocity $\dot{q}^+$, and recalling the identity $M^\dagger M + A^{\dagger T} A = I$,

$$\dot{q}^+ = M^\dagger M \dot{q}^- = (I - A^{\dagger T} A)\dot{q}^- \tag{22}$$

$$\boxed{\dot{q}^+ = \dot{q}^- - A^{\dagger T} A \dot{q}^-} \qquad \text{Plastic Impact Velocity} \tag{23}$$

That is, our new velocity ($\dot{q}^+$) is equal to our old velocity ($\dot{q}^-$) minus the impacting constraint velocity ($A\dot{q}^-$) mapped through $A^{\dagger T}$ (which takes constraint velocities to coordinate velocities).

The contact impulse $\hat{P}$ can similarly be computed from the bottom row of (21), recalling the identity $A^\dagger M + \Lambda^\dagger A = 0$,

$$\boxed{\hat{P} = A^\dagger M \dot{q}^- = -\Lambda^\dagger A \dot{q}^-} \qquad \text{Plastic Impact Impulse} \tag{24}$$

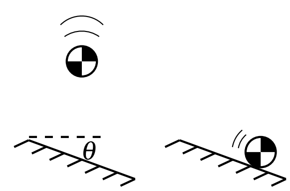
Or, if M is invertible, we can equivalently write $\hat{P} = (AM^{-1}A)^{-1}A\dot{q}^-$.

The body impulse P can be derived from (17) and (23) or (16) and (24), recalling that $MA^{\dagger T} + A^T \Lambda^\dagger = 0$

$$P = -M(\dot{q}^+ - \dot{q}^-) = MA^{\dagger T} A \dot{q}^- = -A^T \Lambda^\dagger A \dot{q}^- = A^T \hat{P} \tag{25}$$

4 2D Plastic Impact Example

Consider now a ball of mass m impacting a sloped ground in the $x-z$ plane. Assume the ball is falling straight down, so $\dot{x}^- = 0$. The ground constraint passes through the origin and slopes down by an angle θ, and is defined by the constraint,



$$a(q) = x\sin(\theta) + z\cos(\theta) \geq 0 \tag{26}$$

The resulting velocity constraint when in contact with the ground is thus,

$$A(q)\dot{q} = \begin{bmatrix} \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = 0 \tag{27}$$

To solve the plastic impact problem, (19), we can set up the set of equations,

$$\text{Given } x, z \text{ and } \dot{x}^-, \dot{z}^-, \text{ solve} \quad \begin{cases} m\dot{x}^+ - m\dot{x}^- + \sin(\theta)\hat{P} = 0 \\ m\dot{z}^+ - m\dot{z}^- + \cos(\theta)\hat{P} = 0 \\ \sin(\theta)\dot{x}^+ + \cos(\theta)\dot{z}^+ = 0 \end{cases} \quad \text{for } \dot{x}^+, \dot{z}^+ \text{ and } \hat{P}. \tag{28}$$

There are many ways to solve this system of equations, first we will substitute the last equation into the middle equation and then the top equation, (and plugging in $\dot{x}^- = 0$),

$$\dot{z}^+ = -\tan(\theta)\dot{x}^+ \tag{29}$$

$$-m\tan(\theta)\dot{x}^+ - m\dot{z}^- + \cos(\theta)\hat{P} = 0 \qquad \Rightarrow \qquad \hat{P} = \frac{m\tan(\theta)\dot{x}^+ + m\dot{z}^-}{\cos(\theta)} \tag{30}$$

$$m\dot{x}^+ - m\dot{x}^- + \tan(\theta)(m\tan(\theta)\dot{x}^+ + m\dot{z}^-) = 0 \quad \Rightarrow \quad \dot{x}^+ = \frac{0 - \tan(\theta)\dot{z}^-}{1 + \tan^2(\theta)} = -\sin(\theta)\cos(\theta)\dot{z}^- \tag{31}$$

$$\dot{z}^+ = \sin^2(\theta)\dot{z}^-, \qquad \hat{P} = m\frac{\sin(\theta)\cos(\theta)\dot{x}^- - \sin^2(\theta)\dot{z}^- + \dot{z}^-}{\cos(\theta)} = m\cos(\theta)\dot{z}^- \tag{32}$$

As we can see, solving (19) by hand can get messy quickly. So instead lets try the block matrix solution, (21),

$$\underbrace{\begin{bmatrix} \dot{x}^+ \\ \dot{z}^+ \\ \hat{P} \end{bmatrix}}_{\begin{bmatrix} \dot{q}^+ \\ \hat{P} \end{bmatrix}} = \underbrace{\begin{bmatrix} m & 0 & \sin(\theta) \\ 0 & m & \cos(\theta) \\ \sin(\theta) & \cos(\theta) & 0 \end{bmatrix}^{-1}}_{\begin{bmatrix} M & A^T \\ A & 0 \end{bmatrix}^{-1}} \underbrace{\begin{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \dot{x}^- \\ \dot{z}^- \end{bmatrix} \\ 0 \end{bmatrix}}_{\begin{bmatrix} M\dot{q}^- \\ 0 \end{bmatrix}} \tag{33}$$

$$= \begin{bmatrix} \cos^2(\theta)/m & -\sin(\theta)\cos(\theta)/m & \sin(\theta) \\ -\sin(\theta)\cos(\theta)/m & \sin^2(\theta)/m & \cos(\theta) \\ \sin(\theta) & \cos(\theta) & -m \end{bmatrix} \begin{bmatrix} 0 \\ m\dot{z}^- \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin(\theta)\cos(\theta)\dot{z}^- \\ \sin^2(\theta)\dot{z}^- \\ m\cos(\theta)\dot{z}^- \end{bmatrix} \tag{34}$$

Finally, we can check the simplified equations in (23) and (24),

$$\begin{bmatrix} M & A^T \\ A & 0 \end{bmatrix}^{-1} = \begin{bmatrix} M^\dagger & A^{\dagger T} \\ A^\dagger & \Lambda^\dagger \end{bmatrix} = \begin{bmatrix} \cos^2(\theta)/m & -\sin(\theta)\cos(\theta)/m & \sin(\theta) \\ -\sin(\theta)\cos(\theta)/m & \sin^2(\theta)/m & \cos(\theta) \\ \sin(\theta) & \cos(\theta) & -m \end{bmatrix} \tag{35}$$

$$\underbrace{\begin{bmatrix} \dot{x}^+ \\ \dot{z}^+ \end{bmatrix}}_{\dot{q}^+} = \underbrace{\begin{bmatrix} \dot{x}^- \\ \dot{z}^- \end{bmatrix}}_{\dot{q}^-} - \underbrace{\begin{bmatrix} \sin(\theta) \\ \cos(\theta) \end{bmatrix}}_{A^{\dagger T}} \underbrace{\begin{bmatrix} \sin(\theta) & \cos(\theta) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \dot{x}^- \\ \dot{z}^- \end{bmatrix}}_{\dot{q}^-} = \begin{bmatrix} 0 \\ \dot{z}^- \end{bmatrix} - \begin{bmatrix} \sin(\theta)\cos(\theta)\dot{z}^- \\ \cos^2(\theta)\dot{z}^- \end{bmatrix} = \begin{bmatrix} -\sin(\theta)\cos(\theta)\dot{z}^- \\ \sin^2(\theta)\dot{z}^- \end{bmatrix} \tag{36}$$

$$\hat{P} = -\underbrace{\begin{bmatrix} -m \end{bmatrix}}_{\Lambda^\dagger} \underbrace{\begin{bmatrix} \sin(\theta) & \cos(\theta) \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 \\ \dot{z}^- \end{bmatrix}}_{\dot{q}^-} = m\cos(\theta)\dot{z}^- \tag{37}$$

As we can see, all of these solution strategies give us the same answer.

5 Elastic Impact

Elastic, or partially elastic, impact can be more complicated than plastic impact. However, here we will start by treating elastic impact as you learned it in physics class, followed by a discussion of the limitations of this approach.

Newtonian impact says that the velocity after impact is equal to $-e$ times the velocity before impact. Here e is the *coefficient of restitution*, where $e = 0$ is plastic impact, $e = 1$ is elastic impact, and anything in between is partially elastic. For the 1D bouncing ball, this is simply $\dot{z}^+ = -e\dot{z}^-$. Extending this idea to our general problem we get,

$$\boxed{A\dot{q}^+ = -eA\dot{q}^-} \qquad \text{Elastic Impact Law} \tag{38}$$

As with plastic impact we must solve this equation along with our impulse-momentum equation (17).

$$\text{Given } q \text{ and } \dot{q}^-, \text{ solve} \quad \begin{cases} M\dot{q}^+ - M\dot{q}^- + A^T\hat{P} = 0 \\ A\dot{q}^+ = -eA\dot{q}^- \end{cases} \quad \text{for } \dot{q}^+ \text{ and } \hat{P}. \tag{39}$$

To do this by hand we might write out all of the equations and substitute to find an expression for each element in $\dot{q}^+$ and $\hat{P}$. Computationally, it is easier to use our block matrix solution strategy,

$$\begin{bmatrix} M & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \dot{q}^+ \\ \hat{P} \end{bmatrix} = \begin{bmatrix} M\dot{q}^- \\ -eA\dot{q}^- \end{bmatrix} \tag{40}$$

$$\begin{bmatrix} \dot{q}^+ \\ \hat{P} \end{bmatrix} = \begin{bmatrix} M & A^T \\ A & 0 \end{bmatrix}^{-1} \begin{bmatrix} M\dot{q}^- \\ -eA\dot{q}^- \end{bmatrix} = \begin{bmatrix} M^\dagger & A^{\dagger T} \\ A^\dagger & \Lambda^\dagger \end{bmatrix} \begin{bmatrix} M\dot{q}^- \\ -eA\dot{q}^- \end{bmatrix} \tag{41}$$

Looking at the top row of the solution to find the velocity $\dot{q}^+$, and recalling the identity $M^\dagger M + A^{\dagger T}A = I$,

$$\dot{q}^+ = M^\dagger M\dot{q}^- - eA^{\dagger T}A\dot{q}^- = (I - (1+e)A^{\dagger T}A)\dot{q}^- \tag{42}$$

$$\boxed{\dot{q}^+ = \dot{q}^- - (1+e)A^{\dagger T}A\dot{q}^-} \qquad \text{Elastic Impact Velocity} \tag{43}$$

The contact impulse $\hat{P}$ can similarly be computed from the bottom row of (41), recalling the identity $A^\dagger M + \Lambda^\dagger A = 0$,

$$\boxed{\hat{P} = A^\dagger M\dot{q}^- - e\Lambda^\dagger A\dot{q}^- = -(1+e)\Lambda^\dagger A\dot{q}^-} \qquad \text{Elastic Impact Impulse} \tag{44}$$

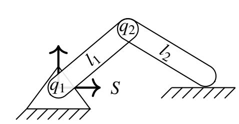
Or, if M is invertible, we can equivalently write $\hat{P} = (1+e)(AM^{-1}A)^{-1}A\dot{q}^-$.

The body impulse P can be derived from (17) and (43) or (16) and (44), recalling that $MA^{\dagger T} + A^T\Lambda^\dagger = 0$

$$P = -M(\dot{q}^+ - \dot{q}^-) = (1+e)MA^{\dagger T}A\dot{q}^- = -(1+e)A^T\Lambda^\dagger A\dot{q}^- = A^T\hat{P} \tag{45}$$

6 Practice Problems

1) What is the impulse and resulting velocity if a 2-link manipulator arm impacts the ground with plastic impact? Assume the ground is along the x axis of the S frame at the base of the robot. Each link i has length l_i, mass m_i, and inertia I_i about its center.



2) Consider a point mass in the plane with state $q = [x,y]^T$ and mass m. There are two constraints on the point, $a_1(q) = y \geq 0$ and $a_2(q) = 3 - x - y \geq 0$. The point mass is initially on the first constraint plasticly impacting the second constraint at configuration $q = [3,0]^T$ with velocity $\dot{q}^- = [\dot{x}^-, 0]^T$. What are $\dot{q}^+$ and $\hat{P}$?

3) How does the solution to the example in Section 4 change if we have elastic impact with coefficient e? Find $\dot{q}^+$ and $\hat{P}$. You may still assume the initial velocity is directly down, so $\dot{x}^- = 0$.

4) Prove that the kinetic energy is maintained through an elastic impact with coefficient $e = 1$, and that it is strictly reduced with a coefficient $e < 1$. You may assume that M is invertible (though a solution exists without this assumption).