Implementation on multiple and multi-dimensional single-peaked domains

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Abstract

We consider settings with transfers where agent preferences belong to either the domain of multiple single-peaked preferences or the domain of multi-dimensional single-peaked preferences on posets. For these two domains, we show that an allocation rule is implementable if and only if it is monotone.

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1 Introduction

Situated at the intersection of the mechanism design and social choice literatures, this paper contributes to the analysis of voting mechanisms with transferable utilities, an area that has attracted considerable research attention recently (see, e.g., Goeree and Zhang, 2017; Lalley and Weyl, 2018; Posner and Weyl, 2018). The major concern of the mechanism design literature is to characterize *implementable allocation rules.* An allocation rule is called implementable if it produces, when combined with a payment rule, a direct mechanism

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where truth-telling is in the best interest of each agent. Mishra et al. (2014) showed that for an important voting domain, the domain of single-peaked preferences, an allocation rule is implementable if and only if it is monotone.

The main result of this paper is to extend Mishra et al. (2014)'s characterization to two important multi-dimensional environments: the domain of *multiple single-peaked preferences* and the domain of *multi-dimensional single-peaked preferences* on partially ordered sets (posets). Each of these domains is an extension of the domain of single-peaked preferences to settings where there is no natural linear order among alternatives.

An agent preference profile is called *single-peaked* if there is an order of alternatives and an alternative, called a peak, such that agent preferences are increasing with respect to the linear order before the peak and agent preferences are decreasing after the peak. In many cases, voters' preferences cannot be described as single-peaked in a common linear order over alternatives (e.g., Niemi and Wright, 1987; Feld and Grofman, 1988; Pappi and Eckstein, 1998). The data are more consistent with the presence of multiple linear orders, where each voter forms a single-peaked preference with respect to a particular order. This observation led Reffgen (2015) to introduce the *multiple single-peaked preference domain,* which is the union of single-peaked preference domains with respect to several linear orders.[1]

The multiple single-peaked preference domain also allows us to describe the typical preferences of voters over multi-dimensional alternatives. Often, political parties take positions on many issues such as trade policy and climate change. The multi-dimensional nature of party positions does not allow for a complete order over alternatives. Instead, the set is only partially ordered. The domain of *multi-dimensional single-peaked preferences* on a poset is defined as consisting of all preference profiles that are single-peaked with respect to some linear order consistent with the given partial order (see also Reffgen, 2015).

The paper proceeds as follows. Section 2 presents notations and definitions. The main results are presented in Section 3. Finally, Section 4 concludes. The technical proofs are given in the Appendix.

2 Notations and Definitions

Consider a setting with a finite set of outcomes A, $N = |A|$ and the set of possible agent types $T \subset \mathbb{R}^N$. Type $t \in T$ is interpreted as a vector of the agent's payoffs for all possible outcomes. We consider settings with transferable utilities such that the agent's utility from

[1] See also Barberà et al. (1993) and Barberà et al. (1997).

outcome $a \in A$ can be written as $u(t, a, p) = t_a - p$, where p is the agent's payment. For convenience, we sometimes write this as $t_a = t(a)$.

We analyze deterministic direct mechanisms characterized by two functions: an allocation rule $f : T \to A$ and a payment rule $p : T \to \mathbb{R}$. The agent's utility from reporting type t' when her true type is t can be conveniently written as

$$t(f(t')) - p(t').$$

We consider the following definition of implementability:

Definition 1. *Allocation rule $f : T \to A$ is implementable if there exists a payment rule $p : T \to A$ such that mechanism (f, p) is incentive compatible; that is,*

$$t(f(t)) - p(t) \geq t(f(t')) - p(t') \quad \forall t, t' \in T.$$

The main contribution of this paper is to provide a characterization of implementable allocation rules on the domain of multiple single-peaked preferences and on the domain of multi-dimensional single-peaked preferences on posets. To define these domains, we begin by introducing *the domain of single-peaked preferences.* Assume there is a linear order $\prec$ over set A.

Definition 2. *The domain of single-peaked preferences $T^{(A,\prec)}$ is defined as*

$$T^{(A,\prec)} = \{t \in \mathbb{R}^N \,:\, \forall p, q, r \in A,\, p \prec q \prec r,\, t_q \geq t_p \;\; or \;\; t_q \geq t_r\}.$$

In other words, if $p \prec q \prec r$, then situation $t_p > t_q < t_r$ is forbidden. This definition, which goes back to Sen (1966), is equivalent to a more standard definition: For each type, there is an alternative, called a peak; the agent's utility increasing in $\prec$ for alternatives that precede the peak and decreasing in $\prec$ for alternatives that follow the peak (see Black, 1948).[2]

Let us denote the set of all possible linear orders over set A as Σ_A. Then, we have the following definition of the multiple single-peaked domain:

Definition 3. *Let $\mathcal{S} = \{\prec_1, ..., \prec_M\}$ be a family of linear orders with $\prec_m \in \Sigma_A, m = 1, ..., M$. The multiple single-peaked domain with respect to $(A, \mathcal{S})$ is defined as $T^{(A,\mathcal{S})} = \bigcup_{m=1}^{M} T^{(A,\prec_m)}$.*

A particular case of the domain of multiple single-peaked preferences is *the domain of single-peaked preferences on a poset.* To define it, consider a partial order $\lhd$ and a poset $(A, \lhd)$.

[2] See Reffgen (2015) for an extended discussion of the relation between these two definitions.

Definition 4. *The domain of multi-dimensional single-peaked preferences on poset* $(A, \lhd)$ *is defined as* $T^{(A,\lhd)} \equiv T^{(A,\mathcal{S}(\Delta))}$, *where* $\mathcal{S}(\Delta)$ *is the set of all linear orders on* A *that are consistent with* $\lhd$.

Monotone allocation rules will play an important role in our analysis.

Definition 5. *Function* $f : T \to A$ *is monotone if for all* $t, t' \in T$

$$t(f(t)) - t(f(t')) + t'(f(t')) - t'(f(t)) \geq 0.^{3} \tag{1}$$

The above definition provides a generalization of the one-dimensional monotonicity condition to multi-dimensional settings. It has been extensively used in convex analysis (see Rockafellar, 1966). The monotonicity condition appears also to be a relaxation of a more demanding cyclic monotonicity condition.

Definition 6. *Function* $f : T \to A$ *is* ***cyclically monotone*** *if for any integer* M *and any points* $t^0, t^1, \ldots, t^M = t^0$ *in* T

$$\sum_{k=0}^{M-1} t^k(f(t^k) - f(t^{k+1})) \geq 0. \tag{2}$$

The latter condition is important in light of Rochet (1987)'s result establishing that cyclic monotonicity is necessary and sufficient for implementability in quasi-linear environments.

Theorem (Rochet, 1987)**.** *An allocation rule is implementable iff it is cyclically monotone.*

Though the cyclic monotonicity condition characterizes the set of implementable allocation rules, this condition is often tedious to verify. In an important contribution, Saks and Yu (2005) showed that for convex domains the characterization of implementable allocation rules with a finite set of outcomes can be greatly simplified. They established that an allocation rule on a convex domain is cyclically monotone if and only if it is monotone. Unfortunately, neither the domain of multiple single-peaked preferences nor the domain of multi-dimensional single-peaked preferences on a poset is typically convex. Recently, Kushnir and Lokutsievskiy (2020) provided sufficient conditions for non-convex domains to guarantee that if f is monotone, it is also cyclically monotone. We will exploit these conditions together with Rochet (1987)'s theorem to establish our main results in the next section.

[3] The definition considers monotonically non-decreasing functions. We call such functions monotone to be consistent with the previous literature on convex analysis.

3 Main Results

This section presents our main results on the characterization of the implementable allocation rules on the domain of multiple single-peaked preferences and the domain of multi-dimensional single-peaked preferences on a poset.

Theorem 1. *Let A be a finite set of outcomes, $\mathcal{S} \subset \Sigma_A$ be a family of linear orders, and $T^{(A,\mathcal{S})}$ be the domain of multiple single-peaked preferences. Then, function $f : T^{(A,\mathcal{S})} \to A$ is implementable iff it is monotone.*

The proof of Theorem 1 uses Rochet (1987)'s result together with the sufficient conditions obtained by Kushnir and Lokutsievskiy (2020). To explain the latter conditions, we introduce some notations. For allocation rule $f : T \to A$, we consider a cover of T by a finite number of subsets. To define these subsets for any ordered pair $a, b \in A$, we consider

$$\ell_{ab} = \inf_{t \in T : f(t) = a} t(a - b).$$

Using these lower bounds, for each $a \in A$ we consider a subset

$$T_a^f = \{t \in T : t(a - b) \geq \ell_{ab}, \forall b \in A\}.$$

System $\{T_a^f\}_{a \in A}$ forms a cover of set T, $T = \bigcup_{a \in A} T_a^f$. Using the above notation, we consider the following property:

Definition 7. *Function $f : T \to A$ satisfies the **local-to-global** condition if for any two outcomes $a, b \in f(T)$ with $T_a^f \cap T_b^f = \emptyset$, there exists a path $\{a \equiv a_0, ..., a_M \equiv b\}$ such that $T_{a_m}^f \cap T_{a_{m+1}}^f \neq \emptyset$, $m = 0, ..., M - 1$ and $\ell_{ab} \geq \sum_{m=0}^{M-1} \ell_{a_m a_{m+1}}$.*

Note that lower bound ℓ_{ab} can be interpreted as the minimum benefit to accrue to the agent from revealing true type compared to lying when lying leads to outcome $b \in A$ (excluding transfers). Hence, $-\ell_{ab}$ are the maximum gains from lying. We interpret $-\ell_{ab}$, $T_a^f \cap T_b^f = \emptyset$ as the gains from global deviations, and $-\ell_{ab}$, $T_a^f \cap T_b^f \neq \emptyset$ as the gains from local deviations. Then, the local-to-global condition ensures that the gains from global deviations are smaller than the total gains from deviations along some path connecting $t \in T_a^f$ and some type in T_b^f.[4]

We call set S *path-connected* if it is non-empty and any two points $x \in S$ and $y \in S$ can be connected with a continuous curve lying inside S. A set S is called *simply connected* if

[4]See Carroll (2012) and Sato (2013) for related notions.

it is path-connected and any loop on S can be continuously contracted to a point. We can now state the following result:

Theorem (Kushnir and Lokutsievskiy, 2020)**.** *Consider a domain $T \subset \mathbb{R}^N$, a finite set A, and a function $f : T \to A$. Suppose that*

1) T is simply connected,

2) T_a^f is either path-connected or empty for each $a \in A$, and

3) f satisfies the local-to-global condition.

Then if f is monotone, it is also cyclically monotone.

Proof of Theorem 1. According to Rochet (1987), any implementable allocation rule is cyclically monotone. Hence, any implementable allocation rule is also monotone as the monotonicity condition is weaker than the cyclic monotonicity condition.

For the reverse implication that any monotone $f : T^{(A,\mathcal{S})} \to A$ is implementable, we show that $(T^{(A,\mathcal{S})}, A, f)$ satisfies the conditions of Kushnir and Lokutsievskiy (2020)'s theorem. Thus, f is cyclically monotone, and, hence, Rochet (1987) implies that it is implementable.

Denote family $\mathcal{S}$ as $\{\prec_1, ..., \prec_M\}$ and domain $T^{(A,\mathcal{S})} = \bigcup_{m=1}^{M} T^{(A,\prec_m)}$. To establish that domain $T^{(A,\mathcal{S})}$ is simply connected, notice that if $t \in T^{(A,\prec_m)}$ for some $m = 1, ..., M$, then $\alpha t \in T^{(A,\prec_m)} \subset T^{(A,\mathcal{S})}$ for $\alpha \geq 0$. Therefore, $T^{(A,\mathcal{S})}$ is path-connected, as any point can be joined by a segment with 0 within $T^{(A,\mathcal{S})}$. Moreover, any loop on $T^{(A,\mathcal{S})}$ can be continuously contracted to the origin 0. Hence, domain $T^{(A,\mathcal{S})}$ is simply connected.

The following two lemmas establish that every non-empty set T_a^f is path-connected and every monotone $f : T^{(A,\mathcal{S})} \to A$ satisfies the local-to-global property.

Lemma 1. *For any monotone function $f : T^{(A,\mathcal{S})} \to A$ and $a \in A$, every non-empty set T_a^f is path-connected.*

Proof. Take $t, t' \in T_a^f$. Let $t - M$ be a type where we subtract constant M from all the coordinates of t, and let $t' + M$ be a type where we add M to all the coordinates of t'. Notice that both $t - M$ and $t' - M$ still belong to T_a^f. Types t and $t - M$ can be connected with path $t - \lambda M$, $\lambda \in [0, -1]$. Similarly, t' and $t' - M$ can be connected with a path within T_a^f.

Hence, it remains to show that $t - M$ and $t' - M$ can be connected by a continuous path within T_a^f. To do this, we now prove a convenient fact: If $t^1, t^2 \in T_a^f$, then $\min(t^1, t^2) \in T_a^f$.

Define a polygon $\widetilde{T}_a^f = \{t \in \mathbb{R}^N : t_a \geq t_b + \ell_{ab}, \forall b \in A\}$. As $T_a^f = \widetilde{T}_a^f \cap T^{(A,\mathcal{S})}$, we have

$$t_a^1 \geq t_b^1 + \ell_{ab} \geq \min(t_b^1, t_b^2) + \ell_{ab} \quad \text{and} \quad t_a^2 \geq t_b^2 + \ell_{ab} \geq \min(t_b^1, t_b^2) + \ell_{ab} \qquad \text{for any } b \in A.$$

Hence, $\min(t_a^1, t_a^2) \geq \min(t_b^1, t_b^2) + \ell_{ab}$ for any $b \in A$ and $\min\{t^1, t^2\} \in \widetilde{T}_a^f$. To prove the fact, we need to show that $\min\{t^1, t^2\} \in T^{(A,\mathcal{S})}$. Consider linear order $\prec_m \in \mathcal{S}$ and domain $T^{(A,\prec_m)}$. We claim that domain $T^{(A,\prec_m)}$ has a property that for any two types $t^1, t^2 \in T^{(A,\prec_m)}$ their coordinate-wise minimum $\min\{t^1, t^2\}$ also belongs to domain $T^{(A,\prec_m)}$. Indeed, let us consider any three alternatives $p, q, r \in A$ with $p \prec_m q \prec_m r$. Assume that $t_q^1 = \min\{t_q^1, t_q^2\}$. As t^1 is single-peaked, we have that $t_q^1 \geq t_p^1$ or $t_q^1 \geq t_r^1$. This implies that $t_q^1 \geq \min\{t_p^1, t_p^2\}$ or $t_q^1 \geq \min\{t_r^1, t_r^2\}$. Therefore, $\min\{t^1, t^2\}$ is also single-peaked with respect to order $\prec_m$. Hence, if $t^1, t^2 \in T^{(A,\mathcal{S})}$, then $\min\{t^1, t^2\} \in T^{(A,\mathcal{S})}$. Overall, given $T_a^f = \widetilde{T}_a^f \cap T^{(A,\mathcal{S})}$, we obtain that $t^1, t^2 \in T_a^f$ implies $\min\{t^1, t^2\} \in T_a^f$, which proves the fact.

As a consequence, we have that $\min(t - \lambda, t' + \lambda) \in T_a^f$ for all $\lambda \in \mathbb{R}$. Notice that $\min(t - M, t' + M) = t - M$ and that $\min(t + M, t' - M) = t' - M$ when $M > 0$ is large enough. Hence, $t - M$ and $t' - M$ are connected within T_a^f by the path $\min(t - \lambda M, t' + \lambda M)$, $\lambda \in [-1; 1]$. As pairs t and $t - M$, $t - M$ and $t' - M$, and t' and $t' - M$ are each connected with a path in T_a^f, we obtain that t and t' are also connected with a path in T_a^f. $\square$

Lemma 2. *A monotone function $f : T^{(A,\mathcal{S})} \to A$ satisfies the local-to-global condition.*

Proof. To establish that a monotone function $f : T^{(A,\mathcal{S})} \to A$ satisfies the local-to-global condition, we show that f satisfies the following geometric property:

For any pair $a, b \in f(T^{(A,\mathcal{S})})$, any type $x \in T_a^f$, and any type $\varepsilon > 0$, there exist $x^\varepsilon \in T^{(A,\mathcal{S})}$ and $y^\varepsilon \in T_b^f$ such that $||x^\varepsilon - x||| \leq \varepsilon$ and interval $[x^\varepsilon, y^\varepsilon]$ lies in $T^{(A,\mathcal{S})}$.

In Lemma A1 in the Appendix, we prove that the above geometric property implies the local-to-global condition.[5]

Let us take some outcome $a \in A$ and some type $x \in T_a^f$. We also set some linear order $\prec$ in $\mathcal{S}$. We denote a weak version of the order as $\preceq$. We first show that if f is monotone, then we can always find a type in T_b^f that has its peak at b according to $\prec$. Indeed, take $t' = (-M, \ldots, -M, 0, -M, \ldots, -M)$, where 0 stays at position b. Type t' is single-peaked

[5]A version of this property was first suggested by Carroll (2012) to study when local incentive compatibility implies global incentive compatibility in *environments without transfers*. Carroll (2012) also established that every allocation rule defined on the domain of single-peaked preferences satisfies a version of the geometric property. Though his result does not apply to our setting, the proof below follows some of the ideas originally developed in his paper. See also Kushnir and Lokutsievskiy (2020).

with respect to any linear order. For $M \geq \max_{q \in A} \ell_{bq}$, type t' satisfies inequalities $t'_b - t'_q \geq \ell_{bq}$ for any $q \in A$. Therefore, $t' \in T^f_b$.

Let us assume that x has its peak at p according to order $\prec$. We set ε and consider a strict single-peaked type $x' \in T$ with the peak at p such that $||x' - x|| \leq \varepsilon$. In particular, $x'_q < x'_{q'}$ for any q, q' satisfy $q \prec q' \preceq p$ or $p \preceq q' \prec q$.

First, let us consider the case when $b = p$. Then put $y' = t' \in T^f_b$. Hence, both x'_q and y'_q are increasing for $q \preceq p$ and decreasing for $q \succeq p$. Hence, for all $\beta \in [0, 1]$ all types $(1-\beta)x'_q + \beta y'_q$ are increasing for $q \preceq p$ and decreasing for $q \succeq p$. Hence, all types in interval $[x', y']$ are single-peaked and $[x', y'] \subset T^{(A,\prec)}$.

Now consider case $b \succ p$ (a similar argument applies if $b \prec p$). Both x'_q and t'_q are increasing for $q \preceq p$ and decreasing for $q \succeq b$ together with their convex combination. For $p \preceq q \preceq b$, type x'_q is decreasing in q and t'_q is increasing in q.

We now construct a new type $y' \in T^f_b$. We pick y'_b arbitrarily and choose y'_{b-1} such that $y'_b - y'_{b-1} > t'_b - t'_{b-1}$. If $p < b - 1$, we then choose y'_q for $q = b-2, ..., p$ satisfying the inequalities

$$\frac{y'_{q+2} - y'_{q+1}}{x'_{q+1} - x'_{q+2}} < \frac{y'_{q+1} - y'_q}{x'_q - x'_{q+1}} \tag{3}$$

$$t'_{q+1} - t'_q < y'_{q+1} - y'_q \tag{4}$$

This can be done by choosing y'_q at a low enough value at each step. Finally, for $q \prec p$ we choose y'_q such that $t'_{q+1} - t'_q < y'_{q+1} - y'_q$, and for $q \succ b$ we choose y'_q such that $t'_{q-1} - t'_q < y'_{q-1} - y'_q$.

We now show that inequalities (3) and (4) ensure that $(1-\beta)x' + \beta y'$ is single-peaked, i.e., $(1-\beta)x'_q + \beta y'_q$ increases in q up to some peak and it decreases in q following the peak. Note that both x'_q and y'_q are increasing for $q \preceq p$ and decreasing for $q \succeq b$, as is the case for their convex combination. Let us now consider $p \preceq q \preceq b$. Assume that there exists $q \in \{p, ..., b-2\}$ such that

$$(1-\beta)x'_q + \beta y'_q > (1-\beta)x'_{q+1} + \beta y'_{q+1} \tag{5}$$

and

$$(1-\beta)x'_{q+1} + \beta y'_{q+1} < (1-\beta)x'_{q+2} + \beta y'_{q+2}. \tag{6}$$

We can rearrange inequalities (5) and (6) to obtain

$$\frac{1-\beta}{\beta} > \frac{y'_{q+1} - y'_q}{x'_q - x'_{q+1}} \qquad \frac{1-\beta}{\beta} < \frac{y'_{q+2} - y'_{q+1}}{x'_{q+1} - x'_{q+2}}$$

where we used that x' is *strict single-peaked*, i.e., $x'_q > x'_{q+1} > x'_{q+2}$. Note that $\beta \neq 0$; otherwise, inequality (6) must be violated. The above two inequalities contradict (3). Hence, $[x', y'] \subset T^{(A,\prec)}$. Note also that inequalities (4) (together with the choice of y'_q for $q \prec p$ and $q \succ b$) guarantee that for any outcome $c \neq b$ we have $y'_b - y'_c > t'_b - t'_c$. Hence, if outcome c was chosen for type y' that would violate monotonicity. Hence, $y' \in T^f_b$. Overall, any monotone function $f : T^{(A,\mathcal{S})} \to A$ satisfies the geometric property and, hence, it also satisfies the local-to-global condition (see Lemma A1 in the Appendix). $\square$

Overall, the result of Theorem 1 follows from Lemmas 1 and 2. $\square$

Taking into account that every domain of multi-dimensional single-peaked preferences on poset $(A, \lhd)$ is also a domain of multiple single-peaked preferences, we also obtain the following corollary:

Corollary 1. *Assume $T^{(A,\lhd)} \subset \mathbb{R}^N$ is the domain of multi-dimensional single-peaked preferences on poset $(A, \lhd)$. Then, $f : T^{(A,\lhd)} \to A$ is implementable if and only if it is monotone.*

The corollary is an extension of Mishra and Roy (2013)'s result for the domain of single-peaked preferences to multi-dimensional preference domains that have a similar structure. This extension is a contribution to the literature at the intersection of mechanism design and social choice that should advance research into implementable voting mechanisms in multi-dimensional settings with transfers, an area that is drawing a lot of recent research interest (see Goeree and Zhang, 2017; Lalley and Weyl, 2018; Posner and Weyl, 2018).

4 Conclusion

In this paper, we use insights offered by Rochet (1987) and Kushnir and Lokutsievskiy (2020) to characterize the set of implementable allocation rules on the domain of *multiple single-peaked preferences* and on the domain of *multi-dimensional single-peaked preferences* on a poset. In particular, we establish that an allocation rule is implementable if and only if it is monotone. This characterization adds the above domains to the previously established list of only two other non-convex domains where such a characterization is possible: the domain of single-peaked preferences (see Mishra and Roy, 2013) and the domain of gross substitutes (see Kushnir and Lokutsievskiy, 2020).

Appendix

Lemma A1. *Suppose that $f : T^{(A,\mathcal{S})} \to A$ is monotone and for any pair $a, b \in f(T^{(A,\mathcal{S})})$, any type $x \in T^f_a$, and any $\varepsilon > 0$, there exist $x^\varepsilon \in T^{(A,\mathcal{S})}$ and $y^\varepsilon \in T^f_b$ such that $||x^\varepsilon - x||| \leq \varepsilon$ and interval $[x^\varepsilon, y^\varepsilon]$ lies in $T^{(A,\mathcal{S})}$. Then, f satisfies the local-to-global condition.*

Proof. Consider a monotone function $f : T^{(A,\mathcal{S})} \to A$ and outcomes $a, b \in A$ with $T^f_a \cap T^f_b = \emptyset$. Take some $x \in T^f_a$. The geometric condition stated in the lemma then implies that there must exist $x^\varepsilon \in T^{(A,\mathcal{S})}$ and $y^\varepsilon \in T^f_b$ such that $||x^\varepsilon - x||| \leq \varepsilon$ and $[x^\varepsilon, y^\varepsilon]$ lies in $T^{(A,\mathcal{S})}$.

Consider sets $\widetilde{T}^f_q = \{t \in \mathbb{R}^N : t(q - b') \geq \ell_{qb'}, \forall b' \in A\}$. Then, $T^f_q = \widetilde{T}^f_q \cap T^{(A,\mathcal{S})}$. As any set $\widetilde{T}^f_q$ is closed and convex, intersection $[x^\varepsilon, y^\varepsilon] \cap \widetilde{T}^f_q = [x^\varepsilon, y^\varepsilon] \cap T^f_q$ is a closed interval, a point, or an empty set. Therefore, we are able to choose

1) a path $\{a \equiv a_0, ..., a_K \equiv b\}$ such that $T^f_{a_k} \cap T^f_{a_{k+1}} \neq \emptyset$, $k = 0, ..., K-1$, $[x^\varepsilon, y^\varepsilon] \cap T^f_{a_k} \neq \emptyset$;

2) points $z_k \in [x^\varepsilon, y^\varepsilon] \cap T^f_{a_k}$ such that $z_{k+1} - z_k \neq 0$ and vectors $z_{k+1} - z_k$ and $y^\varepsilon - x^\varepsilon$ have the same direction for any $k = 1, ..., K-1$.

This can be done in the following way. We put $a_0 = a$ and $z_0 = x^\varepsilon$. Then, we denote the right end of interval $[x^\varepsilon, y^\varepsilon] \cap T^f_{a_0}$ by z_1. For point z_1, there must exist $a_1 \neq a_0$ such that $T^f_{a_1} \cap [x^\varepsilon, y^\varepsilon]$ is a segment and z_1 belongs to its interior. We denote the right end of the segment by z_2. Note that $T^f_{a_0} \cap T^f_{a_1} \neq \emptyset$, as z_1 belongs to both sets. Moreover, $z_2 \neq z_1$ and vectors $z_2 - z_1$ and $x^\varepsilon - y^\varepsilon$ have the same direction by the choice of a_1. For point z_2, there must exist $a_2 \neq a_0, a_1$ such that $T^f_{a_2} \cap [x^\varepsilon, y^\varepsilon]$ is a segment and z_1 belongs to its interior. We denote the right end of the segment by z_3 and repeat the process until we cover the whole interval $[x^\varepsilon, y^\varepsilon]$. We will finish in a finite number of steps as set A is finite and at each step we pick points from different outcome sets. Hence, we establish properties 1) and 2) stated above. Note that we do not exclude the case $z_0 = z_1$ as $k \geq 1$ in 2).

The conditions of the theorem imply that $x(a - b) \geq x^\varepsilon(a - b) - \varepsilon||a - b||$. In addition, for each $z_k \in [x^\varepsilon, y^\varepsilon] \cap T^f_{a_k}$, $k = 1, ..., K$, we can write

$$x^\varepsilon(a - b) = \sum_{k=0}^{K-1} x^\varepsilon(a_k - a_{k+1}) = x^\varepsilon(a_0 - a_1) + \sum_{k=1}^{K-1} (x^\varepsilon - z_k)(a_k - a_{k+1}) + \sum_{k=1}^{K-1} z_k(a_k - a_{k+1}).$$

As all z_k belong to the same interval $[x^\varepsilon, y^\varepsilon]$, there exists λ_k such that $x^\varepsilon - z_k = \lambda_k(z_k - z_{k+1})$. Moreover, $\lambda_k > 0$ by the choice of z_k, $k \geq 1$. Hence, weak-monotonicity implies

$$(x^\varepsilon - z_k)(a_k - a_{k+1}) = \lambda_k(z_k - z_{k+1})(a_k - a_{k+1}) \geq 0.$$

In addition, $x^\varepsilon(a_0-a_1) \geq \ell_{a_0a_1} - \varepsilon||a_0-a_1||$ and $z_k(a_k-a_{k+1}) \geq \ell_{a_ka_{k+1}}$ for each $k = 1, .., K-1$. Therefore,

$$x(b-a) \geq x_\varepsilon - \varepsilon||a-b|| \geq \sum_{k=0}^{K-1} \ell_{a_ka_{k+1}} - \varepsilon(||a-b|| + ||a_1 - a_0||)$$

or

$$x(b-a) \geq \sum_{k=0}^{K-1} \ell_{a_ka_{k+1}} - 2\varepsilon \max_{c,d\in A} ||c-d||.$$

Notice that path $\{a \equiv a_0, ..., a_K \equiv b\}$ can generally depend on the choice of x and ε. However, the number of such paths is finite because these paths contain no cycles by construction. Taking infimum over all points $x \in T_a^f$ and taking $\varepsilon \to 0$, we obtain that for some path $\{a \equiv a_0, ..., a_K \equiv b\}$ we have $\ell_{ab} \geq \sum_{k=0}^{K-1} \ell_{a_ka_{k+1}}$. □

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