Print Date: 3/25/2010

The Moderating Impact of Promotion on Substitution: An Economic Approach

Elina Petrova and Alan L. Montgomery

Working Draft, Do Not Distribute

March 2010

Elina Petrova is an Assistant Professor of Marketing at Washington State University and Alan L. Montgomery (e-mail: alan.montgomery@cmu.edu) is an Associate Professor at the Tepper School of Business, Carnegie Mellon University, 5000 Forbes Ave., Pittsburgh, PA 15213.

Abstract

We show that there is an economically consistent way to introduce interactions between price and promotion. The model affords a more informative and comprehensive outlook on the impact of promotion on the demand elasticities in the category and enables interesting insights into the associated changes in consumer price sensitivity and spending proclivity. The main effect of promotions is that they increase market share. However, there are many other more subtle effects that depend upon the strength of the product. For example, as the category expands we find that superior products can benefit from promotions over weaker ones. The vulnerability of a product to competitor prices can be lessened if it is superior, while lesser products will experience increased price competition. The often discussed asymmetric competition between products can either be magnified or reduced by promotions. If promotional effects are strong enough then substitution between a promoted product and superior alternative can be depressed enough that we can see sign reversals in the cross-price elasticity between this promoted product and its non-promoted superior-alternative. Negative cross-price elasticities are persistent problems when estimating demand models in marketing applications, and promotions may provide an explanation for why these unexpected signs can occur for apparent substitutes.

Keywords: Economic Theory, Promotions, Substitution

1. Introduction

Promotions are a powerful and ubiquitous tool to spur demand and are almost always accompanied by price discounts. Empirical research shows that promotions reinforce price discounts and generate more in sales than a price discount alone (Blattberg, Briesch and Fox 1995). Marketing models like Wittink et al. (1988) SCAN*PRO model or Guadagni and Little's (1983) logit model frequently incorporate interactions between price and promotions to capture this effect. However, these direct interactions are often not consistent with economic models, and as a consequence we lose the rich knowledge about demand afforded by economic theory. Our objective is to show how promotions can interact with price in an economically consistent manner. Our results yield insights into understanding how promotion moderates competition. Obviously promotions tend to increase own product sales, but they can also confound existing substitution patterns and under certain conditions increase the attractiveness of non-promoted products and led to increases in their purchases.

The idea that promotions may have interact with price changes is empirically well supported by empirical marketing research. Mulhern and Padgett (1995) finds evidence that buying promoted items also induces customers to spend more on non-promoted items in the store. In a similar vein, Heilman et al. (2002) find that unexpected in-store coupons induce customers to make more unplanned purchases in general, as well as more unplanned purchases of treats, and more purchases in categories that are in close physical proximity to the promoted ones. Heilman et al. (2002) call it "the psychological income effect". Using household data, Dreze et al (2004) find that consumer shopping trips to supermarkets that include purchases of promoted dry goods result in households that spend more on non-promoted items in that category than is spent on the entire category when no promoted items are purchased. Finally, promotional elasticities have been shown to be distinctly different from the attendant price elasticities (Lattin and Bucklin, 1989; Blattberg and Neslin, 1990).

Thus, the empirical evidence seems to indicate that promotions are more effective than a commensurate price adjustment may suggest. From a classical economic perspective the interactions between prices and promotions are perplexing, since promotions serve an educational role which is often assumed away by complete information. The traditional assumptions concerning transitivity and fungability of money also indicate that price promotions should have the same impact as a price discount. An alternative behavioral theory is that an interaction between price and promotion occurs because of reference price effects. However, this behavioral explanation may not be consistent with a classical economic framework. If we reject an economic approach then the desirable theoretical foundation provided by demand theory is lost.

Our objective is to understand if a classical economic approach can be extended to incorporate non-traditional roles for promotion in moderating price discounts and changing consumer perceptions of their expenditures. To achieve this aim we show how promotions can be introduced by modifying price and expenditures with functions that depend on promotions, or what we refer to simply as modifying functions. We adapt a method developed by Lewbel (1985) for incorporating demographic variables into demand model to show how promotional variables can also be incorporated. The advantage of this approach is that it generalizes to any demand system and preserves the consistency of the original demand system.

The novelty and substantive value of our approach can also be found in its inferential capabilities about how promotions moderate substitution. The main effect of promotions is that they increase market share. However, there are many other more subtle effects that depend upon the strength of the product. For example, as the category expands we find that superior products can benefit from promotions of weaker ones. The vulnerability of a product to competitor prices can be lessened if it is superior, while lesser products will experience increased price competition. The often discussed asymmetric competition between products can either be magnified or reduced by promotions. If promotional effects are strong enough then substitution between a promoted product and superior alternative can be depressed enough that we can see sign reversals in the cross-price elasticity between this promoted product and its non-promoted superior-alternative. Negative cross-price elasticities are persistent problems when estimating demand models in marketing applications, and promotions may provide an explanation for why these unexpected signs can occur for apparent substitutes.

2. Introducing Price-Promotion Interactions through Modifying Functions

An economic approach to understanding sales response (Deaton and Muellbauer 1980) is characterized by a consumer maximizing their direct utility function, $U(\mathbf{q}^*)$, which is a function of the vector of *n* quantities $\mathbf{q}^* = \left[q_1 \ q_2 \ \cdots \ q_n \right]'$ consumed at their corresponding prices $\mathbf{p}^* = \left[p_1 \ p_2 \ \cdots \ p_n \right]'$, and subject to a constraint on expenditures x^*:

$$\max_{\mathbf{q}^*} \ U(\mathbf{q}^*) \ s.t. \ \mathbf{p}^{*}{}'\mathbf{q}^* = x^* \tag{1}$$

The solution to this problem yields a system of Marshallian demand equations: $q_i^* = D_i^* \left(x, \mathbf{p}^* \right)$. Alternatively we could also consider the dual problem in which a consumer minimizes their expenditures subject to a utility constraint:

$$\min_{x^*} \ x^* = \mathbf{p}^{*}{}'\mathbf{q}^* \ s.t. \ U(\mathbf{q}^*) = u \tag{2}$$

The solution to this problem yields a system of Hicksian demand equations: $q_i^* = H_i^*\left(u, \mathbf{p}^*\right)$. Substitution of Hicksian demands into the expenditure equation yields the cost function: $x^* = C^*\left(u, \mathbf{p}^*\right)$. For the time being we suppress indices for consumer or store and time for all our variables.

A common approach to incorporate exogenous variables, like demographics and promotions, is to assume that the parameters of the utility function are themselves functions of these variables. The problem is that this ad hoc introduction can destroy the consistency of the economic approach. For example, the adding up, homogeneity, or Slutsky symmetry can be destroyed or the concavity of the cost function may not hold. Suppose we employ a log-log demand model (Montgomery and Rossi 1999) and presume that the price elasticity parameters are a function of promotions then the homogeneity constraint is violated. In general the ad hoc introduction of exogenous variables as functions of the parameters is problematic.

An alternative approach is to introduce promotions and other exogenous variables by directly modifying price and expenditures. We posit that promotions operate by transforming consumer's perceptions of prices and expenditures. Lewbel (1985) provides a general treatment for understanding what types of transformations are permissible and how an existing demand system or cost function can be transformed into a new, consistent demand system. Although Lewbel's objective was to introduce demographic variables into a demand system the same methodology can be used to introduce promotions and other marketing variables into a demand system.

The methodology derives a correspondence between the original (and potentially unobserved) decision space and a new decision that incorporates observed variables of interest. For example, in our analysis we could begin with an Almost Ideal Demand System (AIDS) model and modify price and expenditures with promotion effects to derive a new AIDS model with promotion effects. Instead of assuming that quantity, prices, and expenditures are directly observed in the original demand system, we can instead think of these as intermediate goods. Our new variables can be thought of as the observable quantities, and our modifying functions define the transformation of these intermediate goods into the observable quantities.

In our research we focus our attention on the following class of modifying functions:

$$p_i^* = h(p_i, s_i) = p_i \exp(\beta_i s_i)\,, \tag{3}$$

$$x^* = \left(x / \overline{P}\right)^{1/(1-\Sigma\gamma_i s_i)}, \quad \overline{P} = \prod_{i=1}^{n} p_i^{\gamma_i s_i}\,. \tag{4}$$

Notice the variables with asterisk superscripts are assumed to be the shadow or effective variables that are not directly observed. Specifically, x^* denotes the effective expenditure while x is the observed expenditure, p_i^* denotes the effective price while p_i indicates the observed price, the variable s_i is an indicators for the presence of promotions, and $\mathbf{s} = \left[s_1\ s_2\ \cdots\ s_n\right]'$ denotes the vector of promotion variables. We consider the behavioral interpretation of these modifying functions in subsection 2.1. It is trivial to remove promotions from the model by nullifying the parameters ($\gamma_i = 0$ and $\beta_i = 0$). Under this condition our intermediate variables are the same as the observable ones: $x^* = x$ and $p_i^* = p_i$.

Notice that equation (4) implicitly defines our new cost function:

$$x = C\left(u, \mathbf{p}, \mathbf{s}\right) = f\left(C^*\left(u, h(\mathbf{p}, \mathbf{s})\right), \mathbf{p}, \mathbf{s}\right) = C^*\left(u, h(\mathbf{p}, \mathbf{s})\right)^{1-\Sigma\gamma_i s_i} \cdot \overline{P}\,, \tag{5}$$

where the vector is defined componentwise: $\mathbf{p}^* = h(\mathbf{p}, \mathbf{s})$.

Additionally we must impose the following constraints on this transformation to ensure that the cost function implied by our modifying function will be valid:

$$0 \le \sum_i \gamma_i s_i < 1. \tag{6}$$

Since our promotion variables must lie within the interval 0 to 1, this constraint implies that our parameters must be non-negative and less than unity: $0 \le \gamma_i < 1$. This is also a sufficient condition if only one product is promoted. A proof of the validity of our modifying functions is given in Appendix A.

The importance of our modifying function is that we can express our new demand system in terms of our original system. Following Lewbel (1985) we can show (see Appendix B3 for a derivation) that that our modified Marshallian demand is:

$$q_i = \left(1 - \sum_j \gamma_j s_j\right) \exp\{\beta_i s_i\} \frac{x}{\left(x / \overline{P}\right)^{1/(1-\Sigma\gamma_i s_i)}} q_i^* + \frac{\gamma_i s_i}{p_i} x. \tag{7}$$

Hicksian demand is:

$$q_i = \left(1 - \sum_j \gamma_j s_j\right) \exp\{\beta_i s_i\} \frac{C(u, \mathbf{p}, \mathbf{s})}{\left(C(u, \mathbf{p}, \mathbf{s}) / \overline{P}\right)^{1/(1-\Sigma\gamma_i s_i)}} q_i^* + \frac{\gamma_i s_i}{p_i} x. \tag{8}$$

The budget shares or market shares (w_i and w_i^*) are:

$$w_i = \left(1 - \sum_j \gamma_j s_j\right) w_i^* + \gamma_i s_i, \text{ where } w_i = \frac{p_i q_i}{x} \text{ and } w_i^* = \frac{p_i^* q_i^*}{x^*}. \tag{9}$$

Although it appears the share is a linear function of promotions, we point out that quantity and expenditures have nonlinear dependencies on promotions.

Perhaps it is easiest to understand the relationship between the original and modified demand systems using the budget share equation. In equation (9) we observe that promotions translate demand and then rescale the original budget shares to guarantee adding up. Observe that the translation in budget shares due to promotion guarantees that a promoted product will attain a minimum share of $w_i \ge \gamma_i s_i$. Additionally, if a product is not promoted while other products are promoted, then its budget share can never exceed the original budget share: $w_i \le w_i^*$. Alternatively, suppose that only a single competing product *j* is promoted (e.g., $s_i = 0, s_j = 1, i \ne j$), then the share of the remaining products are reduced proportionately: $w_i = (1 - \gamma_j s_j) w_i^*$.

Our choice of a modifying function for price is driven by our desire to rescale price, for example promotions may reinforce price declines and make them appear more substantial. We also considered power transformations of price with functions of promotions in the power, but these transformations violate the homogeneity of the cost function (see Theorem 1 of Lewbel 1985). The choice of modifying function for expenditure is largely driven by our need to deflate expenditures as a function of price and promotions to insure homogeneity of the cost function.

Our modifying function falls within a more general family of modifying functions:

$$x = \tau(\mathbf{d}) x^{*\sigma(\mathbf{d})/\upsilon(\mathbf{d})} \bar{P}, \text{ where } \bar{P} = \prod_k p_k^{\,d_k} \tag{10}$$

$$p_i^* = \psi_i(\mathbf{d}) p_i^{\upsilon(\mathbf{d})} \tag{11}$$

Where $\mathbf{d}$ is the vector of promotional effects, and d_i is the *i*th element of this vector. In our previous example $d_i = \gamma_i s_i$. We have introduced this new variable *d* instead of *s* to generalize the functional dependence on promotions. Appendix A shows that these modifying functions of this form will yield consistent economic models if the following conditions are met:

$$\tau(\mathbf{d}) \geq 0,\ \sigma(\mathbf{d}) > 0,\ \upsilon(\mathbf{d}) \geq 0,\ \ \psi(\mathbf{d}) \geq 0, \exists i \text{ s.t. } \psi_i(\mathbf{d}) \geq 0, d_i \geq 0, \sum_i d_i \leq 1 \qquad (12)$$

If $\psi_i(\mathbf{d}) = \psi(\mathbf{d})$ then Lewbel (1985, Theorem 8) shows that this form of modifying functions are the only ones that satisfy the necessary conditions for ordinary budget share scaling and translation. In other words our observed budget shares (w_i) translate and scale our shadow ones (w_i^*): $w_i = \vartheta_0 + \vartheta_1 w_i^*$, where the translation and scale factors are functions of our demographic and promotional variables.

2.1 Interpretting our Modifying Functions

We believe that behavioral changes in consumer behavior triggered by promotions are related to biases in the perception of prices and committed expenditures. Previous research supports the proposal that promotions may alter consumers' perceptions of prices. Inman et al. (1990) found that some consumers use promotions as a sufficient cue to purchase a product and obviating the need for further deliberation. Inman et al.'s study demonstrates that signals implying good product value may be more important than the actual depth of the price discount. This can be construed as a perceptual bias of the price variable. Dickson and Sawyer (1990) provide further evidence for perceptual changes in prices are due to promotions. They report a systematic downward bias in the recalled prices of just purchased products when they are bought on promotion. Hence, promotions seem to impact customers' price perceptions. Therefore, we believe it is justified to believe that promotional prices are perceived differently than regular prices.

However, it is not sufficient to only consider the direct impact of promotions on price, since it must also have a corresponding change in expenditures to satisfy adding up restrictions. The studies of Dreze et al (2004), Heilman et al. (2002), and Mulhern and Padgett (1995) provide empirical evidence that promotions increase expenditure above the level warranted by the price decrease alone, and also bring about a reallocation of expenditure. This change in behavioral patterns can be attributed to an altered decision-making process whereby the expenditure factor is perceived with bias fostered by promotion.

We conjecture that promotions work through two mechanisms: an incentive role and a perceptual role. The incentive role is to temporarily improve a product's value through price changes either compared with competing products or its non-promoted price. The perceptual role serves to focus consumer attention on the promoted brand and subsequently alter the perception of the other prices. From a classical economic perspective the incentive role should equal the price discount, however the perceptual role is frequently ignored in an economic model which assumes there are no costs associated with the transaction or gathering information. Although we attempt to distinguish between the incentive and perceptual roles of promotions they are clearly dependent upon one another.

The incentive role can be observed directly through the modifying function on price. Specifically, our modifying function shows that when $\beta_i > 0$ then the observed or perceived prices will be lower due to promotion, this follows by inverting equation (3) to $p_i = \exp\{-\beta_i s_i\} p_i^*$. The greater the value of β_i then the greater the incentive created by promotions. In practice we find that promoted products also experience a price decrease, which would suggest that promotions serve to reinforce the price decline.

An interpretation of our modifying function on expenditures suggests that promotions may alter a consumer's perception of their budget. Observe that our observed or perceived budget increases with the values of γ_i. Therefore the stronger the promotional effect then the greater the perceived or observed amount of money that consumers have to expend. (Specifically observed expenditures grow logarithmically with observed prices.) Therefore promotions may lead consumers to inflate their perceived expenditures. This would move consumers further out on their Engle curves and encourage consumers to reallocate their allocations towards superior products. This argument is analogous to the quality argument advocated by Allenby and Rossi (1991) to create a non-homothetic choice model.

2.2 Illustrating our Approach with a Cobb-Douglas Utility Model

To better understand the implications of our modifying functions we consider a simple example with the following bivariate, Cobb-Douglas direct utility:

$$U^*(\mathbf{q}^*) = \sqrt{q_1^* q_1^*}\,. \tag{13}$$

This is a special case of the linear expenditure system in which the exponents are equal to ½ and there are no translation terms on quantity. Equivalently the cost function can be expressed as:

$$x^* = C^*(u, \mathbf{p}^*) = 2u\sqrt{p_1^* p_2^*}\,. \tag{14}$$

The indirect utility is:

$$u = \psi(x^*, \mathbf{p}^*) = \frac{x^*}{2\sqrt{p_1^* p_2^*}}\,. \tag{15}$$

The Marshallian demand functions are:

$$q_i^* = \frac{x^*}{2p_i^*}\,. \tag{16}$$

The Hicksian demand functions are:

$$q_i^* = \frac{u}{\sqrt{p_i p_j}},\ i \neq j\,. \tag{17}$$

Notice that the market shares are equal to ½ due to the symmetry of the problem: $w_i^* = {p_i^* q_i^*}/{x^*} = \frac{1}{2}$.

These statements are proven in Appendix B1.

Our modifying function defines our new cost function:

$$x = f\left(x^*, \mathbf{p}, \mathbf{s}\right) = \left(x^*\right)^{1-\gamma_1 s_1 - \gamma_2 s_2} \cdot \overline{P},\ \ p_i^* = \exp\{\beta_i s_i\} p_i\,. \tag{18}$$

After applying our transformation functions we can derive new expressions for the indirect utility, Marshallian demand, and Hicksian demand functions following Theorem 4 of Lewbel (1985). Specifically, the indirect utility is:

$$u = \frac{\left({x}/{\overline{P}}\right)^{1/(1-\gamma_1 s_1 - \gamma_2 s_2)}}{2\sqrt{\exp\{\beta_1 s_1\} p_1 \cdot \exp\{\beta_2 s_2\} p_2}}\,. \tag{19}$$

The Marshallian demand functions are:

$$q_i = \frac{x}{2p_i}\left(1 + \gamma_i s_i - \gamma_j s_j\right),\ \ i \neq j\,. \tag{20}$$

The Hicksian demand functions are:

$$q_i = 2^{-\gamma_1 s_1 - \gamma_2 s_2}\left(u\sqrt{\exp\{\beta_1 s_1\} p_1 \cdot \exp\{\beta_2 s_2\} p_2}\right)^{1-\gamma_1 s_1 - \gamma_2 s_2} \frac{\overline{P}}{p_i}\left(1 + \gamma_i s_i - \gamma_j s_j\right),\ \ i \neq j\,. \tag{21}$$

The market shares are now impacted by promotion:

$$w_i = \tfrac{1}{2}\left(1 - \gamma_i s_i - \gamma_j s_j\right) + \gamma_i s_i = \tfrac{1}{2}\left(1 + \gamma_i s_i - \gamma_j s_j\right),\ \ i \neq j\,. \tag{22}$$

Notice the advantage of Lewbel (1985) is that we can transform the original system into the new system, without having to solve the utility maximization or the cost minimization problems.

Although as an exercise in Appendix B2 we show that these forms can be derived equivalently by optimizing utility following the usual budget constraint.

The qualitative impact of promotion in this simple model is clear: promotions increase sales at the expense of competitors. If both products are promoted then the stronger one (i.e., the one with a larger promotion parameter: $\gamma_i > \gamma_j$) will have higher sales, while the weaker one's sales will decrease. Also notice that if neither product is promoted we have our original structure. We discuss the implications on the elasticity structure in greater depth for a more flexible demand system in the next section.

This procedure illustrates the use of our modifying functions. The alternative would have been to directly modify the utility or cost functions by introducing terms that depend upon promotions without destroying the homogeneity of the cost function. The primary advantage of modifying functions is that the same form of modifying function can be applied to *any* demand system and yield an economically consistent form. This does imply that our modifying function is unique or correct. However, it does illustrate the primary advantage of this approach is its generality. If we employ a specific utility function then promotional effects could yield quite different results depending upon the properties of the model. Secondly, we can directly transform the cost or demand models of a known system that does not have promotional effects into one that includes such effects. Finally, as discussed in the previous subsection the motivation of promotional effects are interpretable in light of consumer behavior.

2.3 Illustrating our Approach with the AIDS Model

Our modifying functions are quite general and can be applied to any demand model. The previous subsection provided a very elementary example. In general our approach could be applied to a translog, Rotterdam model, linear expenditure system, Stone-Geary utility, or any demand model. The Almost Ideal Demand System (AIDS) model is a good candidate since it can be justified as a second-order approximation to any utility function and has found a number of empirical applications in marketing (Cotterill, Putsis, and Dhar 2000; Putsis and Cotterill 2001; Dreze, Nisol, and Vilcassim 2004).

Following Deaton and Muellbauer (1980) suppose consumers have the following expenditure function:

$$\ln\left(C(u,\mathbf{p}^*)\right) = a(\mathbf{p}^*) + u \cdot b(\mathbf{p}^*) \tag{23}$$

Where

$$a(\mathbf{p}^*) = \alpha_0 + \sum_k \alpha_k \ln(p_k^*) + \tfrac{1}{2}\sum_j \sum_k \overline{\theta}_{kj} \ln(p_k^*)\ln(p_j^*) \tag{24}$$

$$b(\mathbf{p}^*) = \phi_0 \prod_k \left(p_k^*\right)^{\phi_k} \tag{25}$$

As before C is the minimum expenditure necessary to receive utility level u for the given price vector $\mathbf{p}^*$.

Using Shephard's Lemma the budget share (or market share) equations implied by the demand model are:

$$w_i^* = \alpha_i + \phi_i \ln(x^* / P^*) + \sum_j \theta_{ij} \ln(p_j^*) \tag{26}$$

Where P^* is a price index defined by:

$$\ln(P^*) = \alpha_0 + \sum_k \alpha_k \ln(p_k^*) + \tfrac{1}{2}\sum_j \sum_k \theta_{kj} \ln(p_k^*) \ln(p_j^*) \tag{27}$$

and

$$\theta_{ij} = \tfrac{1}{2}\left(\overline{\theta}_{ij} + \overline{\theta}_{ji}\right) = \theta_{ji}\,. \tag{28}$$

The AIDS model is a popular one amongst econometricians since it can be considered a first-order approximation to any demand system, it satisfies the axioms of choice, and it can be aggregated over consumers (Deaton and Muellbauer 1980). The following restrictions on adding up, homogeneity and symmetry can be tested or imposed directly. Adding up requires:

$$\sum_k \alpha_k = 1, \quad \sum_k \phi_k = 0, \quad \sum_k \theta_{kj} = 0 \tag{29}$$

Homogeneity is satisfied if:

$$\sum_k \theta_{jk} = 0 \tag{30}$$

Slutsky symmetry is satisfied if:

$$\theta_{ij} = \theta_{ji} \tag{31}$$

The Standard Approach: Introducing Promotions by Translating Demand

Our goal is to introduce promotions into this model. A common approach for incorporating demographics or promotion variables into the AIDS demand is to assume that the intercept is a linear function of these variables:

$$\alpha_i = \psi_{i0} + \sum_k \psi_{ik} d_k\,. \tag{32}$$

The first adding up restriction now becomes:

$$\sum_i \psi_{i0} = 1 \text{ and } \sum_i \psi_{ik} = 0 . \tag{33}$$

Presently, marketing applications of the AIDS model have incorporated promotion variables in this manner (Cotterill, Putsis, and Dhar 2000; Putsis and Cotterill 2001; Dreze, Nisol, and Vilcassim 2004). Although in these applications the price index is assumed to be well approximated by the Stone price index, hence market shares (or budget shares) depend only linearly upon promotion, and there is no interaction between promotion and price. We could follow a similar approach for allowing the price parameters to be linear functions of price, but empirically this will improve a large number of restrictions that may be difficult to estimate in practice.

The Use of Modifying Functions to Introduce Promotion Into Demand

Using the modifying functions that we defined in equations (3) and (4), along with the specification of the AIDS model in budget form from equations (26) and (27) we can state the demand model for our new demand model in share form:

$$w_i = \gamma_i s_i + \left(1 - \sum \gamma_j s_j\right)\alpha_i + \phi_i \ln(x / \overline{\overline{P}}) + \left(1 - \sum \gamma_j s_j\right)\sum_j \theta_{ij}\left\{\ln(p_j) + \beta_j s_j\right\} \tag{34}$$

Where $\overline{\overline{P}}$ is a price index defined by:

$$\begin{aligned} \ln(\overline{\overline{P}}) &= \sum_k \gamma_k s_k \ln(p_k) \\ &+ \left(1 - \sum_k \gamma_k s_k\right)\left(\alpha_0 + \sum_k \alpha_k\left(\ln(p_k) + \beta_k s_k\right) + \tfrac{1}{2}\sum_k\sum_j \theta_{kj}\left(\ln(p_k) + \beta_k s_k\right)\left(\ln(p_j) + \beta_j s_j\right)\right) \end{aligned} \tag{35}$$

The derivation is shown in Appendix B4.

Notice that equation (34) has a similar form to the original AIDS model in that the budget share is a linear function of the deflated expenditures and the logged price terms. However, the price index and price terms are now functions of promotions. This contrasts with the form of the AIDS model in which only the constants are translated by promotions. Also, adding up, homogeneity, and symmetry still require that the constraints given in equations (29) through (31) be enforced. Additionally, we must enforce the constraint given in equation (6) to ensure the consistency of our system.

3. The Moderating Effects of Promotions on Substitution

The incorporation of promotions into the demand function means that competition is no longer driven only by prices but also through promotions. To understand how promotions can alter the competitive structure we consider the elasticity matrix. Just as we can relate the original and new demand systems through our modifying functions, we can also work to understand the elasticity structure in terms of these modifying functions. In our problem the original system provides a baseline for our elasticity structure when no promotions occur. We discuss the properties of the modified elasticity structure in this section.

3.1 Expenditure Elasticities

Expenditure elasticities (μ_i) measure how the product responds to changes in expenditures: $\mu_i = \frac{\partial q_i}{\partial x}\frac{x}{q_i}$. Throughout the remainder of this paper we focus on subset demand systems that include all products within a category. This is consistent with the focus in the marketing on demand within a category. It means that expenditures refer to category expenditures, and not total store expenditures or income (e.g., expenditures over all goods). Although we could replace category expenditures with income and think about a full system of demand equations if this is desirable. Category expenditures may increase due to market influences (e.g., new product additions or product improvements), increased income, or heavy promotional activity. Following usual economics nomenclature we define products as superior if $\mu_i > 1$, necessities if $0 < \mu_i < 1$, and inferior if $\mu_i < 0$.

The expenditure elasticity is given by (see Appendix C1 for its derivation):

$$\mu_i = \left(1 - \frac{\gamma_i s_i}{w_i}\right)\left(\frac{1}{1 - \sum_j \gamma_j s_j}\right)\left(\mu_i^* - 1\right) + 1 \tag{36}$$

Notice that promotions will not effect homothetic demand functions: $\mu_i^* = 1 \Rightarrow \mu_i = 1$. The impact of promotions is mediated by its superiority or inferiority. We offer a series of observations about how promotions impact expenditure elasticities (proofs are given in Appendix C2 and C3).

Promotions of promoted products shrink their expenditure elasticities towards unity. The effect of promotions on the expenditure elasticity depends upon whether the product is superior, a necessity, or inferior. If a product is superior then promotions will draw the expenditure elasticity towards the minimum value of unity. This dampening effect is enough to diminish the superiority of the product, but it will remain superior. If a product is a necessity, it will remain a necessity, however the promotion will increase its value towards the maximum of unity. In the more extreme case that a product is inferior, it is possible that a promotion can turn the product into a necessity (e.g., the expenditure can change from negative to positive).

Promotions of other products amplify the expenditure elasticities of non-promoted items. If product *i* is not promoted but another product *j* is promoted ($s_i = 0$ and $i \neq j$) then the impact on its expenditure elasticity depends upon its position in the market. If the product is superior, $\mu_i^* > 1$, then it will become more superior: $\mu_i > \mu_i^* > 1$. If the product is inferior, $\mu_i^* < 1$, then it will become more inferior: $\mu_i < \mu_i^* < 1$. Potentially, it can even push a product that was considered a necessity to become inferior. For example, suppose there are two products and the first product is the market leader (e.g., high market share: $w_1 = .9$, superior: $\mu_1^* = 1.2$, and promotions have a strong positive influence: $\gamma_1 = .5$), then the promotion of the market leader can make the non-promoted product move from necessary ($\mu_2^* = .2$) to inferior ($\mu_1^* = -.6$). In summary, strong products can benefit from competitor promotions, while weak products are hurt.

Promotions that expand the consumer base will benefit superior products. If promotions simply reallocate share across products within the category, then promotions will clearly result in a loss of market share for non-promoted products. As discussed in section 2 we know that if product *i* is not promoted that $w_i \leq w_i^*$. However, it is possible that promotions may bring new consumers and increase expenditures in a product category. The marginal increase of an additional dollar of expenditures will be allocated proportional to the expenditure elasticities. Hence, promotions can benefit strong products within a category through increased quantity sales.

3.2 Price Elasticities

A common measure of price competition is through the measure of price elasticities. Uncompensated price elasticities (η_{ij}) measure the effect of a change in the Marshallian demand of product *i* from a change in the price of product *j*: $\eta_{ij} = \frac{\partial q_i}{\partial p_j} \frac{p_j}{q_i}$. Compensated elasticities (ε_{ij})

similarly measure the effect of a change in the Hicksian demand of product *i* from a change in the price of product *j*: $\varepsilon_{ij} = \left.\frac{\partial q_i}{\partial p_j}\frac{p_j}{q_i}\right|_{u=u_0}$. The uncompensated own-price elasticities of our modified system in terms of the original system are:

$$\eta_{ij} = \begin{cases} \left(1-\frac{\gamma_i s_i}{w_i}\right)\left(\eta_{ij}^* + 1\right) - 1 - \gamma_j s_j \left(\mu_i - 1\right) & \text{if } i = j \\ \left(1-\frac{\gamma_i s_i}{w_i}\right)\eta_{ij}^* - \gamma_j s_j \left(\mu_i - 1\right) & \text{if } i \neq j \end{cases} \tag{37}$$

The uncompensated cross-price elasticities are:

$$\varepsilon_{ij} = \begin{cases} \left(1-\frac{\gamma_i s_i}{w_i}\right)\left(\varepsilon_{ij}^* + 1\right) - 1 + w_j - \frac{\left(w_i - s_i\right)}{w_i}\frac{\left(w_j - \gamma_j s_j\right)}{\left(1-\sum_k \gamma_k s_k\right)} & \text{if } i = j \\ \left(1-\frac{\gamma_i s_i}{w_i}\right)\varepsilon_{ij}^* + w_j - \frac{\left(w_i - s_i\right)}{w_i}\frac{\left(w_j - \gamma_j s_j\right)}{\left(1-\sum_k \gamma_k s_k\right)} & \text{if } i \neq j \end{cases} \tag{38}$$

We refer the reader to Appendix C4 and C5 for the derivation of these equations and proofs of the statements that follow.

We focus upon the uncompensated price elasticity matrix, since this is what usually receives more emphasis in marketing research. Our price elasticities are driven by promotions of the own product and competing products. To help better understand our price elasticity structure we compute the price elasticities by varying whether the own product is promoted ($s_i = 1$) or not ($s_i = 0$) and whether the competing product is promoted ($s_j = 1$) or not ($s_j = 0$), and provide the results in Table 1. Trivially notice that under the condition of no promotions the own and cross price elasticity structure is left unchanged.

		Own Promotion	
	Competitor Promotion	Yes ($s_i = 1$)	No ($s_i = 0$)
Own-Price Elasticities ($i = j$)	Yes ($s_j = 1$) No ($s_j = 0$)	$\left(1 - \frac{\gamma_i}{w_i}\right)\left(\eta_{ii}^* + 1\right) - 1 - \gamma_i\left(\mu_i - 1\right)$	η_{ii}^*
Cross-Price Elasticities ($i \neq j$)	Yes ($s_j = 1$)	$\left(1 - \frac{\gamma_i}{w_i}\right)\eta_{ij}^* - \gamma_j\left(\mu_i - 1\right)$	$\eta_{ij}^* - \gamma_j\left(\mu_i - 1\right)$
	No ($s_j = 0$)	$\left(1 - \frac{\gamma_i}{w_i}\right)\eta_{ij}^*$	η_{ij}^*

Table 1. Uncompensated Cross Price Elasticity Matrix under varying own and cross promotions.

Own promotions increase the own-price sensitivity of the promoted product, but do not directly impact the own-price sensitivity of other products. Most products sold in supermarkets tend to be quite own-price sensitive ($\eta_{ii}^* \ll -1$) usually with own-price elasticities in the range of -2 to -8. Notice that for superior products we see the price elasticity becomes more price sensitive, since the contribution of the expenditure effect reduces its price sensitivity further ($-1 - \gamma_i\left(\mu_i - 1\right) < 0$). Necessities also experience a decrease since $-1 < \gamma_i\left(\mu_i - 1\right) < 0$. It is possible for an inferior product to experience a positive own-price elasticity. However, this would require an inferior product with high budget share and strong promotional lift. For example, consider if $\mu_i^* = -.5$, $w_i^* = .9$, $\varepsilon_{ii}^* = -1.45$, and $\eta_{ii}^* = -1$, $\gamma_i = .8$, $s_i = 1$ and $s_j = 0$ then $\mu_i = -.378$, $w_i = .98$, $\varepsilon_{ii} = -.268$, and $\eta_{ii} = .102$. Notice that in this example the product is inferior, but it is not a Giffen good in the original space. However, after the promotion the product would realize a positive own-price elasticity and exhibit Giffen-like behavior. Given the debates in economics about whether any Giffen goods have been observed, it is not clear if this observation can explain positive own-price elasticities that are sometimes estimated or whether it is more of theoretical curiosity.

Own promotions lessen a product's clout, while competitive promotions decrease our vulnerability if our product is superior and increase our vulnerability otherwise. Notice that there are two components of the cross price elasticity: $\left(1-\frac{\gamma_i s_i}{w_i}\right)\eta_{ij}^*$ and $-\gamma_j s_j\left(\mu_i - 1\right)$. If we assume that products in the original space are substitutes then the first component will remain positive, albeit smaller. Hence, own promotions will lessen the impact of the promoted products price on other products (e.g., its clout is diminished). The second component will be negative if own product is superior and positive otherwise. Hence, superior products are less vulnerable to competitive price changes while necessities and inferior products are more vulnerable. The intuition behind this result is that a competitive promotion will apparently increase a consumer's expenditures. These increased expenditures will result in the consumer reallocating them towards superior products, while reducing expenditures of necessities and inferior ones.

Promotions magnify asymmetric competition for superior products, and diminish the asymmetries for necessities and inferior products. One implication of the previous result is that the asymmetry in price competition between products (Blattberg and Wisniewski 1989) is increased. In our framework asymmetries between products are caused by differences in the market shares and expenditure elasticities. In our problem we can think of "quality" as being measured by the expenditure elasticity. Therefore as either the budget shares or expenditure elasticity is increased the asymmetry between products are increased. To illustrate this point consider the elasticities and budget shares in the original and modified systems as show in Table 2 when the first product which is superior is promoted and the second product which is a necessity is not promoted. The asymmetry between the products has been increased due to the promotion.

	Product	*Market Share*	*Expenditure Elasticities*	*Compensated Elasticity*		*Uncompensated Elasticity*	
				1	2	1	2
Original	1	.60	1.2	-2.02	2.02	-2.74	1.54
	2	.40	0.7	3.02	3.02	2.60	-3.30
Modified	1	.80	1.15	-.81	.81	-1.73	.58
	2	.20	0.4	3.22	-3.22	2.90	-3.30

Table 2. Example to illustrate increased asymmetry for two products system ($\gamma_1 = \gamma_2 = \frac{1}{2}$). The first product is promoted and a superior, while the second is not promoted and a necessity.

To illustrate what happens when the first product is not promoted while the second product is promoted consider the results given in Table 3. Notice that the promotion of the weaker product has result in an increase in its share, a decrease in its own-price sensitivity, and a reduction in the vulnerability of price changes to the first product. In summary, the asymmetry between the products has been reversed.

	Product	*Market Share*	*Expenditure Elasticities*	*Compensated Elasticity*		*Uncompensated Elasticity*	
				1	2	1	2
Original	1	.60	1.2	-2.02	2.02	-2.74	1.54
	2	.40	0.7	3.02	-3.02	2.60	-3.30
Modified	1	.30	1.4	-2.32	2.32	-2.74	1.34
	2	.70	0.83	.99	-.99	.74	-1.57

Table 3. Example to illustrate increased asymmetry for two products system ($\gamma_1 = \gamma_2 = \frac{1}{2}$). The first product is promoted and a necessity, while the second is not promoted and superior.

Competitive promotions can result in sign-reversal for cross-price elasticities. A vexing problem for applied researchers is negative cross-price elasticities. In general products within a category are assumed to be substitutes, which means that the cross price elasticities should be small positive numbers. One explanation is that these negative cross-price elasticities can result from the second component, $-\gamma_j s_j \left(\mu_i - 1\right)$, in equation (37). Notice that if the promotional effect of the competing product *j* is strong and the superiority of product *i* is high, then apparent substitutes can become complements. Consider another example in Table 4, where the first product is strongly superior but the second and weaker product is promoted. Notice that the cross-price effect of the second product on the first has gone from .30 to -.20. Apparently the second product is a complement to the first. Again the intuition is that the promotion of the second brand has resulted in consumers perceiving that their expenditures are worth more than they were previously. As a consequence of consumer's perceiving greater expenditures, superior products like the first one benefit.

	Product	*Market Share*	*Expenditure Elasticities*	*Compensated Elasticity*		*Uncompensated Elasticity*	
				1	2	1	2
Original	1	.60	1.5	-.90	.90	-1.80	.30
	2	.40	0.25	1.35	-1.35	1.20	-1.45
Modified	1	.30	2.0	-2.32	2.32	-1.80	-.20
	2	.70	0.57	.99	-.99	.34	-.91

Table 4. Example to illustrate sign reversals of cross price elasticities due to promotions ($\gamma_1 = \gamma_2 = \frac{1}{2}$). The first product is promoted and a necessity, while the second is not promoted and superior.

4. Empirical AIDS Model using Dominicks Finer Foods Price Promotions

In this section we illustrate our theoretical model of modifying functions based upon promotions by applying them to supermarket scanner data collected from Dominick's Finer Foods (Dominicks). Dominicks is a major supermarket chain in the Chicago area and this data has been employed in a number of other studies. The data spans about two years of weekly category sales from 83 Dominicks stores in the Chicago area. We select two categories to illustrate our model: refrigerated juice and paper towels. There are 8 products in the refrigerated juice category and 10 in the paper towels category. Descriptive statistics for the market shares, price, and promotions are presented in Table 1.

Theoretically, the modified demand system in equations (7) through (9) can be applied to any proper demand model. We choose to apply our model using the functional form implied by an AIDS model. Alternatively we could have chosen any demand system and we choose the AIDS models since it is theoretically consistent and has been employed both other researchers using scanner data. Up to this point we have not discussed that promotions may take different forms in supermarket environments. The two predominate forms are feature promotions and display promotions. Feature promotions or more simply features refer to out of store advertisements that are placed in printed weekly circulars that are distributed to households directly or in-store or sometimes features are simultaneously promoted via television or radio. Display promotions or more simply displays refer to in-store advertisements that are placed next to the product. For example, in the Dominicks dataset displays may include signs displaying the text "bonus buy", shelf-talkers, coupon dispensers, large end-of-aisle displays, or other types of special in-store promotional displays (e.g., cardboard cutouts in front of the product).

To account for these two types of promotions we augment our model to think of promotions as a vector. In our notation s_i becomes $\mathbf{s}_i = \begin{bmatrix} f_i & d_i \end{bmatrix}$, where f_i is feature and d_i is display for product *i*. The scalar effects in our model now become dot products, $\gamma_i s_i$ becomes $\boldsymbol{\gamma}_i \bullet \mathbf{s}_i = \gamma_i^f f_i + \gamma_i^d d$ and $\gamma_i s_i$ becomes $\boldsymbol{\beta}_i \bullet \mathbf{s}_i = \beta_i^f f_i + \beta_i^d d$. Our modifying functions are easily generalized except that our restrictions in equation (6) becomes $0 \le \sum_i \boldsymbol{\gamma}_i \bullet \mathbf{s}_i < 1$. Again since our indicators only take values on the unit interval we enforce the following sufficient conditions:

$$0 \le \gamma_i^f < 1,\ 0 \le \gamma_i^d < 1,\ 0 \le \gamma_i^f + \gamma_i^d < 1,\ 0 \le \sum_i \gamma_i^f + \gamma_i^d < 1 \tag{39}$$

Following others who have worked with this data we aggregate individual products that are promoted together (e.g., plain white versus printed patterns, scented versus unscented, or orange juice with pulp versus without) and our promotion indicators become indices that are weighted by individual product sales.

Using the introduction of display and feature promotions we postulate the form of our AIDS model that we employ in our empirical application follows from equations (34) and (35) given in section 2.3. We model the expected market share ($\overline{w}_{ikt}$) for product *i* in store *k* at week *t*:

$$\overline{w}_{ikt} = \boldsymbol{\gamma}_{ik} \bullet \mathbf{s}_{ikt} + \left(1 - \sum_{j=1}^{M} \boldsymbol{\gamma}_{jk} \bullet \mathbf{s}_{jkt}\right)\alpha_{ik} + \phi_{ik} \ln(x_{kt} / \overline{\overline{P_{kt}}}) + \left(1 - \sum_{j=1}^{M} \boldsymbol{\gamma}_{jk} \bullet \mathbf{s}_{jkt}\right)\sum_{j} \theta_{ijk} \left\{\ln(p_{jkt}) + \boldsymbol{\beta}_{jk} \bullet \mathbf{s}_{jkt}\right\} \quad (40)$$

Where $\overline{\overline{P_{kt}}}$ is a price index defined by:

$$\begin{aligned} \ln(\overline{\overline{P_{kt}}}) &= \sum_{j} \boldsymbol{\gamma}_{jk} \bullet \mathbf{s}_{jkt} \ln(p_{jkt}) + \left(1 - \sum_{j} \boldsymbol{\gamma}_{jk} \bullet \mathbf{s}_{jkt}\right) \\ &\cdot \left(\alpha_{0k} + \sum_{j} \alpha_{jk} \left(\ln(p_{jkt}) + \boldsymbol{\beta}_{jk} \bullet \mathbf{s}_{jkt}\right) + \tfrac{1}{2}\sum_{j}\sum_{l} \theta_{jlk} \left(\ln(p_{jkt}) + \boldsymbol{\beta}_{jk} \bullet \mathbf{s}_{jkt}\right)\left(\ln(p_{lkt}) + \boldsymbol{\beta}_{lk} \bullet \mathbf{s}_{lkt}\right)\right) \end{aligned} \quad (41)$$

Throughout this paper we have assumed that economic theory holds, therefore we also enforce the restrictions implied by adding up, homogeneity, and symmetry from equations (29), (30), and (31), respectively. We refer to this as model A.

We assume that these restrictions hold exactly. However, if we so desired we could follow Montgomery and Rossi (1999) and use these restrictions as prior information over which we center our data. Our reasoning for not attempting to relax these restrictions is that we are attempting to see what economic theory has to offer, and it is outside the scope of this paper to test whether economic theory holds.

An alternative model B is to assume that the promotional patterns are nullified in which case we are left with the standard AIDS model:

$$\overline{w}_{ikt} = \alpha_{ik} + \phi_{ik} \ln(x_{kt} / P_{kt}) + \sum_{j} \theta_{ijk} \ln(p_{jkt}) \quad (42)$$

Where P_{kt} is a price index defined by:

$$\ln(P_{kt}) = \alpha_{0k} + \sum_j \alpha_j \ln(p_{jkt}) + \tfrac{1}{2}\sum_j \sum_l \theta_{jlk} \ln(p_{jkt}) \ln(p_{lkt}) \tag{43}$$

A final alternative model C is to assume that the intercepts in model B are translated by promotional variables: $\alpha_{ik} = \psi_{i0k} + \psi_{i1k} f_{ikt} + \psi_{i2k} d_{ikt}$, as suggested in (32).

To complete the stochastic specification of these models we must define the likelihood to relate the observed market shares to the predicted ones given by our model. We follow Woodland (1979) in postulating that observed shares follow a Dirichlet distribution:

$$\Pr\left(w_{1kt}, w_{2kt}, \ldots, w_{Mkt} \,\middle|\, x_{kt}, \overline{w}_{1kt}, \ldots, \overline{w}_{Mkt}, \kappa, \Omega_k \right) = \frac{\Gamma\left(\kappa x_{kt}\right)}{\sum_{j=1}^{M} \Gamma\left(\kappa x_{kt} \overline{w}_{jkt}\right)} \prod_{j=1}^{M} \left(w_{jkt} \right)^{\kappa \overline{w}_{jkt} - 1}, \tag{44}$$

where κ is the scalar parameter of the Dirichlet distribution ($\kappa > 0$) and Ω_k is the vector of parameters. A primary benefit of this specification opposed to an additive error that is estimated with OLS is that our market shares are guaranteed to sum up to one. Notice that the expectation of our market shares is prediction from our AIDS model:

$$E\left[w_{ikt} \,\middle|\, x_{kt}, \overline{w}_{1kt}, \ldots, \overline{w}_{Mkt}, \kappa, \Omega_k \right] = \frac{\kappa x_{kt} \overline{w}_{ikt}}{\sum_{j=1}^{M} \kappa x_{kt} \overline{w}_{jkt}} = \overline{w}_{ikt} \tag{45}$$

Finally, we follow Montgomery (1997) and complete our specification using a hierarchical Bayesian specification, which assumes that the parameter vectors across stores have a multivariate normal distribution:

$$\Omega_k \sim \mathrm{N}\left(\overline{\Omega}, V_\Omega\right), \text{ where } \Omega_k{}' = \begin{bmatrix} \boldsymbol{\alpha}_k{}' & \boldsymbol{\varphi}_k{}' & \boldsymbol{\theta}_k{}' & \boldsymbol{\gamma}_k{}' & \boldsymbol{\beta}_k{}' \end{bmatrix} \tag{46}$$

The priors are assumed to be diffuse and are given in Appendix D along with the MCMC algorithm used to estimate the model.

5. Findings from Empirical Estimates of AIDS Model using Dominicks Data

[discuss results, the key point is to reinforce the findings from the analytical section to the extent that they are present]

6. Discussion and Conclusions

Economic theory is criticized for making unrealistic assumptions about consumer behavior. However, instead of positing a new theory of consumer behavior we have returned to the classical economic approach. This is driven by our belief that economic theory is a powerful device for understanding consumer behavior. It is beyond the scope of this research to prove whether the economic approach is correct. Instead we have simply asked what insights it has to offer about promotions if they operate within standard economic theory. Our desire to rely upon theory is not meant to be an affront to empirical approaches to understanding how promotions work (Blattberg, Briesch and Fox 1995, van Heerde et al 2002), since empirical analysis generalizations can offer insights to both practice and theory.

We have introduced promotional effects so that they interact with price and expenditures. Central to our approach is the premise that promotions may affect customer perceptions of prices, expenditure, and products, and temporarily alter the consumer decision-making process. Our theoretical framework is based upon the work of Lewbel (1985) to demonstrate how promotions can be incorporated into demand systems using modifying functions. Ours is the first within marketing to suggest such an approach. A primary advantage of this approach is that it yields a regular method for incorporating promotions into any demand system. Theoretically we have shown that perceptual changes of prices and expenditures are consistent with classical economic theory. An alternative interpretation of our modifying functions is that they posit the existing of intermediary or shadow goods that enter into a consumer's utility maximization or cost minimization problems. However, the essence of the classical economic approach remains intact.

Our results show that promotions moderate substitution between products. We demonstrate both analytically and numerically a number of findings that we believe are of theoretical and practical importance for empirical research on price promotions. The main effect of promotions is that they increase market share. However, there are many other more subtle effects that depend upon the strength of the product. For example, as the category expands we find that superior products can benefit from promotions over weaker ones. The vulnerability of a product to competitor prices can be lessened if it is superior, while lesser products will experience increased price competition. The often discussed asymmetric competition between products can either be magnified or reduced by promotions. If promotional effects are strong enough then substitution between a promoted product and superior alternative can be depressed enough that we can see sign reversals in the cross-price elasticity between this promoted product and its non-promoted superior-alternative. Negative cross-price elasticities are persistent problems when estimating demand models in marketing applications, and promotions may provide an explanation for why these unexpected signs can occur for apparent substitutes.

A limitation of our current research is that we consider only one family of modifying functions. This family was motivated by our desire to translate demand and to introduce a simple scaling function of prices. However, it is not a unique solution to the problem. There may exist other modifying functions that better capture consumer behavior. We hope our work encourages other to consider the use of modifying functions in understanding how promotions impact demand.

References

Allenby, Gregory M. and Peter E. Rossi (1991), "Quality Perceptions and Asymmetric Switching Between Brands", *Marketing Science*, 10 (3), 185-204.

Blattberg, Robert C., Richard Briesch, and Edward J. Fox (1995), "How Promotions Work", *Marketing Science*, 14 (3), G122-G132.

Blattberg, Robert C. and Scott A. Neslin (1990), *Sales Promotion: Concepts, Methods, and Strategies*, Englewood Cliff, NJ: Prentice Hall.

Blattberg, Robert C. and Kenneth J. Wisniewski (1989), "Price-Induced Patterns of Competition", *Marketing Science*, 8 (4), 291-309.

Cotterill, Ronald W., William P. Putsis, Jr., and Ravi Dhar (2000), Assessing the Competitve Interaction between Private Labels and National Brands", *Journal of Business*, 73 (1), 109-137.

Deaton, Angus and John Muellbauer (1980), "An Almost Ideal Demand System", *American Economic Review*, 70, 312-326.

Dickson, Peter R. and Alan G. Sawyer (1990), "The Price Knowledge and Search of Supermarket Shoppers", *Journal of Marketing*, 54, 42-53.

Dreze, Xavier, Patricia Nisol, and Naufel J. Vilcassim (2004), "Do Promotions Increase Store Expendtirue?" A Descriptive Study of Household Shopping Behavior", *Quantitative Marketing and Economics*, 2, 59-92.

Guadagni, Peter M. and John D.C. Little (1983), "A Logit Model of Brand Choice Calibrated on Scanner Data", *Marketing Science*, 27 (1), 29-48.

Heilman, Carrie M., Kent Nakamoto, and Ambar Rao (2002), "Please Surprises: Consumer Respond to Unexepcted In-Store Coupons", *Journal of Marketing Research*, 39, 242-252.

Inman, J. Jeffrey, Leigh McAlister, and Wayne D. Hoyer (1990), "Promotion Signal: Proxy for a Price Cut?", *Journal of Consumer Research*, 17, 74-81.

Lattin, James M. and Randolph E. Bucklin (1989), "Reference Effects of Price and Promotionon Brand Choice Behavior", *Journal of Marketing Research*, 26, 299-309.

Lewbel, Arthur (1985), "A Unified Approach to Incorporating Demographic of Other Effects into Demand Systems", *Review of Economic Studies*, 52 (1), 1-18.

Montgomery, Alan L. (1997), "Creating Micro-Marketing Pricing Strategies Using Supermarket Scanner Data", *Marketing Science*, 16 (4), 315-337.

Montgomery, Alan L. and Peter E. Rossi (1999), "Estimating Price Elasticities with Theory-based Priors", *Journal of Marketing Research*, 36 (4), 413-423.

Mulhern, Francis J. and Daniel Padgett (1995), "The Relationship Between Retail Price Promotions and Regular Price Purchases", *Journal of Marketing*, 59, 83-90.

Putsis and Cotterill 2001

van Heerde, Harald J., Peter S. H. Leeflang, and Dick R. Wittink (2002), "How Promotions Work: SCAN*PRO-Based Evolutionary Model Building", Schmalenbach Business Review (54), 198-220.

Wittink, Dick R., M. J. Addona, W.J. Hawkes, and J.C. Porter (1988), "SCAN*PRO: The Estimation, Validation and Use of Promotion Effects Based on Scanner Data", Cornell University, Working Paper.

Woodland, A. D. (1979), "Stochastic Specification and the Estimation of Share Equations", *Journal of Econometrics*, 10, 361-383.

Appendix A: Consistency of our Modifying Functions

A1. Consistency of General Functional Form

We show the consistency of the general form of our modifying functions by verifying that they satisfy Theorems 1, 2, and 3 of Lewbel (1985). First we restate our modifying definitions from equations (10) and (11):

$$f(m^*, \mathbf{p}, \mathbf{d}) = m = \tau(\mathbf{d}) m^{*\sigma(\mathbf{d})/\alpha(\mathbf{d})} \widetilde{P}(\mathbf{p}, \mathbf{d}), \text{ where } \widetilde{P}(\mathbf{p}, \mathbf{d}) = \prod_{i=1}^{n} p_i^{d_i}$$

$$h_i(\mathbf{p}, \mathbf{d}) = p_i^* = \psi_i(\mathbf{d}) p_i^{\alpha(\mathbf{d})};$$

Compliance with Lewbel's Theorem 1:

$$\sum_j \frac{\partial h_i}{\partial p_j} \frac{p_j}{p_i^*} = \theta(\mathbf{p}, \mathbf{d}) \Rightarrow \frac{\partial h_i}{\partial p_i} \frac{p_i}{p_i^*} = \frac{\psi_i(\mathbf{d}) \alpha(\mathbf{d}) p_i^{\alpha(\mathbf{d})}}{\psi_i(\mathbf{d}) p_i^{\alpha(\mathbf{d})}} = \alpha(\mathbf{d}) = \theta(\mathbf{p}, \mathbf{d});$$

$$1 - \sum_j \frac{\partial f}{\partial p_j} \frac{p_j}{f} = 1 - \frac{\sum_j d_j \tau(\mathbf{d}) m^{*\sigma(\mathbf{d})/\alpha(\mathbf{d})} \prod_{i=1}^{n} p_i^{d_i}}{f} = 1 - \sum_j d_j;$$

$$\frac{\partial f}{\partial m^*} \frac{m^*}{f} \theta(\mathbf{p}, \mathbf{d}) = \frac{\sigma(\mathbf{d})}{\alpha(\mathbf{d})} \theta(\mathbf{p}, \mathbf{d}) = \frac{\sigma(\mathbf{d})}{\alpha(\mathbf{d})} \alpha(\mathbf{d}) = \sigma(\mathbf{d});$$

$$\Rightarrow 1 - \sum_j d_j = \sigma(\mathbf{d}).$$

Compliance with Lewbel's Theorem 2:

$$(a)\ f(m^*, \mathbf{p}, \mathbf{d}) \geq 0 \Leftrightarrow \tau(\mathbf{d}) \geq 0;$$

$$(b)\ \frac{\partial f}{\partial m^*} > 0 \Leftrightarrow \frac{\sigma(\mathbf{d})}{\alpha(\mathbf{d})} > 0;$$

$$(c)\ \frac{\partial f}{\partial p_i} \geq 0 \Leftrightarrow d_i \geq 0;$$

(d) for all i and $j, \frac{\partial h_j}{\partial p_i} \geq 0 \Leftrightarrow \alpha(\mathbf{d})\psi_i(\mathbf{d}) \geq 0$;

(e) at least one $\psi_i(\mathbf{d}) > 0$

Compliance with Lewbel's Theorem 3:

$$\frac{\partial f^2(m^*, \mathbf{p}, \mathbf{d})}{\partial m^{*2}} \leq 0 \Leftrightarrow \frac{\sigma(\mathbf{d})}{\alpha(\mathbf{d})}\left[\frac{\sigma(\mathbf{d})}{\alpha(\mathbf{d})} - 1\right] \leq 0 \Leftrightarrow \frac{\sigma(\mathbf{d})}{\alpha(\mathbf{d})} \leq 1$$

$$\frac{\partial f^2(m^*, \mathbf{p}, \mathbf{d})}{\partial p_j^2} \leq 0 \Leftrightarrow d_i(d_i - 1) \leq 0 \Leftrightarrow d_i \leq 1$$

$$\frac{\partial h_i^2(\mathbf{p}, \mathbf{d})}{\partial p_j^2} \leq 0 \Leftrightarrow \alpha(\mathbf{d})[\alpha(\mathbf{d}) - 1] \leq 0$$

A2. Consistency of Simplified Modifying Functional Form

For our modifying functions given in equations (3) and (4) we can simplify the more general functional form given above as following:

$\alpha(\mathbf{d}) = \tau(\mathbf{d}) = 1,\ d_i = \gamma_i s_i$, where $s_i \in \{0,1\}$ is the indicator for promotion,
$\sigma(\mathbf{d}) = 1 - \sum_j d_j = 1 - \sum_j \gamma_j s_j$ and $\psi_i(\mathbf{d}) = \exp(\beta_i s_i)$.
With these substitutions, the conditions from Lewbel's Theorems 1-3 are reduced to the following restrictions on the parameters:

$$0 \leq \gamma_i \leq 1 \text{ and } 0 \leq \sum_i \gamma_i s_i \leq 1 .$$

Appendix B: Cobb-Douglas Demand Example

First, we provide some definitions and their specifications that we shall use or prove in this appendix:

Direct utility: $u(q) = q_1^{*1/2} q_2^{*1/2}$, s.t. $p_1^* q_1^* + p_2^* q_2^* = m^*$

Marshallian Demand: $q_1^* = \dfrac{m^*}{2p_1^*}$; $q_2^* = \dfrac{m^*}{2p_2^*}$;

Indirect Utility: $u = \sqrt{\dfrac{m^*}{2p_1^*}}\sqrt{\dfrac{m^*}{2p_2^*}} = \dfrac{1}{2}\dfrac{m^*}{\sqrt{p_1^* p_2^*}}$

Cost Function: $m^* = 2u\sqrt{p_1^* p_2^*}$

Hicksian Demand: $q_1^* = up_1^{*(-1/2)} p_2^{*1/2}$; $q_2^* = up_1^{*1/2} p_2^{*(-1/2)}$

Market Share: $q_1^* = \dfrac{m^*}{2p_1^*} \Rightarrow w_1^* = \dfrac{p_1^* q_1^*}{m^*} = \dfrac{1}{2}$; $w_2^* = \dfrac{1}{2}$.

B1. Derivation of Demand Model Using Cobb-Douglas Utility

Utility Maximization Subject to a Budget Constraint

We start with a Cobb-Douglas utility specification for two brands, $u = q_i^{1/2} q_j^{1/2}$. The budget constraint is $m = p_i q_i + p_j q_j$. Hence, our utility maximization problem is:

$$\max_{q_1, q_2} U = q_1^{1/2} q_2^{1/2}, \text{ s.t. } m = p_1 q_1 + p_2 q_2$$

We formulate the Lagrangean that is associated with our problem:

$$\Lambda = q_1^{1/2} q_2^{1/2} - \lambda(m - p_1 q_1 - p_2 q_2)$$

The first order conditions for this optimization problem are met when:

$$\frac{\partial\Lambda}{\partial q_1} = \tfrac{1}{2} q_1^{-1/2} q_2^{1/2} - \lambda p_1 = 0 \Rightarrow \tfrac{1}{2} q_1^{-1/2} q_2^{1/2} = \lambda p_1;$$
$$\frac{\partial\Lambda}{\partial q_2} = \tfrac{1}{2} q_1^{1/2} q_2^{-1/2} - \lambda p_2 = 0 \Rightarrow \tfrac{1}{2} q_1^{1/2} q_2^{-1/2} = \lambda p_2;$$
$$\frac{\partial\Lambda}{\partial \lambda} = m - p_1 q_1 - p_2 q_2 = 0; \qquad \Rightarrow \frac{q_2}{q_1} = \frac{p_1}{p_2},\ \text{or } q_1 = \frac{p_2 q_2}{p_1} \text{ and } q_2 = \frac{p_1 q_1}{p_2};$$

Hence, $q_1 = \dfrac{m - p_2 q_2}{p_1} = \dfrac{p_2 q_2}{p_1}$ and $q_2 = \dfrac{m - p_1 q_1}{p_2} = \dfrac{p_1 q_1}{p_2}$, which yields the optimum solution:

$$q_1 = \frac{m}{2p_1};\ q_2 = \frac{m}{2p_2}.$$

Cost Minimization Subject to Utility Constraint

The dual of our previous utility maximization problem is to minimize our expenditures subject to a utility constraint:

$$\min\{p_1 q_1 + p_2 q_2\},\ \text{s.t. } u = q_1^{1/2} q_2^{1/2},$$

Which yields the Lagrangean:

$$\Lambda = p_1 q_1 + p_2 q_2 - \lambda(u - q_1^{1/2} q_2^{1/2})$$

The first order conditions imply the following:

$$\frac{\partial\Lambda}{\partial q_1} = p_1 + \tfrac{1}{2} u q_1^{-1/2} q_2^{1/2} = 0 \Rightarrow p_1 = -\tfrac{1}{2} u q_1^{-1/2} q_2^{1/2};$$
$$\frac{\partial\Lambda}{\partial q_2} = p_2 + \tfrac{1}{2} u q_1^{1/2} q_2^{-1/2} = 0 \Rightarrow p_2 = -\tfrac{1}{2} u q_1^{1/2} q_2^{-1/2};$$
$$\frac{\partial\Lambda}{\partial \lambda} = -u + q_1^{1/2} q_2^{1/2} = 0; \quad \Rightarrow \frac{p_1}{p_2} = \frac{q_2}{q_1},\ \text{or } q_1 = \frac{p_2 q_2}{p_1};$$

Hence $u = q_2\sqrt{\dfrac{p_2}{p_1}}$ and $u = q_1\sqrt{\dfrac{p_1}{p_2}}$ which yields the Hicksian demand functions:

$$q_1 = up_1^{-1/2}p_2^{1/2} \text{ and } q_2 = up_1^{1/2}p_2^{-1/2}.$$

B2. Demand for Modified Cobb-Douglas Utility Model

In this section we deduce the demand models for the Cobb-Douglas Utility with our modifying functions using Lewbel's Theorems and directly through substitution and optimization.

Deduction of Demand using Lewbel's Theorems

First we restate our modifying functions for price and expenditures:

$$p_i^* = h(p_i, s_i) = p_i \exp(\beta_i s_i);$$

$$m^* = \left(m / \widetilde{P}\right)^{1/(1-\Sigma\gamma_i s_i)}, \quad \widetilde{P} = \prod_{i=1}^{n} p_i^{\gamma_i s_i}, \text{ or } m = m^{*(1-\Sigma\gamma_i s_i)}\widetilde{P}$$

The derivatives of these transformations are:

$$\frac{\partial h_i}{\partial p_j} = \begin{cases} \exp(\beta_i s_i), \text{ if } i = j \\ 0, \text{ if } i \neq j \end{cases}$$

$$\frac{\partial f}{\partial p_i} = m^{*(1-\Sigma\gamma_j s_j)}\frac{\gamma_i s_i}{p_i}\widetilde{P}$$

$$\frac{\partial f}{\partial m^*} = (1 - \sum_j \gamma_j s_j) m^{*-\Sigma\gamma_j s_j}\widetilde{P}$$

Following Lewbel the indirect utility can be found by equating the indirect utility from the original system through substitution:

$$V(m,p,r)=V^*(m^*,p^*)=\frac{m^*}{2\sqrt{p_1^* p_2^*}}=\frac{\left(m/\widetilde{P}\right)^{1/(1-\gamma_1 s_1-\gamma_2 s_2)}}{2\sqrt{p_1\exp(\beta_1 s_1)\,p_2\exp(\beta_2 s_2)}}$$

We can derive the Hicksian Demand using Lewbel's Theorem 4:

$$q_i=\frac{\partial f}{\partial m^*}\frac{\partial h_i}{\partial p_i}q_i^*+\frac{\partial f}{\partial p_i}=(1-\Sigma\gamma_j s_j)m^{*(-\Sigma\gamma_j s_j)}\widetilde{P}\exp(\beta_i s_i)q_i^*+m^{*(1-\Sigma\gamma_j s_j)}\widetilde{P}\frac{\gamma_i s_i}{p_i}=$$

$$=(1-\Sigma\gamma_j s_j)m^{*(-\Sigma\gamma_j s_j)}\widetilde{P}\exp(\beta_i s_i)up_i^{*(-1/2)}p_j^{*(1/2)}+m^{*(1-\Sigma\gamma_j s_j)}\widetilde{P}\frac{\gamma_i s_i}{p_i}=$$

$$=m^{*(-\Sigma\gamma_j s_j)}\widetilde{P}\left[(1-\Sigma\gamma_j s_j)\exp(\beta_i s_i)u\sqrt{\frac{p_j\exp(\beta_j s_j)}{p_i\exp(\beta_i s_i)}}+m^*\frac{\gamma_i s_i}{p_i}\right]=$$

$$=\left(2u\sqrt{p_i^* p_j^*}\right)^{(-\Sigma\gamma_j s_j)}\widetilde{P}\left[(1-\Sigma\gamma_j s_j)\exp(\beta_i s_i)u\sqrt{\frac{p_j\exp(\beta_j s_j)}{p_i\exp(\beta_i s_i)}}+2u\sqrt{p_i^* p_j^*}\frac{\gamma_i s_i}{p_i}\right]=$$

$$=\left(2u\sqrt{p_i^* p_j^*}\right)^{(-\Sigma\gamma_j s_j)}\widetilde{P}u\sqrt{p_j\exp(\beta_j s_j)}\left[(1-\Sigma\gamma_j s_j)\exp(\beta_i s_i)\sqrt{\frac{1}{p_i\exp(\beta_i s_i)}}+2\sqrt{p_i\exp(\beta_i s_i)}\frac{\gamma_i s_i}{p_i}\right]=$$

$$=\left(2u\sqrt{p_i^* p_j^*}\right)^{(-\Sigma\gamma_j s_j)}\widetilde{P}u\sqrt{p_j\exp(\beta_j s_j)}\sqrt{p_i\exp(\beta_i s_i)}\frac{1}{p_i}\left[(1-\Sigma\gamma_j s_j)+2\gamma_i s_i\right]=$$

$$=2^{(-\Sigma\gamma_j s_j)}\left[u\sqrt{p_i\exp(\beta_i s_i)\,p_j\exp(\beta_j s_j)}\right]^{(1-\Sigma\gamma_j s_j)}\frac{\widetilde{P}}{p_i}\left[(1+\gamma_i s_i-\gamma_j s_j)\right]$$

The Marshallian demand can be found following the same theorem:

$$q_i=\frac{\partial f}{\partial m^*}\frac{\partial h_i}{\partial p_i}q_i^*+\frac{\partial f}{\partial p_i}=(1-\Sigma\gamma_j s_j)m^{*(-\Sigma\gamma_j s_j)}\widetilde{P}\exp(\beta_i s_i)\frac{m^*}{2p_i^*}+m^{*(1-\Sigma\gamma_j s_j)}\widetilde{P}\frac{\gamma_i s_i}{p_i}=$$

$$=m^{*(1-\Sigma\gamma_j s_j)}\widetilde{P}\left[(1-\Sigma\gamma_j s_j)\exp(\beta_i s_i)\frac{1}{2p_i^*}+\frac{\gamma_i s_i}{p_i}\right]=$$

$$=m^{*(1-\Sigma\gamma_j s_j)}\frac{\widetilde{P}}{p_i}\left[\frac{1}{2}(1-\Sigma\gamma_j s_j)+\gamma_i s_i\right]=\frac{m}{p_i}\left[\frac{1}{2}(1-\Sigma\gamma_j s_j)+\gamma_i s_i\right]=$$

$$= \frac{m}{2p_i}(1 + \gamma_i s_i - \gamma_j s_j)$$

Finally, we can compute the budget shares:

$$w_i = \frac{q_i p_i}{m} = \frac{1}{2}(1 + \gamma_i s_i - \gamma_j s_j)$$

Alternative Derivations directly from the Cost Function

Instead of applying Lewbel's Theorems to deduce the demand model we can instead use first principles to derive the demand model. First notice that our cost function is given directly from our modifying function:

$$c(u, p, r) = f(c^*(u, h(p, r), p, r);$$

$$m = f(m^*, p, r)$$

$$m = \widetilde{P}\left[2u\sqrt{p_i \exp(\beta_i s_i) p_j \exp(\beta_j s_j)}\right]^{(1-\Sigma\gamma_j s_j)}$$

From here, we invert this equation to yield the indirect utility:

$$u = \frac{m^{1/(1-\Sigma\gamma_j s_j)}}{2\widetilde{P}^{1/(1-\Sigma\gamma_j s_j)}\sqrt{p_i \exp(\beta_i s_i) p_j \exp(\beta_j s_j)}}$$

We can apply Roy's identity to obtain the demand function:

$$-\frac{\partial u / \partial p_i}{\partial u / \partial m} = \frac{m}{2p_i}\left(1 + \gamma_i s_i - \gamma_j s_j\right)$$

B3. General Form of Demand Using Our Modifying Functions

Here we derive demand using Lewbel's Theorem 4. Again we restate our modifying functions:

$$p_i^* = p_i \exp(\beta_i s_i)$$

$$m^* = (m/\widetilde{P})^{\frac{1}{1-\Sigma\gamma_j s_j}}, \qquad \widetilde{P} = \prod_{j=1}^{n} p_j^{\gamma_j s_j}$$

$$\Leftrightarrow m = m^{*(1-\Sigma\gamma_j s_j)}\,\widetilde{P}$$

Marshallian demand is:

$$q_i = \frac{\partial f}{\partial m^*}\frac{\partial h_i}{\partial p_i} q_i^* + \frac{\partial f}{\partial p_i} = (1-\Sigma\gamma_j s_j) m^{*(-\Sigma\gamma_j s_j)}\,\widetilde{P} \exp(\beta_i s_i) q_i^* + m^{*(1-\Sigma\gamma_j s_j)}\,\widetilde{P}\,\frac{\gamma_i s_i}{p_i} =$$

$$= (1-\Sigma\gamma_j s_j) m^{*(1-\Sigma\gamma_j s_j)}\,\widetilde{P}\left[m^{*-1} \exp(\beta_i s_i) q_i^* + \frac{\gamma_i s_i}{p_i}\frac{1}{(1-\Sigma\gamma_j s_j)} \right] =$$

$$= (1-\Sigma\gamma_j s_j)\frac{m}{p_i}\left[\frac{q_i^* p_i^*}{m^*} + \frac{\gamma_i s_i}{(1-\Sigma\gamma_j s_j)} \right]$$

$$\Leftrightarrow w_i = (1-\Sigma\gamma_j s_j)\left[w_i^* + \frac{\gamma_i s_i}{(1-\Sigma\gamma_j s_j)} \right] = (1-\Sigma\gamma_j s_j) w_i^* + \gamma_i s_i$$

Market share is:

$$w_i = \frac{\partial f}{\partial m^*}\frac{m^*}{m}\frac{\partial h_i}{\partial p_i}\frac{p_i}{p_i^*} w_i^* + \frac{\partial f}{\partial p_i}\frac{p_i}{m} = (1-\Sigma\gamma_j s_j)\frac{m^{*(1-\Sigma\gamma_j s_j)}\,\widetilde{P}}{m} w_i^* + m^{*(1-\Sigma\gamma_j s_j)}\,\widetilde{P}\,\frac{\gamma_i s_i}{p_i}\frac{p_i}{m}$$

$$\Leftrightarrow w_i = (1-\Sigma\gamma_j s_j) w_i^* + \gamma_i s_i$$

Hicksian demand is:

$$\hat{q}_i = \frac{\partial f}{\partial m^*}\frac{\partial h_i}{\partial p_i} \hat{q}_i^* + \frac{\partial f}{\partial p_i} = (1-\Sigma\gamma_j s_j) m^{*(-\Sigma\gamma_j s_j)}\,\widetilde{P} \exp(\beta_i s_i) q_i^* + m^{*(1-\Sigma\gamma_j s_j)}\,\widetilde{P}\,\frac{\gamma_i s_i}{p_i} =$$

$$= (1-\Sigma\gamma_j s_j) m^{*(1-\Sigma\gamma_j s_j)}\,\widetilde{P}\left[\exp(\beta_i s_i)\frac{\hat{q}_i^*}{m^*} + \frac{\gamma_i s_i}{p_i}\frac{1}{(1-\Sigma\gamma_j s_j)} \right] =$$

$$= (1-\Sigma\gamma_j s_j)\frac{m\exp(\beta_i s_i)}{(m/\widetilde{P})^{\frac{1}{1-\Sigma\gamma_j s_j}}}\hat{q}_i^* + \frac{\gamma_i s_i}{p_i}m$$

B4. AIDS Model with Modifying Functions

The market share form of the AIDS model is given as:

$$w_i^* = \alpha_i + \phi_i \ln(m^*/P^*) + \sum_j \theta_{ij} \ln p_j^*, \quad \text{where } \ln P^* = \alpha_0 + \sum_i \alpha_i \ln p_i^* + \frac{1}{2}\sum_i \sum_j \theta_{ij} \ln p_i^* \ln p_j^*$$

Using the previous subsection results we can state the market share of the new model as:

$$w_i = (1-\Sigma\gamma_j s_j)w_i^* + \gamma_i s_i$$

This can be simplified with the following operations:

$$w_i = (1-\Sigma\gamma_j s_j)[\alpha_i + \phi_i \ln(m^*/P^*) + \sum_j \theta_{ij} \ln p_j^*] + \gamma_i s_i$$

$$= (1-\Sigma\gamma_j s_j)\{\alpha_i + \phi_i \ln[(m/\widetilde{P})^{\frac{1}{1-\Sigma\gamma_j s_j}} / P^*)] + \sum_j \theta_{ij} \ln p_j^*\} + \gamma_i s_i$$

$$= (1-\Sigma\gamma_j s_j)[\alpha_i + \frac{\phi_i}{1-\Sigma\gamma_j s_j}\ln(m/\widetilde{P}) - \phi_i \ln P^* + \sum_j \theta_{ij} \ln p_j^*] + \gamma_i s_i$$

$$= \alpha_i(1-\Sigma\gamma_j s_j) + \phi_i \ln(m/\widetilde{P}) - \phi_i(1-\Sigma\gamma_j s_j)\ln P^* + (1-\Sigma\gamma_j s_j)\sum_j \theta_{ij} \ln p_j^* + \gamma_i s_i$$

$$= \alpha_i(1-\Sigma\gamma_j s_j) + \phi_i \ln m - \phi_i \sum_j \gamma_j s_j \ln p_j - \phi_i(1-\Sigma\gamma_j s_j)\ln P^* + (1-\Sigma\gamma_j s_j)\sum_j \theta_{ij}(\ln p_j + \beta_j s_j) + \gamma_i s_i$$

$$= \alpha_i(1-\Sigma\gamma_j s_j) + \phi_i \ln(m/\hat{P}) + (1-\Sigma\gamma_j s_j)\sum_j \theta_{ij}(\ln p_j + \beta_j s_j) + \gamma_i s_i$$

where $\ln \hat{P} = \sum_j \gamma_j s_j \ln p_j + (1 - \Sigma \gamma_j s_j) \ln P^*$ and

$$\ln P^* = \alpha_0 + \sum_i \alpha_i (\ln p_i + \beta_i s_i) + \frac{1}{2} \sum_i \sum_j \theta_{ij} (\ln p_i + \beta_i s_i)(\ln p_j + \beta_j s_j)$$

Notice that the price index of this new modifying form is an average of a price index based upon promoted prices and the original price index with modified prices.

Appendix C: Deriving the Elasticity Structure

First we begin with the market shares:

$$w_i = (1-\Sigma\gamma_j s_j)w_i^* + \gamma_i s_i = (1-\Sigma\gamma_j s_j)(w_i^* + \frac{\gamma_i s_i}{1-\Sigma\gamma_j s_j})$$

$$\ln w_i = \ln p_i + \ln q_i - \ln m = \ln(1-\Sigma\gamma_j s_j) + \ln(w_i^* + \frac{\gamma_i s_i}{1-\Sigma\gamma_j s_j})$$

$$\ln p_i + \ln q_i - \ln m = \ln(1-\Sigma\gamma_j s_j) + \ln\left[\exp\left\{\ln p_i^* + \ln q_i^* - \ln m^*\right\} + \frac{\gamma_i s_i}{1-\Sigma\gamma_j s_j}\right]$$

The total derivative with respect to the natural logarithms is:

$$d\ln w_i = \frac{1}{w_i^* + \frac{\gamma_i s_i}{1-\Sigma\gamma_j s_j}}\left[\exp\left\{\ln q_i^* + \ln p_i^* - \ln m^*\right\}\left(d\ln q_i^* + d\ln p_i^* - d\ln m^*\right)\right]$$

$$d\ln w_i = \frac{w_i^*}{w_i^* + \frac{\gamma_i s_i}{1-\Sigma\gamma_j s_j}}\left(d\ln q_i^* + d\ln p_i^* - d\ln m^*\right)$$

After substitution of $w_i^* = \frac{w_i - \gamma_i s_i}{1-\Sigma\gamma_j s_j}$ we can simplify this further:

$$d\ln w_i = \frac{w_i - \gamma_i s_i}{w_i}(d\ln q_i^* + d\ln p_i^* - d\ln m^*)$$

Other useful relationships we employ are:

$$\ln m^* = \frac{1}{1-\Sigma\gamma_j s_j}(\ln m - \ln \widetilde{P}) \Rightarrow \frac{\partial \ln m^*}{\partial \ln m} = \frac{1}{1-\Sigma\gamma_j s_j}$$

$$w_i = (1-\Sigma\gamma_j s_j)w_i^* + \gamma_i s_i \Rightarrow \frac{\partial w_i}{\partial \gamma_i} = -s_i w_i^* + s_i = (1-w_i^*)s_i$$

C1. Expenditure Elasticity

Denote $\dfrac{\partial \ln q_i}{\partial \ln m} \equiv \mu_i$; $\dfrac{\partial \ln q_i^*}{\partial \ln m^*} \equiv \mu_i^*$

$$\frac{\partial \ln w_i}{\partial \ln m} = \frac{\partial (\ln q_i + \ln p_i - \ln m)}{\partial \ln m} = \frac{\partial \ln q_i}{\partial \ln m} - \frac{\partial \ln m}{\partial \ln m} = \frac{\partial \ln q_i}{\partial \ln m} - 1 = \mu_i - 1$$

$$\frac{\partial \ln q_i}{\partial \ln m} - 1 = \frac{\partial \ln w_i}{\partial \ln m} = \frac{w_i^*}{w_i^* + \frac{\gamma_i s_i}{1-\Sigma \gamma_j s_j}} \left(\frac{\partial \ln q_i^*}{\partial \ln m} - \frac{\partial \ln m^*}{\partial \ln m}\right) = \frac{w_i^*}{w_i^* + \frac{\gamma_i s_i}{1-\Sigma \gamma_j s_j}} \left(\frac{\partial \ln q_i^*}{\partial \ln m^*} \frac{\partial \ln m^*}{\partial \ln m} - \frac{\partial \ln m^*}{\partial \ln m}\right)$$

$$\mu_i - 1 = \frac{w_i^*}{w_i^* + \frac{\gamma_i s_i}{1-\Sigma \gamma_j s_j}} \frac{1}{1-\Sigma \gamma_j s_j} (\mu_i^* - 1)$$

$$\Leftrightarrow \mu_i = \frac{w_i^*}{w_i^* + \frac{\gamma_i s_i}{1-\Sigma \gamma_j s_j}} \frac{1}{1-\Sigma \gamma_j s_j} (\mu_i^* - 1) + 1$$

$$\Leftrightarrow \mu_i = \frac{w_i - \gamma_i s_i}{w_i} \frac{1}{1-\Sigma \gamma_j s_j} (\mu_i^* - 1) + 1$$

C2. Properties of the Expenditure Elasticity (Own Promotion)

Assume that Product i is promoted, so that $s_i = 1$. We can show that

(A) if $\mu_i^* < 1$ then $\mu_i < 1$ and $\mu_i \uparrow$ as $\gamma_i s_i \to 1$;

(B) if $\mu_i^* > 1$ then $\mu_i > 1$ and $\mu_i \downarrow$ as $\gamma_i s_i \to 1$.

The expenditure elasticity is:

$$\mu_i = \frac{w_i - \gamma_i s_i}{w_i} \frac{1}{1-\Sigma \gamma_j s_j} (\mu_i^* - 1) + 1$$

$$\frac{\partial\mu_i}{\partial\gamma_i}=[\frac{(\frac{\partial w_i}{\partial\gamma_i}-s_i)w_i-\frac{\partial w_i}{\partial\gamma_i}(w_i-\gamma_i s_i)}{w_i^2}\frac{1}{1-\Sigma\gamma_j s_j}-\frac{w_i-\gamma_i s_i}{w_i}\frac{s_i}{(1-\Sigma\gamma_j s_j)^2}](\mu_i^*-1)=$$

$$=[\frac{-s_i w_i+\frac{\partial w_i}{\partial\gamma_i}\gamma_i s_i}{w_i}-\frac{(w_i-\gamma_i s_i)s_i}{(1-\Sigma\gamma_j s_j)}]\frac{(\mu_i^*-1)}{w_i(1-\Sigma\gamma_j s_j)}=[-s_i+\frac{\partial w_i}{\partial\gamma_i}\frac{\gamma_i}{w_i}s_i-\frac{(w_i-\gamma_i)s_i}{(1-\Sigma\gamma_j s_j)}]\frac{(\mu_i^*-1)}{w_i(1-\Sigma\gamma_j s_j)}$$

Substituting $s_i=1$,

$$\frac{\partial\mu_i}{\partial\gamma_i}=[-1+\frac{\partial w_i}{\partial\gamma_i}\frac{\gamma_i}{w_i}-\frac{(w_i-\gamma_i)}{(1-\Sigma\gamma_j s_j)}]\frac{(\mu_i^*-1)}{w_i(1-\Sigma\gamma_j s_j)}=[1-\frac{\partial w_i}{\partial\gamma_i}\frac{\gamma_i}{w_i}+\frac{(w_i-\gamma_i)}{(1-\Sigma\gamma_j s_j)}]\frac{(1-\mu_i^*)}{w_i(1-\Sigma\gamma_j s_j)}$$

Let's consider the first condition:

(A) Let $\mu_i^*<1$ so that $\mu_i^*-1<0$, or $1-\mu_i^*>0$.

From $w_i=(1-\Sigma\gamma_j s_j)w_i^*+\gamma_i s_i$, $(1-\Sigma\gamma_j s_j)>0$, and $w_i^*>0$

we obtain $w_i>\gamma_i$. Hence, $\frac{(w_i-\gamma_i)}{(1-\Sigma\gamma_j s_j)}>0$.

Next, from $\mu_i=\frac{w_i-\gamma_i}{w_i}\frac{1}{1-\Sigma\gamma_j s_j}(\mu_i^*-1)+1$, note that $\frac{w_i-\gamma_i}{w_i}\frac{1}{1-\Sigma\gamma_j s_j}(\mu_i^*-1)<0$ as $\mu_i^*<1$;

$\Rightarrow\mu_i<1$.

We also know that $w_i(1-\Sigma\gamma_j s_j)>0$.

Note that $\frac{\partial w_i}{\partial\gamma_i}\frac{\gamma_i}{w_i}=(1-w_i^*)\frac{\gamma_i}{w_i}$,

$\Rightarrow 0<\frac{\partial w_i}{\partial\gamma_i}\frac{\gamma_i}{w_i}<1$ as $0<(1-w_i^*)<1$ and $0<\frac{\gamma_i}{w_i}<1$.

Also, $1+\frac{(w_i-\gamma_i)}{(1-\Sigma\gamma_j s_j)}>1$. Then $[1-\frac{\partial w_i}{\partial\gamma_i}\frac{\gamma_i}{w_i}+\frac{(w_i-\gamma_i)}{(1-\Sigma\gamma_j s_j)}]$ is always positive, and the direction of change in the own expenditure elasticity will be determined by the sign of $(1-\mu_i^*)$.

$\Rightarrow$ For $1-\mu_i^*>0$, μ_i will be increasing with γ_i, or $\frac{\partial\mu_i}{\partial\gamma_i}>0$.

Notice that if $\mu_i^*=1$, then $\frac{\partial\mu_i}{\partial\gamma_i}=0$, therefore μ_i as a function of γ_i will reach an inflection (saddle) point at $\mu_i^*=1$. However, generally μ_i will not have an extremum at $\mu_i^*=1$.

$\Rightarrow$ if $\mu_i^*<1$ then $\mu_i<1$, and μ_i increases as $\gamma_i s_i \rightarrow 1$;

(B) Let $\mu_i^*>1$ so that $\mu_i^*-1>0$, or $1-\mu_i^*<0$.

From $\mu_i=\frac{w_i-\gamma_i}{w_i}\frac{1}{1-\Sigma\gamma_j s_j}(\mu_i^*-1)+1$, note that $\mu_i>1$ if $\mu_i^*>1$.

Further, analogous to the reasoning above, for $1-\mu_i^*<0$ we conclude that the own expenditure elasticity is decreasing with μ_i, or $\frac{\partial\mu_i}{\partial\gamma_i}<0$.

$\Rightarrow$ if $\mu_i^*>1$ then $\mu_i>1$ and μ_i decreases as $\gamma_i s_i \rightarrow 1$.

Notice that if $\mu_i^*=1$, then $\frac{\partial\mu_i}{\partial\gamma_i}=0$, therefore μ_i as a function of γ_i will reach an inflection saddle) point at $\mu_i^*=1$. However, generally μ_i will not have an extremum at $\mu_i^*=1$.

C3. Properties of the Expenditure Elasticity (Cross Promotion)

If Product i is not promoted, $s_i=0$, and $\mu_i=\frac{1}{1-\Sigma\gamma_j s_j}(\mu_i^*-1)+1$.

Also, note that $0 < (1-\Sigma\gamma_j s_j) < 1,$ so that $\dfrac{1}{1-\Sigma\gamma_j s_j} > 1.$

If $\mu_i^* > 1 \Rightarrow \mu_i^* - 1 > 0 \Rightarrow \mu_i = \dfrac{1}{1-\Sigma\gamma_j s_j}(\mu_i^* - 1) + 1 > \mu_i^* - 1 + 1,$

$\Rightarrow \mu_i > \mu_i^* > 1$, or superior products become more superior.

Conversely, if $\mu_i^* < 1 \Rightarrow \mu_i^* - 1 < 0 \Rightarrow \mu_i = \dfrac{1}{1-\Sigma\gamma_j s_j}(\mu_i^* - 1) + 1 < \mu_i^* - 1 + 1,$

$\Rightarrow \mu_i < \mu_i^* < 1$, or inferior products become more inferior.

Numerical examples:

Let $\begin{cases} \mu_1^* = 1.2; & w_1^* = 0.8; \quad \gamma_1 = 0.5; \\ \mu_2^* = 0.2; & w_2^* = 0.2; \end{cases}$

Then, from $\mu_i = \dfrac{w_i^*}{w_i^* + \dfrac{\gamma_i s_i}{1-\Sigma\gamma_j s_j}} \dfrac{1}{1-\Sigma\gamma_j s_j}(\mu_i^* - 1) + 1$ and $w_i = (1-\Sigma\gamma_j s_j)w_i^* + \gamma_i s_i$ we obtain

$\begin{cases} \mu_1 = 1.178; & w_1 = 0.9; \\ \mu_2 = -0.6; & w_2 = 0.1; \end{cases}$

Example of the expenditure elasticity for the Cobb-Douglas Case:

$$q_i = \frac{m}{2p_i}(1 + \gamma_i s_i - \gamma_j s_j)$$

$$\Rightarrow \ln q_i = \ln m + \ln(1 + \gamma_i s_i - \gamma_j s_j) - \ln 2 - \ln p_i$$

$$\frac{\partial \ln q_i}{\partial \ln m} = 1; \quad \frac{\partial \ln q_i}{\partial \ln p_i} = -1$$

C4. Uncompensated Cross-Price Elasticity

First we use the definition of the cross price-elasticities:

$$\eta_{ij} = \frac{\partial \ln q_i}{\partial \ln p_i}, \ \ \eta_{ij}^* = \frac{\partial \ln q_i^*}{\partial \ln p_i^*},$$

$$\delta_{ij} = 1 \text{ if } i = j, \delta_{ij} = 0 \text{ if } i \neq j.$$

Recall that:

$$\ln p_i^* = \ \beta_i s_i + \ln p_i$$

$$\ln m^* = \frac{1}{1 - \Sigma \gamma_j s_j}(\ln m - \ln \widetilde{P})$$

$$\ln \widetilde{P} = \sum_j \gamma_j s_j \ln p_j$$

Then, $\dfrac{\partial \ln p_i^*}{\partial \ln p_i} = \delta_{ij}$; $\dfrac{\partial \ln \widetilde{P}}{\partial \ln p_j} = \gamma_j s_j$; $\dfrac{\partial \ln m^*}{\partial \ln p_j} = \dfrac{-\gamma_j s_j}{1 - \Sigma \gamma_j s_j}$.

$$\frac{\partial \ln q_i^*}{\partial \ln p_j} = \frac{\partial \ln q_i^*}{\partial \ln p_j^*}\frac{\partial \ln p_j^*}{\partial \ln p_j} + \frac{\partial \ln q_i^*}{\partial \ln m^*}\frac{\partial \ln m^*}{\partial \ln p_j} = \ \eta_{ij}^* - \mu_i^* \frac{\gamma_j s_j}{1 - \Sigma \gamma_j s_j}$$

From the total derivative we have:

$$\frac{\partial \ln w_i}{\partial \ln p_j} = \frac{\partial(\ln q_i + \ln p_i - \ln m)}{\partial \ln p_j} = \frac{\partial \ln q_i}{\partial \ln p_j} + \frac{\partial \ln p_i}{\partial \ln p_j} - \frac{\partial \ln m}{\partial \ln p_j} = \frac{\partial \ln q_i}{\partial \ln p_j} + \delta_{ij}$$

$$\frac{\partial \ln q_i}{\partial \ln p_j} + \delta_{ij} = \frac{\partial \ln w_i}{\partial \ln p_j} = \frac{w_i^*}{w_i^* + \dfrac{\gamma_i s_i}{1 - \Sigma \gamma_j s_j}}\left(\frac{\partial \ln q_i^*}{\partial \ln p_j} + \frac{\partial \ln p_i^*}{\partial \ln p_j} - \frac{\partial \ln m^*}{\partial \ln p_j}\right).$$

$$\eta_{ij} + \delta_{ij} = \frac{w_i^*}{w_i^* + \dfrac{\gamma_i s_i}{1 - \Sigma \gamma_j s_j}}(\eta_{ij}^* - \mu_i^* \frac{\gamma_j s_j}{1 - \Sigma \gamma_j s_j} + \delta_{ij} + \frac{\gamma_j s_j}{1 - \Sigma \gamma_j s_j}) = \frac{w_i^*}{w_i^* + \dfrac{\gamma_i s_i}{1 - \Sigma \gamma_j s_j}}[\eta_{ij}^* + \delta_{ij} + (1 - \mu_i^*)\frac{\gamma_j s_j}{1 - \Sigma \gamma_j s_j}] =$$

$$= \frac{(1-\Sigma\gamma_j s_j)w_i^*}{(1-\Sigma\gamma_j s_j)w_i^* + \gamma_i s_i}[\eta_{ij}^* + \delta_{ij} + (1-\mu_i^*)\frac{\gamma_j s_j}{1-\Sigma\gamma_j s_j}] = \frac{w_i^*}{w_i}[(1-\Sigma\gamma_j s_j)(\eta_{ij}^* + \delta_{ij}) + (1-\mu_i^*)\gamma_j s_j];$$

$$\eta_{ij} + \delta_{ij} = \frac{w_i - \gamma_i s_i}{w_i}[(\eta_{ij}^* + \delta_{ij}) + (1-\mu_i^*)\frac{\gamma_j s_j}{1-\Sigma\gamma_j s_j}] = (1-\frac{\gamma_i s_i}{w_i})(\eta_{ij}^* + \delta_{ij}) + (1-\frac{\gamma_i s_i}{w_i})(1-\mu_i^*)\frac{\gamma_j s_j}{1-\Sigma\gamma_j s_j}.$$

We know that $\mu_i^* - 1 = \dfrac{(\mu_i - 1)}{(1-\frac{\gamma_i s_i}{w_i})\frac{1}{1-\Sigma\gamma_j s_j}}$,

$$\Leftrightarrow \; \eta_{ij} + \delta_{ij} = (1-\frac{\gamma_i s_i}{w_i})(\eta_{ij}^* + \delta_{ij}) - (\mu_i - 1)\gamma_j s_j$$

C5. Compensated Cross-Price Elasticities

The relationship between the compensated and uncompensated elasticities are:

$$\varepsilon_{ij} = \eta_{ij} + \mu_i w_j \Leftrightarrow \eta_{ij} = \varepsilon_{ij} - \mu_i w_j$$

We can simplify this relationship further:

$$\eta_{ij} + \delta_{ij} = (1-\frac{\gamma_i s_i}{w_i})(\eta_{ij}^* + \delta_{ij}) - (\mu_i - 1)\gamma_j s_j$$

$$\varepsilon_{ij} - \mu_i w_j + \delta_{ij} = (1-\frac{\gamma_i s_i}{w_i})(\eta_{ij}^* + \delta_{ij}) - (\mu_i - 1)\gamma_j s_j$$

$$\varepsilon_{ij} - \mu_i w_j + \delta_{ij} = (1-\frac{\gamma_i s_i}{w_i})(\varepsilon_{ij}^* - \mu_i^* w_j^* + \delta_{ij}) - (\mu_i - 1)\gamma_j s_j$$

$$\varepsilon_{ij} + \delta_{ij} = (1-\frac{\gamma_i s_i}{w_i})(\varepsilon_{ij}^* + \delta_{ij}) - (1-\frac{\gamma_i s_i}{w_i})\mu_i^* w_j^* + \mu_i w_j - (\mu_i - 1)\gamma_j s_j$$

Substituting with $\mu_i^* = \dfrac{(\mu_i - 1)(1-\Sigma\gamma_j s_j)}{(1-\frac{\gamma_i s_i}{w_i})} + 1$ and $w_j^* = \dfrac{w_j - \gamma_j s_j}{(1-\Sigma\gamma_j s_j)}$ yields

$$\varepsilon_{ij} + \delta_{ij} = (1 - \frac{\gamma_i s_i}{w_i})(\varepsilon_{ij}^* + \delta_{ij}) - (1 - \frac{\gamma_i s_i}{w_i})[\frac{(\mu_i - 1)(1 - \Sigma\gamma_j s_j)}{(1 - \frac{\gamma_i s_i}{w_i})} + 1]\frac{w_j - \gamma_j s_j}{(1 - \Sigma\gamma_j s_j)} + \mu_i w_j - (\mu_i - 1)\gamma_j s_j$$

$$\varepsilon_{ij} + \delta_{ij} = (1 - \frac{\gamma_i s_i}{w_i})(\varepsilon_{ij}^* + \delta_{ij}) - (\mu_i - 1)(w_j - \gamma_j s_j) - \frac{(w_i - \gamma_i s_i)(w_j - \gamma_j s_j)}{w_i(1 - \Sigma\gamma_j s_j)} + \mu_i w_j - (\mu_i - 1)\gamma_j s_j$$

$$\varepsilon_{ij} + \delta_{ij} = (1 - \frac{\gamma_i s_i}{w_i})(\varepsilon_{ij}^* + \delta_{ij}) + w_j - \frac{(w_i - \gamma_i s_i)(w_j - \gamma_j s_j)}{w_i(1 - \Sigma\gamma_j s_j)}.$$