Complementarity

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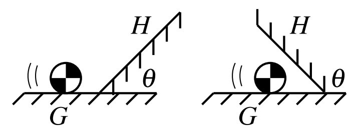
1 Simple Impact Example



Consider a point particle sliding along a planar ground when it hits a second planar surface (that we'll call a hill) which slopes up at an angle of θ. Assume plastic, frictionless impact. If the second surface makes an acute angle with the ground ($\theta < 90°$), then we would expect the point particle to transition off of the ground to sliding up the surface. If, however, the second surface makes an obtuse angle with the ground ($\theta > 90°$), then we would expect the point particle to get "stuck" in the corner. While it may be obvious that there is a difference in this simple case, for a general problem **how do we determine which transition a system will take? And what happens if we "guess" wrong?**

To see what is happening here, let's first assume that the particle makes the transition onto the hill. This means that the ground contact constraint is removed through the impact process. For our state variables $q = [x,z]^T$, let contact constraints from the ground G and the hill H be,

$$a_G(q) = z \geq 0, \qquad a_H(q) = -x\sin(\theta) + z\cos(\theta) \geq 0 \tag{1}$$

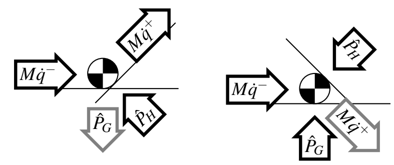
When in contact with each contact surface, the velocity constraints on the system are,

$$A_G(q)\dot{q} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = 0 \qquad A_H(q)\dot{q} = \begin{bmatrix} -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = 0 \tag{2}$$

Let the mass of the particle be m, and its initial velocity be in the positive x direction at the time it reaches the intersection at $q = [0,0]^T$. Using the block matrix solution to the impact process,

$$\begin{bmatrix} \dot{x}^+ \\ \dot{z}^+ \\ \hat{P}_H \end{bmatrix} = \begin{bmatrix} M & A_H^T \\ A_H & 0 \end{bmatrix}^{-1} \begin{bmatrix} M\dot{q}^- \\ 0 \end{bmatrix} = \begin{bmatrix} m & 0 & -\sin(\theta) \\ 0 & m & \cos(\theta) \\ -\sin(\theta) & \cos(\theta) & 0 \end{bmatrix}^{-1} \begin{bmatrix} \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \dot{x}^- \\ \dot{z}^- \end{bmatrix} \\ 0 \end{bmatrix} \tag{3}$$

$$= \begin{bmatrix} \cos^2(\theta)/m & \sin(\theta)\cos(\theta)/m & -\sin(\theta) \\ \sin(\theta)\cos(\theta)/m & \sin^2(\theta)/m & \cos(\theta) \\ -\sin(\theta) & \cos(\theta) & -m \end{bmatrix} \begin{bmatrix} m\dot{x}^- \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos^2(\theta)\dot{x}^- \\ \sin(\theta)\cos(\theta)\dot{x}^- \\ -\sin(\theta)m\dot{x}^- \end{bmatrix} \tag{4}$$



This says that if we have an acute angle hill that the exit velocity will be away from the ground surface, e.g. if $\theta = 45^\circ$, $\dot{q}^+ = [\dot{x}^-/2, \dot{x}^-/2]^T$. Checking the velocity constraint on the ground for a hill at angle θ, $A_G\dot{q}^+ = \sin(\theta)\cos(\theta)\dot{x}^-$. This will be positive when $\theta < 90^\circ$, indicating a separating velocity as the point moves away from the surface. However, we see that when $\theta > 90^\circ$, then $A_G\dot{q}^+$ becomes negative, indicating that the particle will travel into the surface!

If instead we assume that particle maintains contact with both constraints, we get that,

$$\begin{bmatrix}\dot{x}^+\\ \dot{z}^+\\ \hat{P}_G\\ \hat{P}_H\end{bmatrix} = \begin{bmatrix} M & A_{G,H}^T\\ A_{G,H} & 0\end{bmatrix}^{-1}\begin{bmatrix}M\dot{q}^-\\ 0\end{bmatrix} = \begin{bmatrix} m & 0 & 0 & -\sin(\theta)\\ 0 & m & 1 & \cos(\theta)\\ 0 & 1 & 0 & 0\\ -\sin(\theta) & \cos(\theta) & 0 & 0\end{bmatrix}^{-1}\begin{bmatrix}\begin{bmatrix} m & 0\\ 0 & m\end{bmatrix}\begin{bmatrix}\dot{x}^-\\ \dot{z}^-\end{bmatrix}\\ 0\\ 0\end{bmatrix} \tag{5}$$

$$= \begin{bmatrix} 0 & 0 & \cos(\theta)/\sin(\theta) & -1/\sin(\theta)\\ 0 & 0 & 1 & 0\\ \cos(\theta)/\sin(\theta) & 1 & -m/\sin(\theta)^2 & -m\cos(\theta)/\sin^2(\theta)\\ -1/\sin(\theta) & 0 & -m\cos(\theta)/\sin^2(\theta) & -m/\sin^2(\theta)\end{bmatrix}\begin{bmatrix} m\dot{x}^-\\ 0\\ 0\\ 0\end{bmatrix} = \begin{bmatrix} 0\\ 0\\ \cos(\theta)m\dot{x}^-/\sin(\theta)\\ -m\dot{x}^-/\sin(\theta)\end{bmatrix} \tag{6}$$

As expected, the system comes to rest, $\dot{q}^+ = 0$, and both velocity constraints are satisfied. However, we must also consider the sign of $\hat{P}$. The impulse must satisfy the same unilateral constraint cone as the contact forces, i.e. nonadhesive normal impulse and any friction constraints. Recall that for these frictionless contacts, based on the sign conventions we are using here, $U(\lambda_n) = -\lambda_n \geq 0$ for each contact normal force λ_n. For this impact problem, we see that $U(\hat{P}_H) = -\hat{P}_H \geq 0$ satisfies this condition for all $0 < \theta < 180^\circ$, and so there is no problem with the impulse from the hill. However, when checking the impulse from the ground, $U(\hat{P}_G) = -\hat{P}_G \geq 0$ satisfies this condition only for $90^\circ < \theta < 180^\circ$. When $\theta < 90^\circ$, computing the impact into the contact mode with both constraints active requires an adhesive impulse from the ground in order for the particle to "stick" there!

2 Impulse-Velocity Complementarity

Generalizing this analysis to more complicated systems, we see that we have two basic constraints. First, the resulting velocity must not violate any of the constraints j that the system is touching (i.e. whose corresponding $a_j(q) = 0$)

$$\boxed{A_j\dot{q}^+ \geq 0} \qquad \text{Non-penetrating Velocity Constraint} \tag{7}$$

Recall that a positive $A\dot{q}$ corresponds with a separating velocity, i.e. $a(q)$ is getting bigger, while a negative $A\dot{q}$ is an approaching or penetrating velocity, i.e. $a(q)$ is getting smaller, and so if it is already at zero we will go through the constraint.

Second, the impulse produced must satisfy the unilateral constraint cone U which ensures that forces and impulses are non-adhesive and satisfy any friction conditions,

$$\boxed{U_j(\hat{P}) \geq 0} \qquad \text{Non-adhesive and Frictional Impulse Constraints} \tag{8}$$

Unfortunately, these two constraints alone are not sufficient to determine which contact mode will result. In the example, the problem setup enforced another condition: that we can have no impulse from a contact that is separating. Looking back at (3), note that we did not even include $\hat{P}_G$ corresponding to the ground contact that was being removed. This condition is an assumption, and only holds for plastic impact (as elastic impact will have both an impulse and a separating velocity). The simplest way to express mathematically this condition that we cannot have both a separating velocity and an impulse from the same constraint is what is called a *complementarity* condition, which states that their product must be zero (and therefore at least one of the terms must be zero),

$$\boxed{U_j(\hat{P})\left(A_j\dot{q}^+\right) = 0} \qquad \text{Impulse-Velocity (IV) Complementarity} \tag{9}$$

These three conditions, (7)–(9), must hold for our resulting impact process to result in a valid mode transition. They are sometimes written together using the notation,

$$0 \leq U_j(\hat{P}) \perp A_j \dot{q}^+ \geq 0 \tag{10}$$

(which means exactly the conditions (7)–(9)).

3 Mode Selection Problem

It is often convenient to think about the IV complementarity conditions as a *mode selection problem*, that is we want to see what contact mode (i.e. collection of contact constraints) is the correct mode after an impact. Recall that the set of all possible contact constraints is $\mathcal{K}$. If we start in mode $I \subseteq \mathcal{K}$ and undergo an impact, then we want to determine the resulting contact mode $J \subseteq \mathcal{K}$.

To do this we must check the complementarity conditions for any constraint we are touching, i.e. in the normal direction we have $a_j(q) = 0$. For the tangential (friction) constraints, we must check the *corresponding normal*, $i = \alpha(j) \in \mathcal{K}$, that is the normal direction constraint at the same contact point as the tangential constraint. Thus we can define the *scope* $\mathcal{I}_{IV}$ for the IV complementarity problem as the set of contact constraints we must check,

$$\boxed{\mathcal{I}_{IV}(q, \dot{q}) = \{i \in \mathcal{K} : a_{\alpha(i)}(q) = 0\}} \qquad \text{IV Complementarity Scope} \tag{11}$$

Note that the prior mode is a subset of this scope, $I \subset \mathcal{I}_{IV}$, but we must also include any new constraints that the system is impacting as well as any constraints that were not active but are at zero distance.

Within this scope, we can think about the complementarity conditions as determining which constraints are going to be included in $J \subseteq \mathcal{I}_{IV}$ and which are not. Rewriting (7)–(9) in this way,

$$\begin{array}{lll} U_j(\hat{P}_J) \geq 0 & A_j \dot{q}^+ = 0 & \forall j \in J \subseteq \mathcal{I}_{IV} \\ U_k(\hat{P}_J) = 0 & A_k \dot{q}^+ > 0 & \forall k \in \mathcal{I}_{IV} \setminus J \end{array} \tag{12}$$

Typically, the equality constraints in (12) are enforced algebraically, i.e. by the way we calculate $\hat{P}$ and $\dot{q}^+$. As such, we do not need to check them numerically (which can be difficult).

Without knowing anything about how $\hat{P}$ or $\dot{q}^+$ are calculated, then there does not need to be a unique solution mode J that satisfies the mode selection problem in (12). Luckily, for plastic frictionless impact there always exists a unique solution. This is because the separating velocity condition $A_k \dot{q}^+$ for constraint k is equivalent to a negative contact impulse if k were to be added to the mode J, $U_k(\hat{P}_{J \cup \{k\}}) < 0$ (as shown in [1, Thm. 2]). That is, if we add k back into the active mode J when calculating the impulses $\hat{P}$ then it would require an adhesive impulse. Thus we can rewrite (12) as,

$$\begin{array}{lll} U_j(\hat{P}_J) \geq 0 & A_j \dot{q}^+ = 0 & \forall j \in J \subseteq \mathcal{I}_{IV} \\ U_k(\hat{P}_J) = 0 & U_k(\hat{P}_{J \cup \{k\}}) < 0 & \forall k \in \mathcal{I}_{IV} \setminus J \end{array} \tag{13}$$

This reformulation is especially useful if we have a massless link whose separating velocity is not well defined if it is not in contact but where the impulse when in contact is well defined.

Furthermore, considering that the equality constraints are enforced algebraically, we can simplify this further to,

$$(k \in J) \Leftrightarrow U_k(\hat{P}_{J \cup \{k\}}) \geq 0 \qquad \forall k \in \mathcal{I}_{IV} \tag{14}$$

So a constraint is included in J if and only if including it in the impulse calculation results in a nonadhesive impulse.

Note that we now have 5 equivalent expressions for the IV complementarity problem: (7)–(9), (10), (12), (13), and (14). Whichever formulation we use, we can define a predicate (i.e. a condition that can be true or false) $IV(J, q, \dot{q}^-)$ that is true if and only if the contact mode J satisfies the IV complementarity conditions at q, $\dot{q}^-$, so that we don't have to write out all of the conditions every time.

4 Trending Conditions

There is a second complementarity problem we will need to consider at times other than impacts, but in order to set that problem up we need to introduce some new notation. So far we have discussed position constraints, $a(q) \geq 0$. Then, if we exactly satisfy the position constraint, $a(q) = 0$, the velocity constraint says there must be a non-penetrating velocity, $\frac{d}{dt}a(q) = A\dot{q} \geq 0$. Then, if we exactly satisfy both the position and velocity constraint, $a(q) = 0$ and $A\dot{q} = 0$, the acceleration constraint says there must be a non-penetrating acceleration, $\frac{d^2}{dt^2}a(q) = A\ddot{q} + \dot{A}\dot{q} \geq 0$. This pattern continues for higher derivatives,

$$
\begin{aligned}
& & a(q) &\geq 0 & &\text{Position constraint} & (15)\\
&\text{If } a(q) = 0, & \frac{d}{dt}a(q) &\geq 0 & &\text{Velocity constraint} & (16)\\
&\text{If } a(q) = 0, \frac{d}{dt}a(q) = 0, & \frac{d^2}{dt^2}a(q) &\geq 0 & &\text{Acceleration constraint} & (17)\\
& & &\vdots & & &\\
&\text{If } \frac{d^l}{dt^l}a(q) = 0 \forall 0 < l < m, & \frac{d^m}{dt^m}a(q) &\geq 0 & &\text{General constraint} & (18)
\end{aligned}
$$

To capture this general constraint, we define a new relation called **trending positive**, which says that the first nonzero derivative is positive,[1]

$$\boxed{a(q) \succ 0 \text{ if } \exists m : \frac{d^l}{dt^l}a(q) = 0 \; \forall 0 < l < m \text{ and } \frac{d^m}{dt^m}a(q) > 0} \qquad \text{Trending positive} \tag{19}$$

The trending positive condition can be interpreted as the value of $a(q)$ will be positive immediately after the current time. So either $a(q)$ is positive now, or if it is zero then the first non-zero derivative is positive and so for a small enough step forward in time the value will be positive. More formally, this says that [1, Lemma 3],

$$a(q(t)) \succ 0 \Leftrightarrow \exists \delta > 0 : \forall s \in (0, \delta) : a(q(t+s)) > 0 \tag{20}$$

Indeed this property of the trending conditions suggests a computationally efficient method for checking if a function is trending positive – simply simulate the system until it reaches a zero crossing.

If all derivatives are zero, we say that the constraint is **identically zero**,

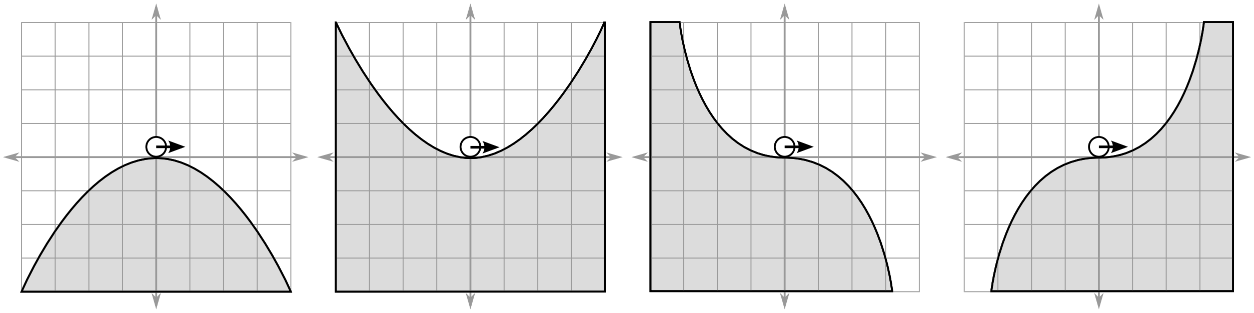
$$\boxed{a(q) \equiv 0 \text{ if } \frac{d^l}{dt^l}a(q) = 0 \; \forall l \in \mathbb{N}} \qquad \text{Identically zero} \tag{21}$$

Similarly, we can define,

$$
\begin{aligned}
\boxed{a(q) \prec 0 \text{ if } \; -a(q) \succ 0} & \qquad \text{Trending negative} & (22)\\
\boxed{a(q) \succeq 0 \text{ if } \; a(q) \succ 0 \text{ or } a(q) \equiv 0} & \qquad \text{Trending non-negative} & (23)\\
\boxed{a(q) \preceq 0 \text{ if } \; a(q) \prec 0 \text{ or } a(q) \equiv 0} & \qquad \text{Trending non-positive} & (24)
\end{aligned}
$$

With this notation we can summarize the conditions (15)–(18) as simply $a(q) \succeq 0$. Fig. 1 has several examples of constraints that are trending positive and trending negative.

[1] Formally, we should interpret the derivative as a Lie derivative with respect to the vector field F, i.e. $a(q) \succ 0$ if $\exists m : (\mathcal{L}_F^l a)(q) = 0 \; \forall l < m$, and $(\mathcal{L}_F^m a)(q) > 0$, where $\mathcal{L}_F^m$ is the m-th Lie derivative with respect to F. See also [1, Sec. 2.1].



(a) (b) (c) (d)

Figure 1: Four examples of a planar point particle ($q = [x,\ y]^T$) with a single constraint, defined as (a) $a(q) = x^2 + 4y$, (b) $a(q) = -x^2 + 4y$, (c) $a(q) = x^3 + 8y$, and (d) $a(q) = -x^3 + 8y$. Note that if the particle velocity is moving to the right ($\dot{q} = [v,\ 0]^T,\ v > 0$), as illustrated, then: the constraint function is trending positive ($a(q) \succ 0$) in (a) and (c); the constraint function is trending negative ($a(q) \prec 0$) in (b) and (d). Figure reproduced from [1, Fig. 2].

These trending relations act similarly to $>$ and $=$, and the notational similarity is intentional. In fact, it is easy to show that $\succ$ or $\succeq$ include more states than $>$ but not as many as $\geq$,

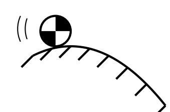
$$\{q : a(q) > 0\} \subset \{q : a(q) \succ 0\} \subset \{q : a(q) \succeq 0\} \subset \{q : a(q) \geq 0\} \tag{25}$$

We will also use the property that,

$$a(q)b(q) \equiv 0 \Leftrightarrow a(q) \equiv 0 \text{ or } b(q) \equiv 0 \tag{26}$$

that is, if a product of two functions is identically zero then one of the two functions must be identically zero (as with equality).

5 Force-Acceleration Complementarity



Consider a point particle that is sliding along a curved surface. This means that there is one active contact constraint, so $a(q) = 0$ and $A\dot{q} = 0$. As the particle goes over a hill it might leave the surface for a moment, depending on the speed and the angle of the surface at that point. **How do we decide when to remove active contact constraints?**

We saw in the last section how we can represent a contact constraint using the trending condition, $a(q) \succeq 0$, to capture (15)–(18). This overall condition is true for all contact constraints at all time – we can never be trending into a contact. However, for constraints we are already touching, the 0^{th} and 1^{st} derivatives, i.e. the relative position and velocity, are already zero (and if they are not, we will apply an impact process). Thus, we only need to ensure that the acceleration and higher terms continues to satisfy this condition,

$$\boxed{A_j\ddot{q} + \dot{A}_j\dot{q} \succeq 0} \qquad \text{Non-penetrating Acceleration Constraint} \tag{27}$$

For constraints that we are keeping, this condition is met as the acceleration constraint is applied as an equality when calculating the dynamics. However, this leaves open the question as to which constraints need to be removed. For this, we use our unilateral constraint cone U, which ensures that forces are non-adhesive and satisfy any friction constraints,

$$\boxed{U_j(\lambda) \succeq 0} \qquad \text{Non-adhesive and Frictional Force Constraint} \tag{28}$$

Any constraint that would require a contact force that violates this condition will need to be removed. Once it is removed, we no longer include the acceleration constraint in the dynamics and allow it separate from (but not go into) the surface. This sets up a Force-Acceleration (FA) complementarity condition: we cannot have both a contact force and a separating acceleration from the same constraint,

$$\boxed{U_j(\lambda)(A_j\ddot{q}+\dot{A}_j\dot{q}) \equiv 0} \qquad \text{Force-Acceleration (FA) Complementarity} \tag{29}$$

Where note that by (26), this condition requires that either the force is identically zero or the acceleration is.

Taken together, we can write these three constraints (27)–(29) as,

$$0 \preceq U_j(\lambda) \perp A_j\ddot{q}+\dot{A}_j\dot{q} \succeq 0 \tag{30}$$

The scope for the FA complementarity problem, i.e. the set of constraints that we must check these conditions against, is,

$$\boxed{\mathcal{I}_{FA} = \{i \in \mathcal{K} : a_{\alpha(i)}(q) = 0, A_i(q)\dot{q} = 0, A_{\alpha(i)}(q)\dot{q} = 0\}} \qquad \text{FA Complementarity Scope} \tag{31}$$

Compared to $\mathcal{I}_{IV}$ in (11), we have two additional conditions here. First, we must check the velocity of the constraint i to see if it is satisfied. In the normal direction, if we have an impacting velocity then we must first apply the IV complementarity conditions, while if we have a separating velocity (e.g. after an impact) we do not need to check the forces or the accelerations. In the tangential (friction) direction, we can only have static friction if there is zero relative velocity. In addition, the final condition says that the corresponding normal direction must have zero constraint velocity in order for the tangential constraint to follow FA complementarity.

Typically, any constraint that would be included in $\mathcal{I}_{FA}$ is already in the active set $I \subseteq \mathcal{I}_{FA}$. Thus, usually it is sufficient to check only the current active constraints I. This has the advantage of avoiding the need to check the equality constraints of (31). Though some grazing contact constraints may be missed, e.g. in Fig. 1(d), such cases are unlikely to come up in a numerical simulation.

With this scope, we can now write out the FA complementarity (27)–(29) as a mode selection problem,

$$\begin{array}{lll} U_j(\lambda_J) \succeq 0 & A_j\ddot{q}+\dot{A}_j\dot{q} \equiv 0 & \forall j \in J \subseteq \mathcal{I}_{FA} \\ U_k(\lambda_J) \equiv 0 & A_k\ddot{q}+\dot{A}_k\dot{q} \succ 0 & \forall k \in \mathcal{I}_{FA} \setminus J \end{array} \tag{32}$$

As with IV complementarity, the calculations for checking the force and acceleration conditions are related (as shown in [1, Thm. 1]), and so we can rewrite these conditions to be in terms of the forces,

$$\begin{array}{lll} U_j(\lambda_J) \succeq 0 & A_j\ddot{q}+\dot{A}_j\dot{q} \equiv 0 & \forall j \in J \subseteq \mathcal{I}_{FA} \\ U_k(\lambda_J) \equiv 0 & U_k(\lambda_{J\cup\{k\}}) \prec 0 & \forall k \in \mathcal{I}_{FA} \setminus J \end{array} \tag{33}$$

That is, instead of checking that a constraint k that is removed has a separating acceleration we can instead check that if we added k to the mode J that it would have resulted in a force that is trending negative. This reformulation is especially useful if we have a massless link whose separating acceleration is not well defined if it is not in contact but where the force when in contact is well defined.

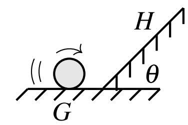
Furthermore, considering that the equality constraints are enforced algebraically, we can simplify this further to,

$$(k \in J) \Leftrightarrow U_k(\lambda_{J\cup\{k\}}) \succeq 0 \qquad \forall k \in \mathcal{I}_{FA} \tag{34}$$

So a constraint is included in J if and only if including it in the force calculation results in a nonadhesive force.

Note that we now have 5 equivalent expressions for the FA complementarity problem: (27)–(29), (30), (32), (33), and (34). Whichever formulation we use, we can define a predicate (i.e. a condition that can be true or false) $FA(J,q,\dot{q})$ that is true if and only if the contact mode J satisfies the FA complementarity conditions at q, $\dot{q}$, so that we don't have to write out all of the conditions every time.

6 Practice Problems



1) Similar to the example in Sec. 1, consider a wheel of radius r rolling along a ground surface when it reaches a second "hill" surface which makes an angle of θ with the ground. The wheel has mass m and uniform density. The coefficient of friction of the wheel with each surface is μ. What is the resulting contact mode, velocity $\dot{q}$, and impulse $\hat{P}$ when it impacts the hill? How does this depend on θ?

2) Prove the condition in (26),

$$a(q)b(q) \equiv 0 \Leftrightarrow a(q) \equiv 0 \text{ or } b(q) \equiv 0$$

3) Consider the examples shown in Fig. 1. Will there be an impact in any of these cases? What will the resulting contact mode be? Does your answer require you to know which contact mode the system was in before the current moment?

4) Consider a planar point particle ($q = [x, y]^T$) sliding on a frictionless surface described by $a(q) = y - cos(x)$. The particle has mass $m = 2$ and gravity $g = 10$. At the point $q = [0, 1]$, for what velocities will the particle stay on the surface and for what velocities will it leave the surface?

References

[1] A. M. Johnson, S. E. Burden, and D. E. Koditschek, “A hybrid systems model for simple manipulation and self-manipulation systems,” *International Journal of Robotics Research*, vol. 35, no. 11, pp. 1354–1392, September 2016.