

When is a Monotone Function Cyclically Monotone?

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Abstract

We provide sufficient conditions for a monotone function with a finite set of outcomes to be cyclically monotone. Using these conditions, we show that any monotone function defined on the domain of gross substitutes is cyclically monotone. The result also extends to the domain of generalized gross substitutes and complements.

Keywords: *monotone, cyclically monotone, non-convex domain, gross substitutes, gross substitutes and complements, mechanism design, algebraic topology, homology, nerve theorem*

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1 Introduction

One of the major goals of mechanism design is to study the properties of optimal mechanisms maximizing a given objective such as revenue or welfare maximization. The difficulty in deriving such mechanisms results from the designer lacking information about the agents' preferences. Hence, a well-designed mechanism should take into account the agents' ability to hide their privately held information, often called *incentive compatibility constraints*.

Previous work provides important insights into these constraints. [Myerson \(1981\)](#) showed that in standard private value settings with one-dimensional types, any *non-decreasing* allocation rule can be implemented; that is, there exists a payment rule that when combined with the allocation rule produces a direct mechanism where truth-telling is in the best interests of the agents. In multi-dimensional settings, [Rochet \(1987\)](#) showed that an allocation rule is implementable if and only if it is *cyclically monotone*. To define a cyclically monotone allocation rule $f : \mathbb{R}^N \rightarrow \mathbb{R}^N$, $N \geq 1$ consider a weighted graph with points in the domain of f being vertices and directed edges from any point t to any other point t' . With each directed edge, we associate weight $t \cdot (f(t) - f(t'))$. Allocation f is then cyclically monotone if along any cycle the sum of edge weights is non-negative. If the weight of any cycle with two edges is non-negative, the allocation rule satisfies a weaker condition called *monotone*.

Though the Rochet's characterization and its modifications have been successfully used in auction theory, computer science, and matching theory, it is often tedious to verify.¹ An important contribution by [Saks and Yu \(2005\)](#) (also [Bikhchandani et al., 2006](#)) establishes an equivalence between cyclically monotone and monotone conditions for convex domains with a finite set of outcomes. Their result greatly simplifies checking whether an allocation rule is implementable.²

Does the equivalence result extend to non-convex domains? [Ashlagi et al. \(2010\)](#) showed that Saks and Yu's result cannot be extended beyond domains with convex closure if one requires equivalence between two conditions for every finite-valued randomized allocation. However, randomized allocations might not always be plausible.

¹See [Lavi and Swamy \(2009\)](#), [Mishra and Roy \(2013\)](#), and [Carbajal and Mu'alem \(2020\)](#). The cyclic monotonicity condition has also applications in revealed preference theory, producer theory, and spatial allocation (see [Chambers and Echenique, 2018](#); [Kushnir and Lokutsievskiy, 2019](#)).

²The result with a similar flavor for environments without transfers also appeared in [Pycia \(2012\)](#).

Further, [Ashlagi et al. \(2010\)](#)’s result does not preclude equivalence between monotone and cyclically monotone conditions on non-convex domains for a given set of possible outcomes.

In this paper, we provide sufficient conditions on *a domain*, on *a set of possible outcomes*, and on *a function* that guarantee that if the function is monotone, then it is also cyclically monotone. Our two main conditions require the domain to be *simply connected* and the function to satisfy the *local-to-global* condition. The former condition ensures that the domain does not contain “holes” of a certain type. The latter condition ensures that if a function is a solution to a local optimization problem, it also delivers the global optimal.³

We then apply these conditions to study deterministic demand functions on the domain of *gross substitutes*. The domain of gross substitutes is an important non-convex domain of agent preferences that has been extensively exploited in the mechanism design, matching, equilibrium, and algorithmic literatures (e.g., [Ausubel and Milgrom, 2002](#); [Roth, 1991](#); [Gul and Stacchetti, 1999](#); [Paes Leme, 2017](#)). We establish that any monotone demand function defined on the domain of *gross substitutes* is also cyclically monotone. We further extend the equivalence between monotone and cyclically monotone conditions to the domain of *generalized gross substitutes and complements*, the domain that allows for multiple objects of the same type and some complementarities across objects ([Sun and Yang, 2006](#); [Shioura and Yang, 2015](#)).

On the methodological side, we introduce some novel techniques to economics. The proof of our main result uses a version of the *Nerve theorem*—a classical result in algebraic topology (see [Björner, 1995](#)). To explain the result, let us consider a set that is covered by a finite system of closed subsets. Nerve is then a special weighted hypergraph associated with this system of subsets. The Nerve theorem helps to map the geometrical properties of the set to the geometrical properties of the nerve. This result could be of special interest to economists working in the areas of mechanism design, social choice, network theory, and operations research.

³As explained later, we also require an additional technical condition.

Related Literature. The cyclic monotonicity condition was introduced by [Rockafellar \(1966\)](#) to characterize the subdifferentials of convex functions. For mechanism design applications, [Rochet \(1987\)](#) was the first to show that in quasi-linear environments an allocation rule can be implemented if and only if it is cyclically monotone. He also drew a parallel between the cyclic monotonicity condition and the strong axiom of revealed preferences (see also [Brown and Calsamiglia, 2007](#); [Makowski and Ostroy, 2013](#)). [Saks and Yu \(2005\)](#) simplified the characterization of implementable allocation rules by establishing that any monotone function is cyclically monotone on convex domains with a finite set of outcomes (see also [Bikhchandani et al., 2006](#)).⁴ Importantly, [Ashlagi et al. \(2010\)](#) showed that Saks and Yu’s characterization cannot be extended beyond domains with convex closure if the equivalence is required to hold for all finite-valued randomized allocation rules. For an infinite set of outcomes, [Müller et al. \(2007\)](#), [Archer and Kleinberg \(2014\)](#), and [Carbajal and Müller \(2015, 2017\)](#) provided various additional conditions to guarantee that the cyclical monotonicity condition is satisfied.

For non-convex domains, the literature is scarce. For single-peaked preferences, [Mishra et al. \(2014\)](#) showed the equivalence of monotone and cyclically monotone conditions. [Vohra \(2011\)](#) provided an inspiring example of a simple domain with two objects and the agent’s valuation of a bundle of objects equals the maximum value of objects in the bundle. For the setting, he established that any monotone function is also cyclically monotone. This is an example of a domain where every valuation satisfies the gross substitutes condition. Ever since, it has been an open question as to whether the equivalence between the monotone and cyclically monotone conditions can be extended to the whole domain of *gross substitutes*.⁵

The gross substitutes condition was introduced by [Kelso and Crawford \(1982\)](#) in the context of labor matching markets. [Sun and Yang \(2006, 2009\)](#) and [Shioura and Yang \(2015\)](#) extended the gross substitutes condition to allow for some com-

⁴[Jehiel et al. \(1999\)](#) also contains the proof of a geometric lemma that is the main step in [Saks and Yu \(2005\)](#). See also [Cuff et al. \(2012\)](#) and [Edelman and Weymark \(2020\)](#) for the cases when every monotone function is cyclically monotone.

⁵The only progress in that direction was made in a concurrent paper by [Agarwal and Roy \(2019\)](#), who extended [Vohra \(2011\)](#)’s example to the case of an arbitrary number of objects.

[Mishra and Roy \(2013\)](#) also showed that the non-negativity of any three-cycle is sufficient for implementability in dichotomous domains. The conditions of [Carbajal and Müller \(2015, 2017\)](#) also apply formally to non-convex domains.

plementarities and multiple objects of the same type, a domain they referred to as *generalized gross substitutes and complements*. They also designed a dynamic auction for efficiently allocating the objects to the agents. Our most general results in Section 4 apply to the latter domain.

Our main theorem exploits the local-to-global condition that relates local incentive compatibility to global incentive compatibility constraints in convex and non-convex domains. This condition is closely connected to the decomposition monotonicity condition first proposed by Müller et al. (2007) to study Bayesian incentive compatible allocation rules on convex domains.⁶ In a related paper, Archer and Kleinberg (2014) considered convex domains and showed that if a function with a finite or infinite set of outcomes is locally monotone and its loop is integral over every sufficiently small triangle vanishes, then it is also incentive compatible. Carroll (2012) also thoroughly studied local and global incentive compatibility constraints. He showed that local incentive compatibility always implies global incentive compatibility for convex domains *with transferable utility*, the single-peaked preference domain, and the single-crossing domain *without transferable utility*.⁷ Though Carroll (2012) did not study non-convex domains with transferable utilities, one of his geometric characterizations has proved to be very useful for our purposes (see Section 3).

One of our results (Proposition 1) is also closely related to the Helmholtz decomposition of Jiang et al. (2011). Candogan et al. (2011) used these techniques to decompose any finite game into potential, harmonic, and nonstrategic components. In a recent paper, Caradonna (2020) also used the decomposition to analyze when the weak axiom of revealed preferences implies the rationalizability of choice functions.

The paper proceeds as follows. Section 2 introduces notations and definitions. Section 3 presents our main results. We use these results in Section 4 to study functions defined on the domain of gross substitutes and the domain of generalized gross substitutes and complements. Section 5 concludes the paper.

⁶See also Berger et al. (2009) and Berger et al. (2017). The condition is also related to the reverse triangle inequality in Mishra et al. (2014).

⁷See also Gibbard (1977), Mishra et al. (2016), Pycia and Ünver (2015), and Sato (2013) for related results.

2 Notations and Definitions

We begin by introducing some notations and definitions. Then we motivate them from the perspective of mechanism design. Consider a domain $T \subseteq \mathbb{R}^N$, a finite set $A \subset \mathbb{R}^N$ for $N \geq 1$, and some function $f : T \rightarrow A$. The vector product of $t \in T$ and $a \in A$ is denoted as both $t \cdot a$ and ta . We consider two monotonicity conditions.

Definition 1. *Function $f : T \rightarrow A$ is **monotone** if for all $t, t' \in T$,*

$$t(f(t) - f(t')) + t'(f(t') - f(t)) \geq 0. \quad (1)$$

This is a generalization of the one-dimensional monotonicity condition to multi-dimensional settings. We use the term *monotone function* following [Rockafellar \(1966\)](#). Some recent papers also call such functions *weakly monotone* (see, e.g., [Bikhchandani et al., 2006](#)). Our second and more demanding condition defined as follows.

Definition 2. *Function $f : T \rightarrow A$ is **cyclically monotone** if for any integer M and any points $t^0, t^1, \dots, t^M = t^0$ in T ,*

$$\sum_{k=0}^{M-1} t^k(f(t^k) - f(t^{k+1})) \geq 0. \quad (2)$$

As we mentioned in the introduction, both definitions could be conveniently interpreted using graph theory. Consider a weighted graph with points $t \in T$ being vertices and directed edges from any point t to any other point t' . With each edge, we associate weight $t(f(t) - f(t'))$. Hence, if f is cyclically monotone, then the weight of any cycle has to be non-negative. If f is monotone, then the above condition is restricted to cycles of length two. Note that in order to check whether f is monotone, we need to verify only inequality (1). At the same time, we need to verify a system of inequalities (2) for all integers M to check whether f is cyclically monotone. The latter is a much more demanding task.

For any $f : T \rightarrow A$, we also consider a cover of T by a finite number of subsets.

To define these subsets for any ordered pair $a, b \in A$, we define lower bound

$$\ell_{ab} = \inf_{t \in T: f(t)=a} t(a-b).$$

Using the lower bounds, we construct a cover $\{T_a^f\}_{a \in A}$ of set T , where for each $a \in A$

$$T_a^f = \{t \in T : t(a-b) \geq \ell_{ab}, \forall b \in A\}.$$

Note that T_a^f depends on the choice of function f and $T = \bigcup_{a \in A} T_a^f$. In addition, if outcome $a \notin f(T)$, then we have $\ell_{ab} = +\infty$ and T_a^f is the empty set. For all other outcomes, each set T_a^f is non-empty and contains the set of points that leads to outcome a , i.e., $\{t \in T, f(t) = a\} \subseteq T_a^f$.

In our analysis, we study functions that lead to path-connected subsets T_a^f and functions that are defined on a simply connected domain T . Set T_a^f is *path-connected* if it is non-empty and any two points $x \in T_a^f$ and $y \in T_a^f$ can be connected with a continuous curve lying inside T_a^f . A domain T is *simply connected* if it is path-connected and any loop in T can be continuously contracted to a point.⁸ For example, a triangle without an interior is not simply connected (see Example 1). At the same time, any set with a point that can be connected to each of the set's other points with a line segment within the set is simply connected. Such a set is called *star-shaped* (or *star-convex*).

For a closed line segment connecting two points $x, y \in \mathbb{R}^N$, we use the standard notation $[x, y] = \{z \in \mathbb{R}^N : z = \alpha x + (1 - \alpha)y, \alpha \in [0, 1]\}$. We also employ $[x, y), (x, y]$, and (x, y) throughout the paper depending on whether the boundary points are included. Finally, the following property will be helpful in our analysis.

Definition 3. *Function $f : T \rightarrow A$ satisfies the **local-to-global** condition if for any two outcomes $a, b \in f(T)$ with $T_a^f \cap T_b^f = \emptyset$, there exists a path $\{a \equiv a_0, \dots, a_M \equiv b\}$ such that $T_{a_m}^f \cap T_{a_{m+1}}^f \neq \emptyset$, $m = 0, \dots, M - 1$, and $\ell_{ab} \geq \sum_{m=0}^{M-1} \ell_{a_m a_{m+1}}$.*

The local-to-global condition can be most accurately interpreted through the prism of mechanism design and we slightly postpone its discussion (see p. 9). Here, we

⁸A formal definition of domain T being simply connected is as follows. Let S^1 denote a circle (in \mathbb{R}^2). Then, for any continuous function (a loop) $\gamma : S^1 \rightarrow T$, there must exist a continuous function $F : [0, 1] \times S^1 \rightarrow T$ such that for all $s \in [0, 1]$, $F(0, s) \equiv \gamma(s)$ and $F(1, s) \equiv t_0$ for some $t_0 \in T$.

mention only one of our results related to the condition. At first glance, verifying the local-to-global condition might require a significant effort. However, we show that it is not the case. We present a sufficient geometric property that ensures a monotone function f satisfies the local-to-global condition (see Lemma A1). In particular, the geometric property requires that for any $a, b \in f(T)$, any $x \in T_a^f$ there should exist type $y \in T_b^f$ such that $[x, y] \subset T$. In Section 4, we show that the geometric property is satisfied for any monotone function defined on some important economic domains.

Mechanism design. For mechanism design applications, one could think of N as the number of outcomes, and T as the set of agent types. Type $t \in T$ can be interpreted as a vector of agent’s valuations for all possible outcomes. With each outcome, we associate an indicator $a \in \{0, 1\}^N$ that has one component equal to 1 and all other components equal to 0. The union of these indicators is then a finite set $A \subset \mathbb{R}^N$. The agent’s utility from $a \in A$ can conveniently be written then as $u(t, a, p) = t \cdot a - p$, where p is the agent’s payment.

We consider direct mechanisms characterized by two functions: an allocation rule, $f : T \rightarrow A$, mapping an agent’s reported type to the set of possible outcomes, and a payment rule, $p : T \rightarrow \mathbb{R}$, mapping an agent’s reported type to the set of real numbers. We consider only deterministic allocation rules and do not allow randomizations over outcomes.⁹ We can then write the agent’s utility as

$$tf(t') - p(t')$$

where t' and t refer to the agent’s reported and true types, respectively. We call allocation rule f *implementable* if there exists a payment rule p such that mechanism (f, p) is incentive compatible; that is, if it satisfies the following constraints:

$$tf(t) - p(t) \geq tf(t') - p(t') \quad \forall t, t' \in T.$$

Rochet (1987) proved an important result characterizing the set of implementable allocations as stated in the following theorem.

⁹See Ashlagi et al. (2010) for the study of randomized mechanisms.

Theorem (Rochet (1987)). *An allocation rule is implementable if and only if it is cyclically monotone.*

Though the cyclic monotonicity condition characterizes the set of all implementable allocation rules, this condition is often tedious to verify. Remarkably, Saks and Yu (2005) showed that for convex domains it is enough to check that only two cycles are non-negative. Saks and Yu (2005)’s characterization and its modifications have been successfully used in several important applications (see Lavi and Swamy, 2009; Mishra and Roy, 2013; Carbajal and Mu’alem, 2020; Shi et al., 2018). One of our main results extends Saks and Yu (2005)’s result to important non-convex domains including the domain of gross substitutes (see Section 4).

Before proceeding to our main results, we discuss the interpretation of the lower bounds ℓ_{ab} and the local-to-global condition (Definition 3). Let us consider all agent types that lead to outcome $a \in A$. Lower bound ℓ_{ab} then corresponds to the lowest benefit from revealing its true type compared to lying when lying leads to outcome $b \in A$ (excluding transfers). For a monotone allocation, set T_a^f almost coincides with the set of types that lead to outcome $a \in A$ (up to the boundary points).

The local-to-global condition can then be interpreted as a condition that ensures local incentive compatibility implies global incentive compatibility (see Archer and Kleinberg, 2014; Carroll, 2012). Lower bound ℓ_{ab} is the lowest benefit from revealing true type compared to lying when lying leads to outcome b . Hence, $-\ell_{ab}$ can be regarded as the maximum gains from lying. We interpret $-\ell_{ab}$, $T_a^f \cap T_b^f = \emptyset$ as the gains from global deviations and $-\ell_{ab}$, $T_a^f \cap T_b^f \neq \emptyset$ as the gains from local deviations. Then, the local-to-global condition ensures that the gains from global deviations are smaller than the total gains from deviations along the path connecting $t \in T_a^f$ and some type in T_b^f ; i.e., *local incentive compatibility* implies *global incentive compatibility*. Similar conditions have been considered in the literature before (see Müller et al., 2007; Mishra et al., 2014). The main difference is that our condition is also applicable to non-convex domains.

3 Main Result

The main result of this section, Theorem 1, provides a set of conditions on a domain T , a set A , and a function $f : T \rightarrow A$ that ensure that if f is monotone, it is also cyclically monotone. These conditions are simplified in Corollary 1. The result of Corollary 1 is then used to analyze monotone demand functions on the domain of gross substitutes and the domain of generalized gross substitutes and complements in Section 4.

Theorem 1 (Main result). *Consider a domain $T \subset \mathbb{R}^N$, a finite set $A \subset \mathbb{R}^N$, and a function $f : T \rightarrow A$. Suppose that*

- 1) T is simply connected,
- 2) T_a^f is either path-connected or empty for each $a \in A$, and
- 3) f satisfies the local-to-global condition.

Then if f is monotone, it is also cyclically monotone.

Discussion: The *simply connected* condition is satisfied for most economically relevant models.¹⁰ It ensures that domain T does not contain “holes” of a certain type. The condition on sets T_a^f is technical. For instance, any star-shaped set is path-connected. To check that a monotone function f satisfies the local-to-global condition, we will show that it is enough for f to satisfy the following geometric property: for any $a, b \in f(T)$ and any $x \in T_a^f$ there should exist type $y \in T_b^f$ such that $[x, y] \subset T$ (see Lemma A1). This condition was originally proposed by Carroll (2012) to show that local incentive compatibility implies global incentive compatibility in single-peaked preferences settings *without transfers*.

Proof. First, we establish that it is sufficient to prove the statement for f such that $f(T) = A$. Indeed, if we prove the statement under this assumption, then for an arbitrary $f : T \rightarrow A$ satisfying the theorem conditions, we put $A' = f(T)$ and apply the established result for $f : T \rightarrow A'$. Hence, without loss of generality, we assume from this point on that $f(T) = A$ and, hence, $T_a^f \neq \emptyset$ for each $a \in A$.

The proof of Theorem 1 is based on graph theory and algebraic topology. We associate two graphs with set A . The first graph Γ is the complete directed graph

¹⁰One exception is a circular domain in monopolistic competition models (Salop, 1979).

with vertices corresponding to each outcome in A and directed edges connecting every ordered pair of vertices. To distinguish vertices from outcomes, for each $a, b, c \in A$, we denote the corresponding vertices in Γ by Gothic letters \mathbf{a} , \mathbf{b} , and \mathbf{c} , respectively, and the set of vertices by A_0 . Each directed edge $\mathbf{a} \rightarrow \mathbf{b}$ in Γ has weight equal to ℓ_{ab} .

The second graph Γ^n is a subgraph of Γ with the same set of vertices A_0 , but directed edges connecting only adjacent outcomes; i.e., \mathbf{a} and \mathbf{b} are connected (by both directed edges) in Γ^n if and only if $T_a^f \cap T_b^f \neq \emptyset$. The subgraph's directed edges still have weight ℓ_{ab} . We call graph Γ^n a *neighborhood subgraph*.

We first notice that Definitions 1 and 2 can be reformulated using the weights of cycles in Γ .¹¹ We say that Γ is cyclically monotone if any M -cycle $\mathbf{a}_0 \rightarrow \mathbf{a}_1 \rightarrow \dots \rightarrow \mathbf{a}_{M-1} \rightarrow \mathbf{a}_M \equiv \mathbf{a}_0$ with $\mathbf{a}_i \in A_0$ for $i = 0, \dots, M-1$ and $M \geq 2$ has non-negative weight, i.e., $\sum_{m=0}^{M-1} \ell_{\mathbf{a}_m \mathbf{a}_{m+1}} \geq 0$. On the other hand, we say that Γ is monotone if any 2-cycle has non-negative weight. It is straightforward to verify that f is monotone (cyclically monotone) if and only if Γ is monotone (cyclically monotone) (e.g., [Heydenreich et al., 2009](#)). Therefore, in order to prove that f is cyclically monotone, it is enough to establish that Γ is cyclically monotone.

To prove that Γ is cyclically monotone, we first establish that all cycles in the neighborhood subgraph Γ^n have exactly zero weight using the conditions that domain T is *simply connected* and sets T_a^f are *path-connected* for $a \in A$. Then, we will show that the *local-to-global* condition implies that all cycles in Γ have non-negative weight.

Proposition 1. *If conditions 1) and 2) are satisfied and f is monotone, then any cycle in the neighborhood subgraph Γ^n has exactly zero weight.*

Proof. We first establish two simple facts about subgraph Γ^n . Consider some directed edge $\mathbf{a} \rightarrow \mathbf{b}$ in Γ^n . As f is monotone, graph Γ is also monotone and $\ell_{ba} + \ell_{ab} \geq 0$. At the same time, as $\mathbf{a} \rightarrow \mathbf{b}$ is in Γ^n , there exists $t \in T_a^f \cap T_b^f$ and $t(a-b) \geq \ell_{ab}$ and $t(b-a) \geq \ell_{ba}$ by definition of T_a^f and T_b^f . Hence, $\ell_{ab} + \ell_{ba} \leq 0$. Overall, we have $\ell_{ba} + \ell_{ab} = 0$. Hence, we obtain the following fact.

Fact 1. *If $T_a^f \cap T_b^f \neq \emptyset$, then $\ell_{ab} = -\ell_{ba}$ and $\ell_{ab} = t(a-b)$ for any $t \in T_a^f \cap T_b^f$.*

Second, let us show that if $T_a^f \cap T_b^f \cap T_c^f \neq \emptyset$, then 3-cycle $\mathbf{a} \rightarrow \mathbf{b} \rightarrow \mathbf{c} \rightarrow \mathbf{a}$ in Γ^n has 0 weight. Indeed, for $t \in T_a^f \cap T_b^f \cap T_c^f$ we have $\ell_{ab} = t(a-b)$, $\ell_{bc} = t(b-c)$ and

¹¹The weight of a cycle (or any path) in a graph is the sum of the weights of its directed edges.

$\ell_{ca} = t(c - a)$ by Fact 1. Hence, $\ell_{ab} + \ell_{bc} + \ell_{ca} = 0$. This result is summarized below.

Fact 2. *If $T_a^f \cap T_b^f \cap T_c^f \neq \emptyset$, then $\ell_{ab} + \ell_{bc} + \ell_{ca} = 0$.*

We now investigate whether it is possible to “pave” any cycle in Γ^n by triplets satisfying Fact 2. For example, if $T_a^f \cap T_b^f \cap T_c^f \neq \emptyset$ and $T_a^f \cap T_{b'}^f \cap T_c^f \neq \emptyset$, then 4-cycle $\mathbf{a} \rightarrow \mathbf{b} \rightarrow \mathbf{c} \rightarrow \mathbf{b}' \rightarrow \mathbf{a}$ has zero weight. Indeed, Facts 1 and 2 imply that

$$\ell_{ab} + \ell_{bc} + \ell_{cb'} + \ell_{b'a} = (\ell_{ab} + \ell_{bc} + \ell_{ca}) + (\ell_{ac} + \ell_{cb'} + \ell_{b'a}) = 0.$$

For the general case, we consider a construction in topology called the *nerve* of a cover $\cup_{a \in A} T_a^f$.¹² Nerve $\mathcal{N} = (A_0, A_1, A_2, \dots)$ of the cover $T = \cup_{a \in A} T_a^f$ is formally composed of vertices A_0 , edges A_1 , triangles A_2 , and their k -dimensional counterparts A_k defined as follows:

- Set A_0 consists of *vertices* \mathbf{a} corresponding to sets $T_a^f \neq \emptyset$, $a \in A$.
- Set A_1 consists of unordered pairs $\{a, b\}$ (where a and b are different), such that $T_a^f \cap T_b^f \neq \emptyset$. The elements of A_1 are called *edges* in \mathcal{N} .
- Set A_2 consists of unordered triples $\{a, b, c\}$ (where a , b , and c are different) such that $T_a^f \cap T_b^f \cap T_c^f \neq \emptyset$. The elements of A_2 are called *triangles* in \mathcal{N} .
- Set A_k for $k \geq 2$ is defined similarly. The elements of A_k for any k are generally called *simplices* in \mathcal{N} .

Nerve \mathcal{N} is usually identified with its geometrical realization, which is a polytope $P(\mathcal{N})$ in \mathbb{R}^{A_0} , where the elements of A_0 form a basis and linear space \mathbb{R}^{A_0} consists of formal sums $\sum_{\mathbf{a} \in A_0} x_{\mathbf{a}} \cdot \mathbf{a}$ with $x_{\mathbf{a}} \in \mathbb{R}$. The vertices of $P(\mathcal{N})$ are the end points of the basis vectors in \mathbb{R}^{A_0} . Two vertices \mathbf{a} and \mathbf{b} in \mathbb{R}^{A_0} are connected by a segment in $P(\mathcal{N})$ iff $T_a^f \cap T_b^f \neq \emptyset$ (i.e., $\{a, b\} \in A_1$). Three vertices \mathbf{a} , \mathbf{b} , and \mathbf{c} are the extreme points of a triangle face in $P(\mathcal{N})$ iff $T_a^f \cap T_b^f \cap T_c^f \neq \emptyset$ (i.e., $\{a, b, c\} \in A_2$), etc. Overall, $P(\mathcal{N})$ is contained in the standard simplex $\{\sum_{\mathbf{a} \in A_0} x_{\mathbf{a}} = 1, x_{\mathbf{a}} \geq 0\}$.

At this point, we are able to explain *the main idea of the proof*. We first show that the geometrical properties of domain T imply that polytope $P(\mathcal{N})$ has a special

¹²The definition of the nerve goes back to Alexandroff (1928).

structure. In particular, Lemma 1 below establishes that conditions 1) and 2) of Theorem 1 imply that $P(\mathcal{N})$ is simply connected. We then prove that $P(\mathcal{N})$ being simply connected ensures that any cycle in Γ^n can be “paved” by triples with zero weight.¹³

To relate the geometrical properties of domain T to that of polytope $P(\mathcal{N})$, we use a variation of the nerve theorem from algebraic topology. The nerve theorem has multiple versions (see, e.g., Björner, 1995). The classical one requires each set T_a^f , $a \in A$, and each possible intersection $T_{a_0}^f \cap T_{a_1}^f \cap \dots \cap T_{a_M}^f$ to be either empty or contractible. If these conditions are satisfied, the nerve theorem says, roughly speaking, that the geometrical properties of T and $P(\mathcal{N})$ coincide. We need to establish, however, a weaker conclusion so that T being simply connected implies that $P(\mathcal{N})$ is simply connected. Hence, we use a weaker requirement on sets T_a^f and no requirement on their intersections.

We state and prove the formal result in Lemma 1 below. The result is new and does not follow from the existing versions of the nerve theorem and, hence, requires a separate proof. As the proof is technical, we postpone it to the Appendix.

Lemma 1. *Let \mathcal{N} be the nerve of the cover $T = \bigcup_{a \in A} T_a^f$. If conditions 1) and 2) are satisfied, then $P(\mathcal{N})$ is simply connected.*

We now show that any cycle in Γ^n can be “paved” by triplets with zero weight. For this purpose, we consider the following algebraic construction. Let us enumerate the elements of finite set A in some way. We write $a < b$ if a comes before b . Consider now the linear space \mathbb{R}^{A_1} as follows:

$$\mathbb{R}^{A_1} = \left\{ \sum_{\substack{\{a,b\} \in A_1 \\ a < b}} x_{ab} \cdot \mathbf{ab} \text{ where } x_{ab} \in \mathbb{R} \right\}.$$

We use the order on A to avoid counting \mathbf{ab} and \mathbf{ba} twice. For \mathbf{ba} , we then write $\mathbf{ba} = (-1) \cdot \mathbf{ab} = -\mathbf{ab} \in \mathbb{R}^{A_1}$. Any path $p = (\mathbf{a}_0 \rightarrow \mathbf{a}_1 \rightarrow \dots \rightarrow \mathbf{a}_M)$ in Γ^n has then a

¹³The latter step is related to Theorem 4 in Jiang et al. (2011). However, they consider a more restrictive setting where $\{\mathbf{a}, \mathbf{b}\}, \{\mathbf{b}, \mathbf{c}\}, \{\mathbf{c}, \mathbf{a}\} \in A_1$ implies $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\} \in A_2$. This assumption is natural in their setting, but might not be satisfied in our environment.

representative in \mathbb{R}^{A_1} :

$$r(p) = \mathbf{a}_0\mathbf{a}_1 + \mathbf{a}_1\mathbf{a}_2 + \dots + \mathbf{a}_{M-1}\mathbf{a}_M \in \mathbb{R}^{A_1}.$$

Having an order on A is important here, as p consists of directed edges in Γ^n , and an order on A allows us to distinguish directed edges $\mathbf{a} \rightarrow \mathbf{b}$ and $\mathbf{b} \rightarrow \mathbf{a}$ in \mathbb{R}^{A_1} . Indeed, $r(\mathbf{a} \rightarrow \mathbf{b}) = \mathbf{ab}$ and $r(\mathbf{b} \rightarrow \mathbf{a}) = \mathbf{ba} = -\mathbf{ab}$.

Slightly abusing the notation, we now define a linear function $\ell : \mathbb{R}^{A_1} \rightarrow \mathbb{R}$ as

$$\ell\left(\sum_{\substack{\{a,b\} \in A_1 \\ a < b}} x_{ab} \cdot \mathbf{ab}\right) \stackrel{\text{def}}{=} \sum_{\substack{\{a,b\} \in A_1 \\ a < b}} x_{ab} \ell_{ab}.$$

Function ℓ measures the weight of any path p in Γ^n . Indeed, if $a_i < a_{i+1}$ then $\ell(\mathbf{a}_i\mathbf{a}_{i+1}) = \ell_{a_i a_{i+1}}$, and if $a_i > a_{i+1}$ then Fact 1 implies $\ell(\mathbf{a}_i\mathbf{a}_{i+1}) = \ell(-\mathbf{a}_{i+1}\mathbf{a}_i) = -\ell_{a_{i+1}a_i} = \ell_{a_i a_{i+1}}$. Therefore,

$$\ell(r(p)) = \ell(\mathbf{a}_0\mathbf{a}_1) + \ell(\mathbf{a}_1\mathbf{a}_2) + \dots + \ell(\mathbf{a}_{M-1}\mathbf{a}_M) = \ell_{a_0 a_1} + \ell_{a_1 a_2} + \dots + \ell_{a_{M-1} a_M}.$$

Using the above definitions, we need to prove that if path $p = (\mathbf{a}_0 \rightarrow \mathbf{a}_1 \rightarrow \dots \rightarrow \mathbf{a}_M)$ in Γ^n is a cycle (i.e., $\mathbf{a}_M = \mathbf{a}_0$), then $\ell(r(p)) = 0$. We reformulate the last statement using a linear map $\partial_1 : \mathbb{R}^{A_1} \rightarrow \mathbb{R}^{A_0}$ (called a *boundary operator*):

$$\partial_1\left(\sum_{\substack{\{a,b\} \in A_1 \\ a < b}} x_{ab} \cdot \mathbf{ab}\right) \stackrel{\text{def}}{=} \sum_{\substack{\{a,b\} \in A_1 \\ a < b}} x_{ab} \cdot (\mathbf{b} - \mathbf{a}).$$

Operator ∂_1 maps edge \mathbf{ab} in \mathcal{N} to the difference between its tail and head $\mathbf{b} - \mathbf{a}$. Hence, if path p is a cycle in Γ^n , we must have $\partial_1(r(p)) = 0$ or $r(p) \in \ker \partial_1 \subset \mathbb{R}^{A_1}$. It is then sufficient to show that ℓ vanishes on any $r(p) \in \ker \partial_1$.

To establish the latter result, we use Fact 2. According to Fact 2, function ℓ vanishes on the boundary of any triangle in A_2 . In addition, function ℓ is linear. Hence, to show that any cycle in Γ^n has zero weight, it is sufficient to “pave” the cycle with triangles in A_2 .

To give an exact algebraic meaning to the idea of “paving,” consider the linear space \mathbb{R}^{A_2} (defined as a linear space of formal sums $\sum_{\substack{\{a,b,c\} \in A_2 \\ a < b < c}} x_{abc} \mathbf{abc}$ with notation

$\mathbf{abc} = \mathbf{bca} = \mathbf{cab} = -\mathbf{acb} = -\mathbf{cba} = -\mathbf{bac}$ for $a < b < c$) and a linear map $\partial_2 : \mathbb{R}^{A_2} \rightarrow \mathbb{R}^{A_1}$ defined as

$$\partial_2\left(\sum_{\substack{\{a,b,c\} \in A_2 \\ a < b < c}} x_{\mathbf{abc}} \cdot \mathbf{abc}\right) = \sum_{\substack{\{a,b,c\} \in A_2 \\ a < b < c}} x_{\mathbf{abc}} (\mathbf{ab} + \mathbf{bc} + \mathbf{ca}).$$

Operator ∂_2 maps a triangle \mathbf{abc} to the sum of its boundary edges $\mathbf{ab} + \mathbf{bc} + \mathbf{ca}$. For example, for $a < c < b$, we have $\partial_2(\mathbf{abc}) = \partial_2(-\mathbf{acb}) = -(\mathbf{ac} + \mathbf{cb} + \mathbf{ba}) = \mathbf{ab} + \mathbf{bc} + \mathbf{ca}$.

Hence, if for cycle p we have $r(p) \in \text{Im } \partial_2$, then p can be ‘‘paved’’ by triangles in A_2 . More precisely, Fact 2 allows us to prove that ℓ vanishes on image $\text{Im } \partial_2$. Indeed, for any $\{a, b, c\} \in A_2$, we have $\ell(\partial_2(\mathbf{abc})) = \ell(\mathbf{ab} + \mathbf{bc} + \mathbf{ca}) = 0$. As maps ℓ and ∂_2 are linear, ℓ also vanishes on $\text{Im } \partial_2$.

We know that $r(p) \in \ker \partial_1$ for any cycle p . Hence, it remains to show that $\ker \partial_1 = \text{Im } \partial_2$. It is easy to see that $\text{Im } \partial_2 \subset \ker \partial_1$, as $\partial_1(\partial_2(\mathbf{abc})) = \partial_1(\mathbf{ab} + \mathbf{bc} + \mathbf{ca}) = (\mathbf{b} - \mathbf{a}) + (\mathbf{c} - \mathbf{b}) + (\mathbf{a} - \mathbf{c}) = 0$ for any $\{a, b, c\} \in A_2$. To measure the difference between $\ker \partial_1$ and $\text{Im } \partial_2$, we consider the first homology of \mathcal{N} defined as

$$H_1(\mathcal{N}, \mathbb{R}) = \ker \partial_1 / \text{Im } \partial_2.$$

Note that $H_1(\mathcal{N}, \mathbb{R})$ is a linear space with $\dim H_1(\mathcal{N}, \mathbb{R}) = \dim \ker \partial_1 - \dim \text{Im } \partial_2$. Hence, $\text{Im } \partial_2 = \ker \partial_1$ if and only if $H_1(\mathcal{N}, \mathbb{R}) = 0$. The latter is guaranteed by the Hurewicz theorem, which ensures that if $P(\mathcal{N})$ is simply connected, then $H_1(\mathcal{N}, \mathbb{R}) = 0$ (see Hatcher, 2001). Hence, $\ker \partial_1 = \text{Im } \partial_2$ by Lemma 1. In particular, $r(p) \in \text{Im } \partial_2$ for any cycle p in Γ^n and the weight of any cycle in Γ^n is 0. \square

Finally, to establish the statement of Theorem 1, we show that any cycle in Γ has non-negative weight. Consider some cycle $\mathbf{a}_0 \rightarrow \dots \mathbf{a}_{M-1} \rightarrow \mathbf{a}_0$ in Γ . The *local-to-global* condition 3) then implies that for each $j = 0, \dots, M-1$ there exists a path $\mathbf{a}_j \equiv \mathbf{a}_j^0 \rightarrow \dots \mathbf{a}_j^{M(j)} \equiv \mathbf{a}_{j+1}$ in Γ^n such that $\ell_{\mathbf{a}_j \mathbf{a}_{j+1}} \geq \sum_{m=0}^{M(j)-1} \ell_{\mathbf{a}_j^m \mathbf{a}_{j+1}^{m+1}}$. Therefore,

$$\sum_{j=0}^{M-1} \ell_{\mathbf{a}_j \mathbf{a}_{j+1}} \geq \sum_{j=0}^{M-1} \sum_{m=0}^{M(j)-1} \ell_{\mathbf{a}_j^m \mathbf{a}_{j+1}^{m+1}} = 0.$$

This implies that Γ is cyclically monotone. Hence, f is also cyclically monotone. \square

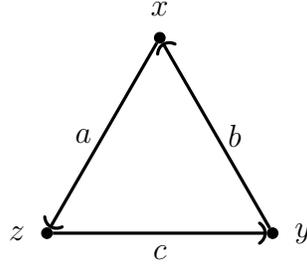


Figure 1: An example of a function that is monotone, but not cyclically monotone.

We now illustrate that the conditions of Theorem 1 are indispensable. For this purpose, we consider the following example.

Example 1. Consider a domain that is the boundary of a triangle with vertices $x = (0, 1, -1)$, $y = (-1, 0, 1)$ and $z = (1, -1, 0)$. Its sides are $[x, z)$, $[z, y)$, and $[y, x)$ (see Figure 1). Assume that $A = \{a, b, c\}$ where $a = (1, 0, 0)$, $b = (0, 1, 0)$, and $c = (0, 0, 1)$. Function f is defined as

$$f(t) = \begin{cases} a & \text{if } t \in [x, z) \\ b & \text{if } t \in [z, y) \\ c & \text{if } t \in [y, x) \end{cases} .$$

A calculation shows that $\ell_{ab} = -\ell_{ba} = \ell_{bc} = -\ell_{cb} = \ell_{ca} = -\ell_{ac} = -1$; hence, f is monotone. There is a negative cycle, $\ell_{ab} + \ell_{bc} + \ell_{ca} = -3$; hence, f is not cyclically monotone.

Example 1 presents a domain $T = [x, z) \cup [z, y) \cup [y, x)$ that is not simply connected and an allocation rule that is monotone, but not cyclically monotone.¹⁴ Note the local-to-global condition is automatically satisfied for the allocation rule of Example 1 as each pair of outcome sets intersects. To present an example of a function and a domain that violates the local-to-global condition, cut a piece from the end of side $[x, z)$. We obtain simply connected domain $T' = [x, z') \cup [z, y) \cup [y, x)$ with $z' = (1/2, 0, -1/2)$. If we keep the same allocation rule on the remaining parts of the domain, lower bounds ℓ do not change. At the same time, sets T_a^f and T_b^f cease to be neighbors. Hence, lower bound ℓ_{ab} should satisfy the local-to-global condition. However, it is not the case, as

¹⁴Note that domain $T = [x, z) \cup [z, y) \cup [y, x)$ is not simply connected as it is a loop that cannot be continuously contracted within the domain to a point.

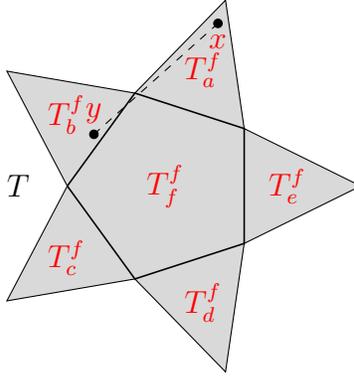


Figure 2: An example of a star-shaped domain T and outcome sets corresponding to $f : T \rightarrow \{a, b, c, d, e, f\}$ that satisfy the conditions of Corollary 1.

$-1 = \ell_{ab} \leq \ell_{ac} + \ell_{cb} = 2$. Finally, we illustrate the importance of sets $T_a^f, a \in A$, being path-connected. Let us take a piece out of the middle of side $[x, z]$ leading to simply connected domain $T'' = [x, x'] \cup [z', z] \cup [z, y] \cup [y, x]$, $x' = (1/4, 1/2, -3/4)$. If we keep the same allocation rule, we obtain an example of an allocation rule that satisfies the local-to-global condition, but with set T_a^f not path-connected. We again observe a domain and a monotone function that is not cyclically monotone.

We now establish a corollary of Theorem 1 that provides easy to check sufficient conditions on a domain T , a finite set A , and a monotone function $f : T \rightarrow A$ that guarantees f is also cyclically monotone.

Corollary 1. *Suppose domain T is star-shaped and $f : T \rightarrow A$ is a monotone allocation rule that satisfies the following property: for every $a \in f(T)$, there exists $t \in T_a^f$ such that $[s, t] \subset T$ for all $s \in T$. Then, f is cyclically monotone.*

A star-shaped domain and sets $T_a^f, a \in A$, that satisfy the conditions of Corollary 1 are illustrated in Figure 2. In the figure, domain T is star-shaped because any point in T_f^f can be connected to any point in T with a line segment. Moreover, for any $a, b \in A$ and for any $x \in T_a^f$, there exists $y \in T_b^f$ such that line segment $[x, y]$ lies in T .

We now present a simple example of a star-shaped domain and a monotone allocation rule that is not cyclically monotone. The example also illustrates that the additional condition on sets T_a^f in Corollary 1 cannot be dropped.

Example 2. *Let us consider a modification of Example 1. We consider the domain consisting of the union of two segments $[x, z]$ and $[x, y]$. This is a star-shaped domain, as x can be connected to any point in the domain with a line segment. There are three alternatives $A = \{a, b, c\}$ and allocation f' that coincides with allocation f in Example 1 on domain $[x, z] \cup [x, y]$. Note that the only preimage of alternative c is point z . The definition of lower bounds then implies that $\ell_{ab}, \ell_{ba}, \ell_{ac}, \ell_{bc}$ remain unchanged. A direct calculation also shows that $\ell_{ca} = z_c - z_a = -1$ and $\ell_{cb} = z_c - z_b = 1$. Hence, f' is monotone, but there is a negative cycle, $\ell_{ab} + \ell_{bc} + \ell_{ca} = -3$.*

Remark 1. *The result of Saks and Yu (2005) follows straightforwardly from Corollary 1, as its conditions are trivially satisfied for convex domains.¹⁵ However, Corollary 1 can be applied to more general settings when a domain can be represented as a union of convex sets C_i , $T = \bigcup_{i=1}^I C_i$, that have non-empty intersection $T_{core} = \bigcap_{i=1}^I C_i$. In such a case, any point in T_{core} can be connected to any point of the domain with a line segment. To ensure that the conditions of Corollary 1 are satisfied for a given monotone function, it remains to show that each non-empty set $T_a^f, a \in A$ contains a point in T_{core} .*

We use the conditions of Corollary 1 and the idea of Remark 1 to study the relationship between monotone and cyclically monotone functions on the domain of gross substitutes and the domain of generalized gross substitutes and complements in the next section.

4 Gross Substitutes and Complements

In this section, we apply our main results to study functions choosing among possible object bundles defined on two important economic domains: the domain of gross substitutes and the domain of generalized gross substitutes and complements. The concept of gross substitutes provides a sufficient condition that ensures the existence of Walrasian equilibria in economies with indivisible objects. The domain of gross substitutes has been explored extensively in the matching, auction, equilibrium, and algorithmic literatures (see, e.g., Paes Leme (2017) and Murota (2016) for extensive

¹⁵In the working paper version (Kushnir and Lokutsievskiy, 2019), we also show how the result for single-peaked preferences by Mishra et al. (2014) follows from Theorem 1.

surveys). The domain of generalized gross substitutes and complements is a generalization of the first domain that also allows for multiple objects of the same type and some complementarities across objects (see [Shioura and Yang, 2015](#)).

To define these domains, we consider a finite set of objects E and $n = |E|$. The set of possible object bundles then equals 2^E and $N = 2^{|E|}$. For each bundle $S \subseteq E$, we denote the agent's value as $t(S)$. Hence, vector t is an element of \mathbb{R}^N . Valuation t is called *modular* if $t(S) = \sum_{e \in S} t(e)$ for all $S \subseteq E$. Finally, we define the demand correspondence for any price $p \in \mathbb{R}^n$ as

$$D(t, p) = \arg \max_{S \subseteq E} \{t(S) - \sum_{e \in S} p(e)\}.$$

We now consider the following condition (see [Kelso and Crawford, 1982](#)).

Definition 4. *Valuation t satisfies the gross substitutes (GS) condition if for any price $p \in \mathbb{R}^n$ and any $S \in D(t, p)$, if p' is a price vector with $p' \geq p$, then there exists $S' \in D(t, p')$ such that $\{e \in S : p(e) = p'(e)\} \subseteq S'$. The domain of all valuations satisfying the GS condition is denoted by T^{gs} .*

In other words, an increase in the price of some goods does not cause a decrease in the demand for other goods. [Reijnierse et al. \(2002\)](#) and [Fujishige and Yang \(2003\)](#) showed that the GS condition can be formulated purely in terms of inequalities on the agent's values. The following example presents their characterization for the case of three objects.

Example 3. *For $|E| = 3$, $t \in T^{gs}$ if and only if for all distinct $i, j, k \in E$ we have*

$$\begin{aligned} t(\{i, j\} \cup \{k\}) + t(\{k\}) &\leq t(\{i, k\}) + t(\{j, k\}), \\ t(\{i, j\}) + t(\emptyset) &\leq t(\{i\}) + t(\{j\}), \\ t(\{i, j\}) + t(\{k\}) &\leq \max\{t(\{i, k\}) + t(\{j\}), t(\{j, k\}) + t(\{i\})\}. \end{aligned}$$

Example 3 illustrates that domain T^{gs} is not convex (because of the third set of inequalities). In general, domain T^{gs} consists of several convex polytopes and, therefore, can be quite complex.

For the domain of gross substitutes, it is natural to consider functions $f : T^{gs} \rightarrow 2^E$. However, 2^E is not a subset of \mathbb{R}^N , and, hence, neither Definitions 1 and 2 in

Section 2 nor our results in Section 3 formally apply. To accommodate the subtlety, with each set $S \subseteq E$ we associate an indicator $\alpha(S) \in \{0, 1\}^N$, with a component corresponding to set S equal to 1 and all other components equal to 0. Hence, for any $t \in T^{gs}$, we have $t(S) = t \cdot \alpha(S)$. We denote then the union of these indicators as $A \subset \mathbb{R}^N$ and the constructed one-to-one function as $\alpha : 2^E \rightarrow A$.¹⁶

This construction allows us to consider functions $f : T^{gs} \rightarrow 2^E$ within our framework. Define $\tilde{f} = \alpha \circ f$. Then, $f : T^{gs} \rightarrow 2^E$ is monotone (or cyclically monotone) if and only if $\tilde{f} : T^{gs} \rightarrow A$ is monotone (or cyclically monotone) according to Definitions 1 and 2. We can then apply the result of Corollary 1 to obtain the following result.

Theorem 2 (Gross Substitutes). *If a function $f : T^{gs} \rightarrow 2^E$ is monotone, then it is cyclically monotone.*

Proof. Let us consider a monotone function $f : T^{gs} \rightarrow 2^E$. Function f is monotone if and only if the associated $\tilde{f} = \alpha \circ f$ is monotone. We will show that every monotone $\tilde{f} : T^{gs} \rightarrow A$ satisfies the conditions of Corollary 1. For this purpose, we establish that domain T^{gs} satisfies two important properties:

- 1) Any modular valuation m belongs to T^{gs} ;
- 2) For any modular m and $\beta \in [0, 1]$, if $t \in T^{gs}$ then $\beta t + (1 - \beta)m \in T^{gs}$.

The first property is well-known (e.g., Paes Leme, 2017) and follows from the definition of demand correspondence. Indeed, for any modular m and $p \in \mathbb{R}^n$, we have

$$S \in D(m, p) \iff \{e \in E : m(e) > p(e)\} \subseteq S \subseteq \{e \in E : m(e) \geq p(e)\}.$$

Therefore, an increase in the price of some goods does not cause a decrease in demand for other goods.

To verify the second property, for any modular valuation $m \in \mathbb{R}^N$, we denote $\tilde{m} \in \mathbb{R}^n$ such that $\tilde{m}(e) = m(e)$ for each $e \in E$. For any price $p \in \mathbb{R}^n$ and any $\beta \in (0, 1]$, we then have

$$D(\beta t + (1 - \beta)m, p) = \arg \max_{S \subseteq E} \{\beta t(S) + (1 - \beta)m(S) - \sum_{e \in S} p(e)\} = D(t, p/\beta - \tilde{m}(1 - \beta)/\beta).$$

¹⁶This construction is similar to the one used in our mechanism design interpretation (see p. 2).

Therefore, if demand $D(t, p)$ satisfies the GS condition for any $p \in \mathbb{R}^n$, so does $D(\beta t + (1 - \beta)m, p)$. For $\beta = 0$, the second property follows from the first one.

Properties 1) and 2) stated above imply that domain T^{gs} is star-shaped. To check the requirement of Corollary 1 on sets $T_a^f, a \in A$, we need to establish that every non-empty set T_a^f contains a point that can be connected to any point in T^{gs} with a line segment within T^{gs} . Using the one-to-one correspondence $\alpha : 2^E \rightarrow A$, consider $S \subseteq E$ such that $a = \alpha(S)$. We then have

$$T_a^f = \{t \in T^{gs} : t(S) - t(G) \geq \ell_{SG}, \forall G \subset E\},$$

where $\ell_{SG} = \ell_{\alpha(S)\alpha(G)}$ for all $G \subset E$. If T_a^f is non-empty, we have $\ell_{SG} < +\infty$ for all $G \subset E$. Consider a modular valuation

$$m(e) = \begin{cases} M & \text{if } e \in S \\ -M & \text{if } e \notin S \end{cases}. \quad (3)$$

For sufficiently large $M > 0$, such modular valuation m satisfies inequalities $m(S) - m(G) \geq \ell_{SG}$ for all $G \subset E$. Therefore, m lies in T_a^f . Hence, property 2) implies that each nonempty T_a^f contains a point that can be connected with a line segment to any point in T^{gs} . Overall, every monotone function $\tilde{f} : T^{gs} \rightarrow A$ satisfies the conditions of Corollary 1. Hence, it is cyclically monotone. Therefore, every monotone function $f : T^{gs} \rightarrow 2^E$ is cyclically monotone. \square

The proof of Theorem 2 has a nice geometric interpretation. As we mentioned before Example 3, [Reijnierse et al. \(2002\)](#) and [Fujishige and Yang \(2003\)](#) showed that domain T^{gs} can be represented as a union of convex polytopes. We established that the intersection of these convex polytopes is non-empty and contains all modular valuations and each non-empty T_a^f contains a modular valuation. Hence, all conditions of Corollary 1 are satisfied (see Remark 1 on page 18).

We now extend the above result to the domain of *generalized gross substitutes and complements (GGSC)*. To define the GGSC condition, consider a finite set of object types E and $n = |E|$. There can be several objects of each type, and we denote the bundle of available objects as $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{Z}_+^E$, where ω_e denotes the available number of objects of type $e \in E$. The types can be divided into two

classes $E = E_1 \cup E_2$ with $E_1 \cap E_2 = \emptyset$. We also denote E_j^c as the complement of set $E_j, j = 1, 2$. The objects are substitutes within each class and complements across the classes. For example, E_1 could be considered the set of left shoes and E_2 the set of right shoes. A more practical example concerns the allocation of spectrum licenses. There are two geographic regions and radio spectra. Radio spectrum licenses are substitutes within each region, but complements across regions.

Denote the set of feasible object bundles as $\Omega = \{z \in \mathbb{Z}_+^n : z \leq \omega\}$ and $N = |\Omega|$. Each agent valuation is then $t : \Omega \rightarrow \mathbb{R}$. A vector $p = (p_1, \dots, p_n) \in \mathbb{R}^n$ indicates the price for each type. For each price and agent valuation, we consider demand correspondence

$$D(p, t) = \arg \max_{z \in \Omega} \{t(z) - p \cdot z\}.$$

Denote $\chi_e \in \mathbb{R}^n$ as the vector with all zeros except one on the place corresponding to object type $e \in E$ and $\chi_0 = 0$. We say also that $C \subseteq \mathbb{Z}^n$ is a *discrete convex set* if it contains all integer vectors in its convex hull. [Shioura and Yang \(2015\)](#) introduced the following definition.

Definition 5. *Valuation t satisfies the generalized gross substitutes and complements (GGSC) condition if*

- (i) *for any price $p \in \mathbb{R}^n$, $D(p, t)$ is a discrete convex set; and*
- (ii) *for any price $p \in \mathbb{R}^n$ and any $e \in E_j, j = 1, 2, \delta > 0$, and $z \in D(p, t)$, there exists $z' \in D(p + \delta\chi_e, t)$ such that*

$$(\forall l \in E_j \setminus \{e\}) z'_l \geq z_l, \quad (\forall l \in E_j^c) z'_l \leq z_l,$$

$$\sum_{l \in E_j} z_l - \sum_{l \in E_j^c} z_l \geq \sum_{l \in E_j} z'_l - \sum_{l \in E_j^c} z'_l.$$

The domain of all the valuations satisfying the GGSC condition is denoted by T^{ggsc} .

The GGSC condition states that the objects of each type in each set E_j are substitutes, but that goods across the two sets E_1 and E_2 are complements. In particular, let us assume that bundle z is demanded for price vector p . If the price of type $k \in E_j$ is increased, then demand for objects of the other types in E_j will not decrease whereas

demand for objects of each type in the other group E_j^c will not increase. In addition, the difference in demand between the two groups at the new prices should not exceed the difference at the old prices.

Note that when there is only one object of each type $\omega = (1, \dots, 1)$, domain T^{ggsc} coincides with the domain of *gross substitutes and complements* introduced by [Sun and Yang \(2006, 2009\)](#). Additionally, if $\omega = (1, \dots, 1)$ and either $E_1 = \emptyset$ or $E_2 = \emptyset$, domain T^{ggsc} coincides with the domain of *gross substitutes* T^{gs} (see [Shioura and Yang, 2015](#)).

Furthermore, any function $f : T^{ggsc} \rightarrow \Omega$ can be put into our environment in a similar way. For any $z \in \Omega$, we again associate an indicator $\alpha(z) \in \{0, 1\}^N$ with a component corresponding to z equal to 1 and all other components equal to 0. Therefore, $t(z) = t \cdot \alpha(z)$. Denote the union of all these indicators as $A_\Omega \subset \mathbb{R}^N$ and $\tilde{f} = \alpha \circ f : T^{ggsc} \rightarrow A_\Omega$. So f is called monotone (or cyclically monotone) if \tilde{f} is monotone (or cyclically monotone). We then establish the following result.

Theorem 3 (Generalized Gross Substitutes and Complements). *If a function $f : T^{ggsc} \rightarrow \Omega$ is monotone, then it is cyclically monotone.*

Theorem 3 subsumes Theorem 2. At the same time, Theorem 3 requires a more involved proof that uses the characterization of domain T^{ggsc} in terms of the GM-concave functions introduced in [Shioura and Yang \(2015\)](#). We postpone the details until the Appendix.

5 Conclusion

In this paper, we provide sufficient conditions for a monotone function with a finite set of outcomes to be cyclically monotone. Using these conditions, we established that for the domain of gross substitutes and the domain of generalized gross substitutes and complements, any monotone function that chooses among possible object bundles is cyclically monotone.

The relationship between the monotone and cyclically monotone conditions has implications beyond mechanism design. In revealed preference theory, [Chambers and Echenique \(2018\)](#) used it to establish that a demand function is strongly rationalizable with a quasi-linear utility if and only if it satisfies a continuity condition and *the law*

of demand (i.e., the negative of the demand function is monotone) (see also [Amir et al., 2017](#)). In producer choice theory, the working paper version ([Kushnir and Lokutsievskiy, 2019](#)) shows that any weakly rationalizable supply functions with a finite range that is positive homogeneous of degree zero is characterized by the *law of supply* (i.e., the supply function is monotone). We also explain how the reduction of cyclically monotonicity to the requirement of an allocation being monotone is helpful in solving spatial allocation problems.

Finally, we want to highlight a limitation of our approach. Our approach is confined to settings with a finite set of outcomes, because our main building block—*the Nerve theorem*—does not hold when the set of outcomes is infinite (see [Björner, 1995](#)). For those interested in an infinite set of outcomes, [Carbajal and Müller \(2015, 2017\)](#) and [Archer and Kleinberg \(2014\)](#) provide conditions when a monotone function is cyclically monotone.¹⁷ Understanding when these conditions are applicable to various convex and non-convex domains is an important direction for future research. This extension will have invaluable implications for some identification problems in econometrics (e.g., [Shi et al., 2018](#)).

¹⁷See also [Berger et al. \(2017\)](#).

A Appendix

A.1 Proof of Lemma 1.

We call a subset of T closed if it is an intersection of T and some closed subset of \mathbb{R}^N . This is a standard convention when one considers the closed subsets of some set.

Let us first prove that $P(\mathcal{N})$ is path-connected. For this purpose, consider the union of all edges in $P(\mathcal{N})$ that we call 1-skeleton $P^1(\mathcal{N})$. We show that $P^1(\mathcal{N})$ is path-connected. Indeed, if we were able to decompose A into two non-intersecting sets $A = A' \cup A''$ such that for any $a' \in A'$ and $a'' \in A''$ vertices \mathbf{a}' and \mathbf{a}'' are not connected in $P^1(\mathcal{N})$, then sets $T' = \bigcup_{a' \in A'} T_{a'}^f$ and $T'' = \bigcup_{a'' \in A''} T_{a''}^f$ would not intersect. This contradicts set T being path-connected, as $T = T' \cup T''$ and both T' and T'' are closed subsets of T . Hence, $P^1(\mathcal{N})$ is path-connected, which implies that $P(\mathcal{N})$ is also path-connected.

Now, we prove that $P(\mathcal{N})$ is simply connected. The proof is based on the following Carrier theorem, which is a standard tool to prove the nerve-type theorems (see Nagórko, 2007; Björner, 1995). In Nagórko (2007), Carrier theorem is proved under very general assumptions. We adopt his statements to our setting.

Definition A1. Let $X = \bigcup_{a \in A} X_a$ and $Y = \bigcup_{b \in B} Y_b$ where A and B are some sets of indices. A carrier is a map $\mathcal{C} : A \rightarrow B$ such that if $\bigcap_{a \in A'} X_a \neq \emptyset$ for some $A' \subset A$, then $\bigcap_{b \in \mathcal{C}(A')} Y_b \neq \emptyset$. We say that a map $f : X' \rightarrow Y$ defined on a closed subset $X' \subset X$ is carried by \mathcal{C} if $f(X_a \cap X') \subset Y_{\mathcal{C}(a)}$ for all $a \in A$.

Definition A2. A topological space Z is an absolute extensor for a topological space W if each continuous map from a closed subset of W into Z extends over the entire W .¹⁸

For example, two-point set $\{0, 1\}$ is not an absolute extensor for interval $[0, 1]$. And any space Z is an absolute extensor for $\{0, 1\}$ as any map $\{0, 1\} \rightarrow Z$ is continuous.

Theorem A1 (Carrier theorem, Nagórko (2007)). Let $X = \bigcup_{a \in A} X_a \subset \mathbb{R}^n$, $Y = \bigcup_{b \in B} Y_b \subset \mathbb{R}^m$ and $\mathcal{C} : A \rightarrow B$ be a carrier. If A and B are finite, X_a is a closed

¹⁸The definition is given for arbitrary topological spaces. Any subset of \mathbb{R}^n is a metric space, and any metric space is a topological space.

subset of X for each $a \in A$, and for any non-empty $B' \subset B$, $\bigcap_{b \in B'} Y_b$ is an absolute extensor for X , then there exists a continuous map $f : X \rightarrow Y$ carried by \mathcal{C} .

To use Carrier theorem, we need some covers of two spaces. We already have cover $T = \bigcup_{a \in A} T_a^f$, where T_a^f are closed subsets of T (since each T_a^f is an intersection of T with a collection of closed half-spaces). The second space is the geometric realization of nerve $P(\mathcal{N})$, and we consider its cover by barycentric stars that can be constructed as follows. For any simplex $\sigma \in \mathcal{N}$, we denote the corresponding face center of mass by $B(\sigma) \in P(\mathcal{N})$, which is also called barycenter $B(\sigma) = 1/|\sigma| \sum_{a \in \sigma} \mathbf{a}$. For a given vertex $\mathbf{a} \in A_0$, we also consider new simplices with vertices in barycenters $B(\sigma_1), \dots, B(\sigma_r)$ and such that $\mathbf{a} \in \sigma_1 \subset \dots \subset \sigma_r \in \mathcal{N}$ (including the case $\sigma_1 = \{\mathbf{a}\}$). The union of all such simplices is called the barycentric star of \mathbf{a} and denoted as

$$\text{bst } \mathbf{a} = \bigcup_{a \in \sigma_1 \subset \dots \subset \sigma_r \in \mathcal{N}} \text{conv} \{B(\sigma_1), \dots, B(\sigma_k)\}.$$

Barycentric stars are closed star-shaped sets and $P(\mathcal{N}) = \bigcup_{\mathbf{a} \in A_0} \text{bst } \mathbf{a}$.

Definition A3. A cover $X = \bigcup_{a \in A} X_a$ with A being finite and X_a being closed subsets of X is called regular for metric spaces if for any non-empty $A' \subset A$ set $\bigcap_{a \in A'} X_a$ is an absolute extensor for any metric space.

Lemma 3.2 in Nagórko (2007) shows that cover $P(\mathcal{N}) = \bigcup_{\mathbf{a} \in A_0} \text{bst } \mathbf{a}$ is a regular cover for metric spaces. Therefore, for any non-empty $A'_0 \subset A_0$, $\bigcap_{\mathbf{a} \in A'_0} \text{bst } \mathbf{a}$ are absolute extensors for T as $T \subset \mathbb{R}^N$.

Consider a carrier $\mathcal{C} : A \rightarrow A_0$ for covers $T = \bigcup_{a \in A} T_a^f$ and $P(\mathcal{N}) = \bigcup_{\mathbf{a} \in A_0} \text{bst } \mathbf{a}$ that sends each $a \in A$ to the corresponding $\mathbf{a} \in A_0$. Note that both \mathcal{C} and \mathcal{C}^{-1} by the definition of the nerve (see Lemma 3.2 in Nagórko (2007)). Moreover, for any non-empty $A'_0 \subset A_0$, $\bigcap_{\mathbf{a} \in A'_0} \text{bst } \mathbf{a}$ is an absolute extensor for T as showed above. By Carrier theorem, there exists a continuous map $\kappa : T \rightarrow P(\mathcal{N})$ carried by \mathcal{C} .

We construct a map $\lambda : P^1(\mathcal{N}) \rightarrow T$ carried by \mathcal{C}^{-1} (see Figure 3). For any $\mathbf{a} \in A_0$, we pick a point $t_{\mathbf{a}}$ in $T_{\mathbf{a}}^f$; and for any $\{\mathbf{a}, \mathbf{b}\} \in A_1$, we pick a point $t_{\mathbf{ab}}$ in $T_{\mathbf{a}}^f \cap T_{\mathbf{b}}^f$. Recall that the definition of nerve \mathcal{N} implies that $T_{\mathbf{a}}^f \neq \emptyset$ for $\mathbf{a} \in A_0$ and $T_{\mathbf{a}}^f \cap T_{\mathbf{b}}^f \neq \emptyset$ for $\{\mathbf{a}, \mathbf{b}\} \in A_1$. We define λ sending each half of edge $[\mathbf{a}, B(\{\mathbf{a}, \mathbf{b}\})]$ into a continuous path connecting $t_{\mathbf{a}}$ and $t_{\mathbf{ab}}$ (note that such path exists as $T_{\mathbf{a}}^f$ is path-connected). This

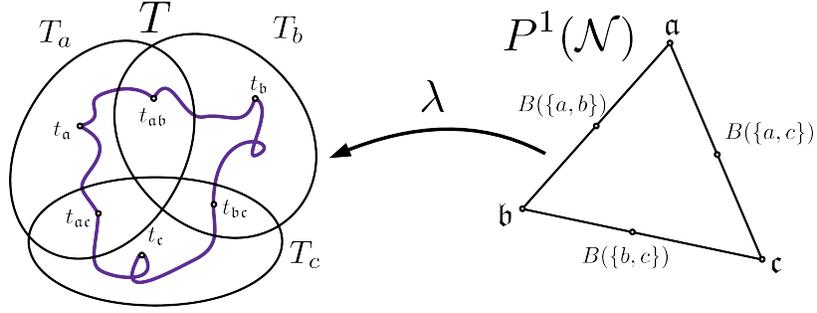


Figure 3: Map $\lambda : P^1(\mathcal{N}) \rightarrow T$ carried by \mathcal{C}^{-1} , where $\mathcal{C} : A \rightarrow A_0$ is a carrier for covers $T = \bigcup_{a \in A} T_a^f$ and $P(\mathcal{N}) = \bigcup_{a \in A_0} \text{bst } \mathbf{a}$.

construction implies that for any $x \in P^1(\mathcal{N}) \cap \text{bst } \mathbf{a} = \bigcup_{b \in A} [\mathbf{a}, B(\{a, b\})]$, we have $\lambda(x) \in T_a^f$. Hence, λ is carried by \mathcal{C}^{-1} .

Recall now some definitions from topology. A continuous map from circle S^1 to Y is called a loop in Y , i.e., $\eta : S^1 \rightarrow Y$. Loop η is called trivial if there exists $y_0 \in Y$ such that $\eta(S^1) \equiv y_0$. Two continuous maps $f_0, f_1 : X \rightarrow Y$ are homotopic if there exists a continuous map $F : [0, 1] \times X \rightarrow Y$ such that $F(0, x) \equiv f_0(x)$ and $F(1, x) \equiv f_1(x)$. Hence, Y is *simply connected* if it is path-connected and any loop in Y is homotopic to some trivial loop.

Using continuous maps $\kappa : T \rightarrow P(\mathcal{N})$ and $\lambda : P^1(\mathcal{N}) \rightarrow T$, we can now prove that $P(\mathcal{N})$ is simply -connected. Consider a loop γ in $P(\mathcal{N})$. By the cellular approximation theorem (see [Hatcher, 2001](#)), γ is homotopic to a loop γ' in 1-skeleton $P^1(\mathcal{N})$, $\gamma' : S^1 \rightarrow P^1(\mathcal{N})$. We consider a loop $\lambda \circ \gamma'$ in T , $\lambda \circ \gamma' : S^1 \rightarrow T$. Loop $\lambda \circ \gamma'$ is homotopic to a trivial loop in T as T is simply connected. Hence, there exists a continuous map $F : [0, 1] \times S^1 \rightarrow T$ such that $F(0, s) \equiv \lambda(\gamma'(s))$ and $F(1, s) \equiv \text{const}$. We claim that $\gamma'' = \kappa \circ \lambda \circ \gamma'$ is homotopic to a trivial loop in $P(\mathcal{N})$. Indeed, $\kappa \circ F$ is a homotopy contracting γ'' to a point, since $\kappa(F(0, s)) \equiv \gamma''(s)$ and $\kappa(F(1, s)) \equiv \text{const}$. To establish that γ is homotopic to a trivial loop in $P(\mathcal{N})$, it remains to prove that γ' and γ'' are homotopic as the homotopic property is transitive.

To prove that γ' and γ'' are homotopic, we first show that they have the following property: for any $s \in S^1$, there exists $\mathbf{a} \in A_0$ such that both $\gamma'(s), \gamma''(s) \in \text{bst } \mathbf{a}$. [Nagórko \(2007\)](#) calls such loops $\{\text{bst } \mathbf{a}\}_{\mathbf{a} \in A_0}$ -close. Indeed, for any $s \in S^1$, there exists $\mathbf{a} \in A_0$ such that $\gamma'(s) \in \text{bst } \mathbf{a}$ as $P(\mathcal{N}) = \bigcup_{\mathbf{a} \in A_0} \text{bst } \mathbf{a}$ is a cover. Since λ is carried by \mathcal{C}^{-1} , we also must have $\lambda(\gamma'(s)) \in T_a^f$. Since κ is carried by \mathcal{C} we also have

$\gamma''(s) = \kappa(\lambda(\gamma'(s))) \in \text{bst } \mathbf{a}$. In addition, cover $P(\mathcal{N}) = \bigcup_{\mathbf{a} \in A_0} \text{bst } \mathbf{a}$ is regular for metric space (see Lemma 3.1 in Nagórko (2007)) and $S^1 \times [0, 1]$ is a metric space. Therefore, Corollary 2.1 in Nagórko (2007) implies that γ' and γ'' are homotopic. \square

A.2 Proof of Corollary 1.

We first note that any star-shaped domain is simply connected. Indeed, take some point $x \in T$ that can be connected with a line segment to any point in T . Such x is called a *base point*. Hence, any loop in T can be continuously contracted to x . Hence, any star-shaped domain is simply connected.

Since any non-empty T_a^f contains a base point of T , the line segment connecting the base point and a point in T_a^f lies in T . The line segment also lies in T_a^f as T_a^f is an intersection of T with some half-spaces. Hence, T_a^f is path-connected.

Finally, we establish that any monotone function f that complies with the conditions of the corollary also satisfies the *local-to-global* condition. In particular, if every non-empty T_a^f contains a base point of T , function f satisfies the following geometric property: for any $a, b \in f(T)$ for any $x \in T_a^f$ there exists $y \in T_b^f$ such line segment $[x, y]$ lies within T . The following lemma establishes that this geometric property ensures that every monotone f also satisfies the local-to-global condition.

Lemma A1. *Consider a domain $T \subset \mathbb{R}^N$, a finite set $A \subset \mathbb{R}^N$, and $f : T \rightarrow A$. Suppose that f is monotone and for any $a, b \in f(T)$ and $x \in T_a^f$, there exists $y \in T_b^f$ such that line segment $[x, y]$ lies within T . Then, f satisfies the local-to-global condition.*

Proof. Consider outcomes $a, b \in f(T)$ with $T_a^f \cap T_b^f = \emptyset$. Take some $x \in T_a^f$ and $y \in T_b^f$ such that line segment $[x, y]$ lies within T . Denote the intersection of closed half-spaces as $\tilde{T}_q^f = \{t \in \mathbb{R}^N : t(q - c) \geq \ell_{qc}, \forall c \in A\}$. Note $T_q^f = T \cap \tilde{T}_q^f$. Since any set \tilde{T}_q^f is closed and convex for any $q \in A$, intersection $[x, y] \cap T_q^f = [x, y] \cap \tilde{T}_q^f$ is either a closed interval, a point, or an empty set. We claim that it is possible to choose

1. a path $\{a \equiv a_0, \dots, a_M \equiv b\}$ such that $T_{a_m}^f \cap T_{a_{m+1}}^f \neq \emptyset$, $m = 0, \dots, M - 1$, $[x, y] \cap T_{a_m}^f \neq \emptyset$;

2. points $z_m \in [x, y] \cap T_{a_m}^f$ such that $z_{m+1} - z_m \neq 0$ and the vectors $z_{m+1} - z_m$ and $x - y$ are co-directed for any $m = 1, \dots, M - 1$.

This can be done in the following way. We put $a_0 = a$ and $z_0 = x$. Then we denote the right end of interval $[x, y] \cap T_{a_0}^f$ by z_1 . Point z_1 must belong to some set T_q^f , $q \neq a_0$. We put $a_1 = q$. The right end of interval $[x, y] \cap T_{a_1}^f$ we denote by z_2 . Point z_2 must belong to some set $T_{q'}^f$, $q' \neq a_0, a_1$. We put $a_2 = q'$. We repeat the process until we cover the whole interval $[x, y]$. We will finish in a finite number of steps as set A is finite and we pick at each step different points. Finally, we eliminate those z_m and a_m for which $z_{m-1} = z_m$ and update the numeration of a_m and z_m preserving the order. Note that $T_{a_m}^f \cap T_{a_{m+1}}^f \neq \emptyset$ because both sets contain z_{m+1} .

For each $z_m \in [x, y] \cap T_{a_m}$, $m = 1, \dots, M$ we could write

$$x(a-b) = \sum_{m=0}^{M-1} x(a_m - a_{m+1}) = x(a_0 - a_1) + \sum_{m=1}^{M-1} (x - z_m)(a_m - a_{m+1}) + \sum_{m=1}^{M-1} z_m(a_m - a_{m+1}).$$

Since all z_m belong to the same interval $[x, y]$ and $z_m \neq z_{m+1}$, there exists λ_m such that $x - z_m = \lambda_m(z_m - z_{m+1})$. Moreover, $\lambda_m > 0$ by the choice of z_m . As $z_m \in T_{a_m}^f$ and $z_{m+1} \in T_{a_{m+1}}^f$, f being monotone implies

$$(x - z_m)(a_m - a_{m+1}) = \lambda_m(z_m(a_m - a_{m+1}) + z_{m+1}(a_{m+1} - a_m)) \geq \lambda_m(\ell_{a_m a_{m+1}} + \ell_{a_{m+1} a_m}) \geq 0.$$

Taking into account that $x(a_0 - a_1) \geq \ell_{a_0 a_1}$ and $z_m(a_m - a_{m+1}) \geq \ell_{a_m a_{m+1}}$, we obtain $x(a - b) \geq \sum_{m=0}^{M-1} \ell_{a_m a_{m+1}}$. Hence, $\ell_{ab} \geq \sum_{m=0}^{M-1} \ell_{a_m a_{m+1}}$, where $T_{a_m}^f \cap T_{a_{m+1}}^f \neq \emptyset$ for each $m = 1, \dots, M - 1$. \square

All three conditions of Theorem 1 are satisfied. Hence, f is cyclically monotone. \square

A.3 Proof of Theorem 3.

Similarly to how the domain of gross substitutes can be characterized in terms of inequalities on the agent's values (see Example 3; Reijnierse et al., 2002; Fujishige and Yang, 2003), Theorem 3.3 in Shioura and Yang (2015) shows that $t \in T^{ggsc}$ if and only if it is *GM-concave*.

To define *GM-concave* valuations, let $U : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a diagonal matrix $U = \text{diag}(1, \dots, 1, -1, \dots, -1)$ that contains 1 as the first $|E_1|$ elements and -1 as the

remaining $|E_2|$ elements. Denote $\text{supp}(z) = \{e : z_e > 0\}$ for $z \in \mathbb{Z}^n$. A valuation $t : \Omega \rightarrow \mathbb{R}$ is called *GM-concave* if

$$\forall z, z' \in \Omega \quad \forall e \in \text{supp}(U(z - z')) \quad \exists l \in \text{supp}(U(z' - z)) \cup 0 :$$

$$t(z) + t(z') \leq t(z - U(\chi_e - \chi_l)) + t(z' + U(\chi_e - \chi_l)). \quad (\text{A.1})$$

In fact, each inequality (A.1) determines a half-space in \mathbb{R}^N . Hence, the set of all *GM-concave* functions (and, hence, domain T^{ggsc}) is the union of some convex polytopes.

Now we construct a generalization of modular valuations. Let $g_e : \{0, 1, \dots, \omega_e\} \rightarrow \mathbb{R}$ be arbitrary concave functions for $1 \leq e \leq n$. Put

$$m : \Omega \rightarrow \mathbb{R}, \quad m(z) = \sum_{e=1}^n g_e(z_e) \quad \text{for } z \in \Omega.$$

Note that valuation m is linear in object types, but concave in the number of objects. So we call such valuations *modular-concave*. We will show below that these valuations satisfy inequalities (A.1) for *each* $e \in \text{supp}(U(z - z'))$ and *each* $l \in \text{supp}(U(z' - z)) \cup 0$. Hence, they belong to the intersection of the convex polytopes shaping domain T^{ggsc} . This will help us to establish that the conditions of Corollary 1 are satisfied (see also Remark 1 on page 18).

Let us show that each modular-concave valuation satisfies inequalities (A.1) for *each* $e \in \text{supp}(U(z - z'))$ and *each* $l \in \text{supp}(U(z' - z)) \cup 0$. Using the definition of modular-concave valuation, we obtain that condition (A.1) is equivalent to

$$g_e(z_e) + g_e(z'_e) + g_l(z_l) + g_l(z'_l) \leq g_e(z_e - U\chi_e) + g_e(z'_e + U\chi_e) + g_l(z_l + U\chi_l) + g_l(z'_l + U\chi_l).$$

Hence, it is sufficient to prove separately the following inequalities for each $e \in \text{supp}(U(z - z'))$ and each $l \in \text{supp}(U(z' - z)) \cup 0$:

$$g_e(z_e) + g_e(z'_e) \leq g_e(z_e - U\chi_e) + g_e(z'_e + U\chi_e), \quad (\text{A.2})$$

$$g_l(z_l) + g_l(z'_l) \leq g_e(z_l + U\chi_l) + g_l(z'_l - U\chi_l). \quad (\text{A.3})$$

We begin with inequality (A.2). If $e \in E_1$, then $z_e > z'_e$ and (A.2) becomes

$$g_e(z_e) - g_e(z_e - 1) \leq g_e(z'_e + 1) - g_e(z'_e),$$

which follows from the concavity of g_e . If $e \in E_2$, then $z_e < z'_e$ and (A.2) becomes

$$g_e(z_e + 1) - g_e(z_e) \geq g_e(z'_e) - g_e(z'_e - 1),$$

which follows from the concavity of g_e . Hence, (A.2) holds for each $e \in \text{supp}(U(z-w))$.

Now we proceed to inequality (A.3). If $l = 0$, then (A.3) is obviously satisfied as equality. If $l \in E_1$, then $z_l < z'_l$ and (A.3) becomes

$$g_l(z_l + 1) - g_l(z_l) \geq g_l(z'_l) - g_l(z'_l - 1),$$

which follows from the concavity of g_l . Finally, if $l \in E_2$, then $z_l > z'_l$ and (A.3) becomes

$$g_l(z_l) - g_l(z_l - 1) \leq g_l(z'_l + 1) - g_l(z'_l),$$

which follows from the concavity of g_l . Hence, (A.3) holds for each $l \in \text{supp}(U(z' - z)) \cup 0$.

Now we use modular-concave valuations to show that any monotone function $\tilde{f} : T^{ggsc} \rightarrow A_\Omega$ satisfies the conditions of Corollary 1. First, let us show that any modular-concave valuation m can be connected with an arbitrary $t \in T^{ggsc}$ by a segment line within T^{ggsc} . In other words, we need to show that for any $\beta \in [0, 1]$, $(1 - \beta)t + \beta m \in T^{ggsc}$. Fix arbitrary $z, w \in \Omega$ and $e \in \text{supp}(U(z - w))$. Since $1 - \beta \geq 0$ valuation $(1 - \beta)t$ satisfies (A.1) for *some* $l = l_0 \in \text{supp}(U(z' - z)) \cup 0$. At the same time, valuation βm is modular-concave and satisfies (A.1) for *each* $l \in \text{supp}(U(z' - z)) \cup 0$, as we showed above. Therefore, $(1 - \beta)t + \beta m$ satisfies (A.1) for $l = l_0$ and, hence, $(1 - \beta)t + \beta m \in T^{ggsc}$. Therefore, domain T^{ggsc} is *star-shaped*.

It remains to establish that any non-empty set T_a^f contains a modular-concave valuation. Using the one-to-one correspondence $\alpha : \Omega \rightarrow A_\Omega$, consider $z \in \Omega$ such that $a = \alpha(z)$. We then have

$$T_a^f = \{t \in T^{ggsc} : t(z) - t(z') \geq \ell_{zz'}, \forall z' \in \Omega\}.$$

where $\ell_{zz'} \equiv \ell_{\alpha(z)\alpha(z')}$ for all $z' \in \Omega$. If T_a^f is non-empty, then $\ell_{zz'} < +\infty$ for all $z' \in \Omega$. For a given $z \in \Omega$, consider concave functions

$$g_e(i) = -M|z_e - i|,$$

for $e = 1, \dots, n$, $i = 1, \dots, \omega_e$, and $M > 0$. These functions define a modular-concave valuation m^* . Let us show that $m^* \in T_a^f$ for sufficiently large $M > 0$. Indeed, for any $z' \in \Omega$, we have

$$m^*(z) - m^*(z') = \sum_{e=1}^n (g_e(z_e) - g_e(z'_e)) = M \sum_{e=1}^n |z_e - z'_e| \geq \ell_{zz'}.$$

For $z' = z$, the above inequality is satisfied as $\ell_{zz} \equiv \ell_{aa} = 0$. For $z' \neq z$ and the above inequalities are satisfied as $\ell_{zz'} < +\infty$ and M is large enough.

Therefore, m^* lies in T_a^f . Hence, T_a^f contains a valuation that can be connected with a line segment to any valuation in T^{ggsc} . Overall, any monotone function $\tilde{f} : T^{ggsc} \rightarrow A_\Omega$ satisfies the conditions of Corollary 1. Hence, it is cyclically monotone. Therefore, every monotone function $f : T^{ggsc} \rightarrow \Omega$ is cyclically monotone.

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