A Geometric Approach to Mechanism Design

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Abstract

Applying basic techniques from convex analysis and majorization theory we develop a novel approach to mechanism design. Using the new approach we provide a simple and unified treatment of the optimal mechanisms for *any* linear social choice problems and *any* linear objectives, including revenue and welfare maximization. We also present further results and applications.

JEL codes: *D44, D82*

Keywords: *mechanism design, optimal mechanisms, convex analysis, support function, majorization, ironing, Hotelling's lemma, reduced-form, BIC-DIC equivalence*

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1. Introduction

Mechanism design concerns the creation of optimal social systems by maximizing well-defined objectives taking into account resource constraints and participants' incentives and hidden information. It provides a framework to address questions like "what auction format assigns goods most efficiently or yields the highest seller revenue" and "when should a public project such as building a highway be undertaken?" The difficulty in answering these questions stems from the fact that the designer typically does not possess detailed information about bidders' valuations for the goods or about voters' preferences for the public project. A well-designed mechanism should therefore elicit participants' private information in a truthful, or incentive compatible, manner and implement the corresponding social optimum accordingly.

The constraints imposed by incentive compatibility are generally treated separately from other more basic constraints, such as resource constraints. As a result, mechanism design theory appears to have developed differently from classical approaches to consumer and producer choice theory despite some obvious parallels. For example, in producer choice theory, the firm also maximizes a well-defined objective: its profit. Given a feasible production set it is a standard, albeit potentially tedious, exercise to compute the firm's profit as a function of input and output prices. In turn, given a firm's profit function its production set can be uniquely recovered, and the firm's optimal production plan follows by taking the subgradient of the profit function – *Hotelling's lemma*.

In this paper, we draw a parallel with classical choice theory to provide a novel geometric approach to mechanism design for *any* linear one-dimensional social choice problems.[1] We observe that the set of feasible allocations – the analogue of the production set – consists of a collection of simplices for which the *support function* – the analogue of the profit function – can be obtained "off the shelf" without doing any calculations. The relationship between the support function and the corresponding convex set then define inequalities that clarify the origin of the "Maskin-Riley-Matthews" conditions for reduced-form auctions (Maskin

[1]For recent interesting approaches to mechanism design, see Kos and Messner (2013)[35], who introduced new techniques building on the concept of extremal transfers to analyze mechanism design problems with arbitrary type spaces. Noldeke and Samuelson (2018)[53] also used the duality techniques to gain novel insights into adverse-selection principal-agent problems and two-sided matching problems without quasilinearity.

and Riley, 1984 [48]; Matthews, 1984 [49]; Border, 1991, 2007 [11, 12]) and allow us to extend reduced-form implementation to social choice settings.

As noted above, a distinguishing role in mechanism design is played by incentive compatibility, which we incorporate using their geometric characterization. Borrowing results from majorization theory due to Hardy, Littlewood, and Pólya (1929) [29] we elucidate the "ironing" procedure introduced by Mussa and Rosen (1978) [51] and Myerson (1981) [52]. We show that the support function for the set of feasible and incentive compatible allocations is simply the support function for the feasible set, evaluated at ironed weights. Furthermore, we establish the equivalence of Bayesian and dominant strategy implementation (Manelli and Vincent, 2010, 2019 [44, 45]; Gershkov et al., 2013 [24]; Kushnir, 2015 [37]; Kushnir and Liu, 2019, 2020 [38] [39]) by showing that the same support function results whether Bayesian or dominant strategy incentive constraints are imposed.

To summarize, the support function for the set of feasible and incentive compatible allocations for *any* linear one-dimensional social choice problems – not just auctions – can be obtained using off-the-shelf results from convex analysis and majorization theory that predate any research in mechanism design. Moreover, the support function is piece-wise linear and it is straightforward to take the subgradient and apply Hotelling's lemma to derive the optimal mechanism for *any* linear objective. Finally, we adapt our approach to include general concave objectives that depend on both allocations and transfers and provide a simple fixed-point condition characterizing the optimal mechanism.

This paper is organized as follows. Section 2 illustrates our approach with a simple auction example. Section 3 considers linear one-dimensional social choice problems: we derive the support function for the set of feasible allocations (Section 3.1), discuss reduced form implementation (Section 3.2), incorporate incentive compatibility (Section 3.3), establish equivalence of Bayesian and dominant strategy implementation (Section 3.4), and derive the optimal mechanism for arbitrary linear objectives (Section 3.5). Section 4 considers concave objectives and incorporates transfers into the support function. We discuss how the geometric approach is related to the classical Myersonian approach and some recent advances in the mechanism design and computer science literatures in Section 5. Section 6 concludes by presenting the most promising research areas where the geometric approach can be applied. The Appendix contains all proofs.

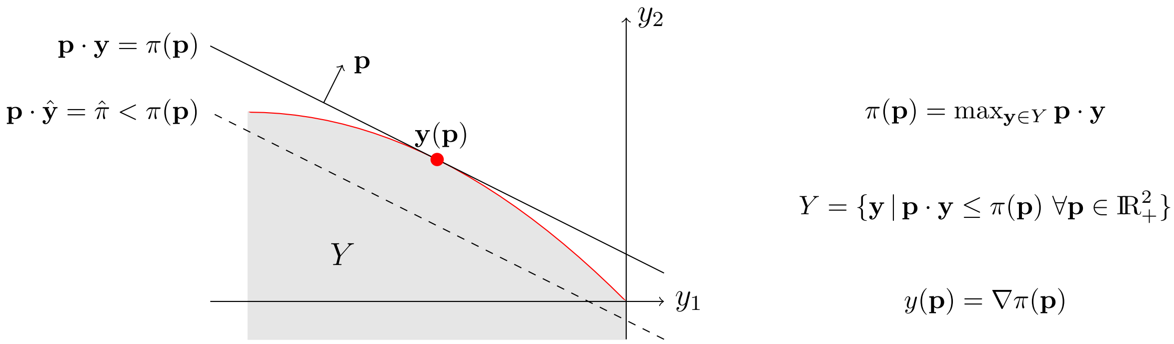

Figure 1. A profit-maximization example to illustrate (i) the relationship between production set $Y=\{(-y_1,y_2)\in\mathbb{R}^2_+\,|\,y_2\le\sqrt{-y_1}\}$ and its profit function $\pi(\mathbf{p})=\frac{p_2^2}{4p_1}$, and (ii) Hotelling's lemma.

2. A Simple Example

Consider a standard producer choice problem $\pi(\mathbf{p})=\max_{\mathbf{y}\in Y}\mathbf{p}\cdot\mathbf{y}$ where the production set is characterized by a square-root production technology $Y=\{(-y_1,y_2)\in\mathbb{R}^2_+\,|\,y_2\le\sqrt{-y_1}\}$, see Figure 1. It is readily verified that the optimal levels of inputs and outputs are given by $y_2(\mathbf{p})=\sqrt{-y_1(\mathbf{p})}=\frac{p_2}{2p_1}$, resulting in profits $\pi(\mathbf{p})=\frac{p_2^2}{4p_1}$. Given a convex production set the profit function is uniquely determined and, in turn, the profit function uniquely determines the production set $Y=\{\mathbf{y}\,|\,\mathbf{p}\cdot\mathbf{y}\le\pi(\mathbf{p})\;\forall\mathbf{p}\in\mathbb{R}^2_+\}$. Moreover, it determines the optimal input and output via *Hotelling's lemma*, $y(\mathbf{p})=\nabla\pi(\mathbf{p})$. The main innovation of this paper is to apply these well-known micro-economics tools to problems in mechanism design, e.g. to derive optimal mechanisms using the subdifferential of the support function.

To this end, we define the *support function* $S^C:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ of a closed convex set $C\subset\mathbb{R}^n$ as

$$S^C(\mathbf{w})\;=\;\sup\{\mathbf{v}\cdot\mathbf{w}\,|\,\mathbf{v}\in C\},$$

with $\mathbf{v}\cdot\mathbf{w}=\sum_{j=1}^n v_jw_j$ the usual inner product. From the support function one can recover the associated convex set, $C=\left\{\mathbf{v}\in\mathbb{R}^n\,|\,\mathbf{v}\cdot\mathbf{w}\;\le\;S^C(\mathbf{w})\;\forall\,\mathbf{w}\in\mathbb{R}^n\right\}$, and the solution to the maximization problem $\sup_{\mathbf{v}\in C}\boldsymbol{\alpha}\cdot\mathbf{v}$ as $\mathbf{v}(\boldsymbol{\alpha})=\nabla S^C(\boldsymbol{\alpha})$.[2] Of course, this approach would be unattractive if computing the support function was tedious or intractable. For a

[2] While the support function may not be everywhere differentiable, it is subdifferentiable as it is a convex function that is the supremum of linear functions. At points of non-differentiability, any $\mathbf{v}\in\nabla S^C(\boldsymbol{\alpha})$, where ∇S^C denotes the subdifferential, is a solution (see Rockafellar, 1997 [54]).

Feasibility constraints	$0 \leq q_{ll} \leq \frac{1}{2}$	$0 \leq q_{lh},\ q_{hl},\ q_{lh} + q_{hl} \leq 1$	$0 \leq q_{hh} \leq \frac{1}{2}$
Support function	$\frac{1}{2}\max(0, w_{ll})$	$\max(0, w_{lh}, w_{hl})$	$\frac{1}{2}\max(0, w_{hh})$

Table 1. Feasibility constraints and associated support functions for a simple example.

broad class of mechanism design problems, however, the underlying feasible set is simply a product of probability simplices for which the support function is well known.

To illustrate, consider a single-unit auction with two ex ante symmetric bidders and two equally likely types, $x_l < x_h$. Assuming a symmetric allocation rule, the probability that a bidder obtains the object is summarized by $\mathbf{q} = (q_{ll}, q_{lh}, q_{hl}, q_{hh})$ where the first (second) subscript denotes the bidder's (rival's) type. The symmetry and feasibility constraints are presented in the first line of Table 1 while the second line shows the associated support functions. The set of feasible allocations is the Cartesian product of the three sets presented in the first line for which the support function is simply the sum of individual support functions

$$S^{\mathbf{q}}(\mathbf{w}) \;=\; \frac{1}{2}\max(0, w_{ll}) + \max(0, w_{lh}, w_{hl}) + \frac{1}{2}\max(0, w_{hh}) \tag{1}$$

A bidder's interim (or expected) allocations $\mathbf{Q} = (Q_l, Q_h)$ are linear transformations of the ex post allocations: $Q_l = \frac{1}{2}(q_{ll} + q_{lh})$ and $Q_h = \frac{1}{2}(q_{hl} + q_{hh})$, which we summarize as $\mathbf{Q} = L\mathbf{q}$ with L being the relevant two-by-four matrix. A basic property of the inner product is that $\mathbf{Q} \cdot \mathbf{W} = L\mathbf{q} \cdot \mathbf{W} = \mathbf{q} \cdot L^T\mathbf{W}$, from which it follows that the support function for the set of feasible interim allocations is

$$S^{\mathbf{Q}}(\mathbf{W}) \;=\; S^{\mathbf{q}}(L^T\mathbf{W}) \;=\; \frac{1}{4}\max(0, W_l) + \frac{1}{2}\max(0, W_l, W_h) + \frac{1}{4}\max(0, W_h) \tag{2}$$

The set of feasible interim allocations follows from $\mathbf{Q} \cdot \mathbf{W} \leq S^{\mathbf{Q}}(\mathbf{W})$ for all $\mathbf{W} \in \mathbb{R}^2$ and is shown in the left panel of Figure 2.[3]

Of course, not all feasible allocations satisfy Bayesian incentive compatibility (BIC), which requires that interim allocations are monotonic in types: $Q_h \geq Q_l$ (see Myerson, 1981 [52]). Graphically, the set of BIC allocations can be seen as the intersection of the

[3] One can easily explain the maximum expected probability of winning $\frac{3}{4}$. Symmetry implies that a bidder wins with probability $\frac{1}{2}$ when facing a rival of the same type, which occurs with probability $\frac{1}{2}$. Hence, the maximum expected probability of winning is $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{3}{4}$.

set of feasible interim allocations and the "above the 45-degree line" half-space (see the middle panel of Figure 2). This half-space can be written as $(1,-1)\cdot\mathbf{Q} \leq 0$ with associated support function

$$S^H(\mathbf{W}) \;=\; \begin{cases} 0 & \text{if} \quad \mathbf{W} \;=\; \Lambda(1,-1) \\ \infty & \text{if} \quad \mathbf{W} \;\neq\; \Lambda(1,-1) \end{cases}$$

for any $\Lambda \geq 0$. The support function for the intersection follows from the convolution

$$S^{BIC}(\mathbf{W}) \;=\; \inf_{\mathbf{W}_1+\mathbf{W}_2=\mathbf{W}} \; S^{\mathbf{Q}}(\mathbf{W}_1) + S^H(\mathbf{W}_2) \;=\; \inf_{\Lambda\geq 0} S^{\mathbf{Q}}(\mathbf{W}-\Lambda(1,-1))$$

The solution to this minimization problem is $\Lambda = \frac{1}{2}\max(0, W_l - W_h)$ so that $S^{BIC}(\mathbf{W}) = S^{\mathbf{Q}}(\mathbf{W}_+)$ where $\mathbf{W}_+$ denote "ironed" weights

$$\mathbf{W}_+ \;=\; \begin{cases} (W_l, W_h) & \text{if} \quad W_l \;\leq\; W_h \\ \frac{1}{2}(W_l+W_h, W_l+W_h) & \text{if} \quad W_l \;>\; W_h \end{cases} \tag{3}$$

Now consider maximization of a linear objective $\boldsymbol{\alpha}\cdot\mathbf{Q} = \alpha_l Q_l + \alpha_h Q_h$ over the set of feasible BIC allocations. For example, revenue maximization corresponds to $\boldsymbol{\alpha} = (2x_l - x_h, x_h)$, see equation (12) in Section 3.5, while welfare maximization correspond to $\boldsymbol{\alpha} = (x_l, x_h)$. In the revenue-maximization case, either $\alpha_l < 0 < \alpha_h$ which yields $\nabla S^{BIC}(\boldsymbol{\alpha}) = (0, \frac{3}{4})$, or $0 < \alpha_l < \alpha_h$ which yields $\nabla S^{BIC}(\boldsymbol{\alpha}) = (\frac{1}{4}, \frac{3}{4})$ as indicated by the small and medium-sized dots in Figure 2.[4] These optimal interim allocations follow by using the symmetric allocation rules $\mathbf{q} = (0, 0, 1, \frac{1}{2})$ and $\mathbf{q} = (\frac{1}{2}, 0, 1, \frac{1}{2})$ respectively. The intuition is that the low type is screened out (e.g. by using a reserve price) when the marginal revenue $2x_l - x_h$ is negative while the allocation rule is efficient when this marginal revenue is positive.

The efficient allocation rule is also optimal for welfare maximization, as this is another example when $0 < \alpha_l < \alpha_h$. A new solution arises when $0 < \alpha_h < \alpha_l$, e.g. when the social objective places higher weight on the low type possibly because of redistributive or fairness concerns. The support function in (2) reduces to $\frac{1}{2}W_l + \frac{1}{2}W_h$ when $0 < W_h < W_l$, since the weights are replaced by their ironed versions, see (3). Hence, $\nabla S^{BIC}(\boldsymbol{\alpha}) = (\frac{1}{2}, \frac{1}{2})$, a solution shown by the large dot in Figure 2. This solution is implemented by the random allocation rule $\mathbf{q} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

[4] Note that $S^{BIC}(\mathbf{W})$ reduces to $\frac{3}{4}W_h$ when $W_l < 0 < W_h$ and to $\frac{1}{4}W_l + \frac{3}{4}W_h$ when $0 < W_l < W_h$.

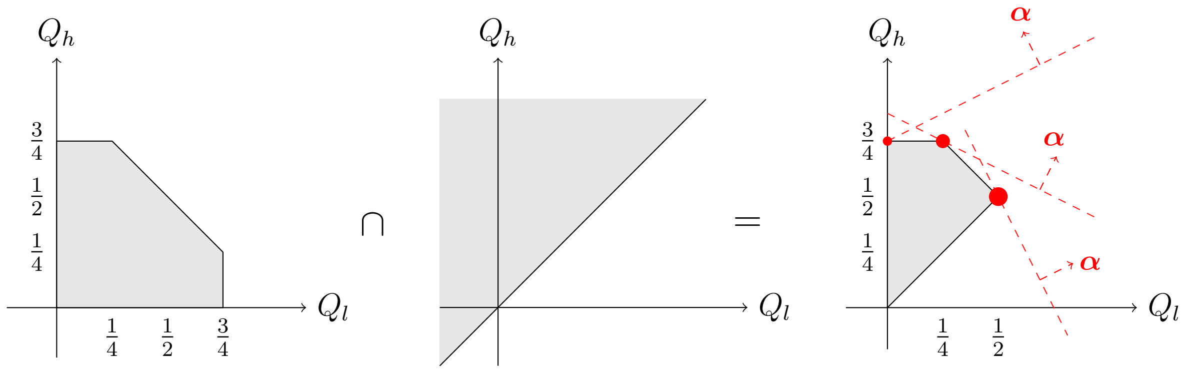

Figure 2. The set of feasible Bayesian incentive compatible interim allocations (right) can be seen as the intersection of the feasible set (left) with the "above the 45-degree line" half-space (middle). On the right, the dashed lines are level-surfaces for the linear objective $\boldsymbol{\alpha} \cdot \mathbf{Q}$. The dots indicate optimal allocations when $\alpha_l < 0 < \alpha_h$ (small), $0 < \alpha_l < \alpha_h$ (medium), $0 < \alpha_h < \alpha_l$ (large).

Overall, the above example illustrates how the support function for the set of feasible and BIC interim allocations can be derived using basic techniques of convex analysis. The optimal mechanisms for any linear objectives then follow from Hotelling's lemma. We generalize these insights to social choice environments and provide more novel results in the next section.

3. Social Choice Implementation

We consider a linear one-dimensional social choice environment with independent private values and quasi-linear utilities. There is a finite set of agents $\mathcal{I} = \{1, 2, \ldots, I\}$ and a finite set of social alternatives $\mathcal{K} = \{1, 2, \ldots, K\}$. When alternative k is selected, agent i's payoff equals $a^i_k x^i$ where $a^i_k \in \mathbb{R}$ is common knowledge and $x^i \in \mathbb{R}_+$ is agent i's privately-known type, which is distributed according to a commonly known probability distribution $f^i(x^i)$ with discrete support $X^i = \{x^i_1, \ldots, x^i_{N^i}\}$, where $x^i_j < x^i_{j+1}$ for $j = 1, ..., N_i - 1$.[5] Let $\mathbf{x} = (x^1, ..., x^I)$ denote the profile of agents' types with $\mathbf{x} \in X = \prod_{i \in \mathcal{I}} X^i$. Without loss of generality we restrict attention to direct mechanisms characterized by $K + I$ functions,

[5]This formulation includes many important applications, e.g. single or multi-unit auctions, public goods provision, bilateral trade, etc.

$\{q_k(\mathbf{x})\}_{k\in\mathcal{K}}$ and $\{t^i(\mathbf{x})\}_{i\in\mathcal{I}}$, where $q_k(\mathbf{x})$ is the probability that alternative k is selected and $t^i(\mathbf{x}) \in \mathbb{R}$ is agent i's payment. We define agent i's value as $v^i(\mathbf{x}) \equiv \sum_{k\in\mathcal{K}} a^i_k q_k(\mathbf{x})$ so that agent i's utility from truthful reporting, assuming others report truthfully as well, is $u^i(\mathbf{x}) = x^i v^i(\mathbf{x}) - t^i(\mathbf{x})$. We use capital letters to indicate interim variables: $V^i(x^i) = E_{x^{-i}}(v^i(\mathbf{x}))$, $T^i(x^i) = E_{x^{-i}}(t^i(\mathbf{x}))$, and $U^i(x_i) = x^i V^i(x^i) - T^i(x^i)$ denote agent i's interim value, interim payment, and interim utility respectively.

3.1. Feasibility

The probabilities with which the alternatives occur satisfy the usual feasibility conditions: they should be non-negative, $q_k(\mathbf{x}) \geq 0$ for $k \in \mathcal{K}$, and sum up to one, $\sum_{k\in\mathcal{K}} q_k(\mathbf{x}) = 1$. In other words, for each type profile, $\mathbf{q}(\mathbf{x}) = \{q_k(\mathbf{x})\}_{k\in\mathcal{K}}$ defines a K-dimensional simplex with support function $S^{\mathbf{q}(\mathbf{x})}(\mathbf{w}(\mathbf{x})) = \max_{k\,\in\,\mathcal{K}} w_k(\mathbf{x})$ and $\mathbf{w}(\mathbf{x}) \in \mathbb{R}^K$. Furthermore, the support function for the Cartesian product of sets equals the sum of support functions (Rockafellar, 1997 [54]) so the support function for the set of all feasible allocations $\mathbf{q} = \{\mathbf{q}(\mathbf{x})\}_{x\in X}$ is given by

$$S^{\mathbf{q}}(\mathbf{w}) \;=\; \sum_{\mathbf{x}\,\in\,X} \max_{k\,\in\,\mathcal{K}} w_k(\mathbf{x})$$

where $\mathbf{w} = \{\mathbf{w}(\mathbf{x})\}_{\mathbf{x}\in X} \in \mathbb{R}^{K|X|}$.

For vector $\mathbf{q} \in \mathbb{R}^{K|X|}$ and any linear transformation A, we have $A\mathbf{q}\cdot\mathbf{w} = \mathbf{q}\cdot A^T\mathbf{w}$ where A^T is the transpose of A. Hence, for set of probability simplicies C, we have $S^{AC}(\mathbf{w}) = S^C(A^T\mathbf{w})$. Therefore, the support function for the set of feasible values $v^i(\mathbf{x}) = \sum_{k\in\mathcal{K}} a^i_k q_k(\mathbf{x})$ equals

$$S^{\mathbf{v}}(\tilde{\mathbf{w}}) \;=\; \sum_{\mathbf{x}\,\in\,X} \max_{k\,\in\,\mathcal{K}} \sum_{i\,\in\,\mathcal{I}} a^i_k \tilde{w}^i(\mathbf{x}) \tag{4}$$

where $\tilde{\mathbf{w}} = \{\tilde{w}^i(\mathbf{x})\}_{\mathbf{x}\in X, i\in\mathcal{I}} \in \mathbb{R}^{\sum_i |X^i|}$. Moreover, interim values are a linear transformation of values: $V^i(x^i) = \sum_{x^{-i}} f^{-i}(x^{-i})v^i(\mathbf{x})$ where $f^{-i}(x^{-i}) = \prod_{j\neq i} f^j(x^j)$. To arrive at expressions symmetric in probabilities we define the support function for interim values using a probability-weighted inner product

$$\mathbf{V}\cdot\mathbf{W} \;=\; \sum_{i\,\in\,\mathcal{I}} \sum_{x^i\,\in\,X^i} f^i(x^i)V^i(x^i)W^i(x^i), \tag{5}$$

where $\mathbf{W} \in \mathbb{R}^{\sum_i |X^i|}$. Under the interim transformation all terms are then multiplied by

$\prod_{i\in\mathcal{I}} f^i(x^i)$ and the sum over type profiles in (4) turns into an expectation.

THEOREM 1. *The support function for the set of feasible interim values is*

$$S^{\mathbf{V}}(\mathbf{W}) \;=\; E_{\mathbf{x}}\,\big(\max_{k\,\in\,\mathcal{K}} \sum_{i\,\in\,\mathcal{I}} a_k^i W^i(x^i)\big) \tag{6}$$

and the feasible interim values $\mathbf{V}$ *satisfy* $\mathbf{V}\cdot\mathbf{W} \le S^{\mathbf{V}}(\mathbf{W})$ *for all* $\mathbf{W} \in \mathbb{R}^{\sum_i |X^i|}$.[6]

The derivations of Theorem 1 readily extend to environments with continuous types, e.g., $X = [0,1]^I$, if one considers allocations to be integrable functions and the support functional of the set of feasible interim values defined on the dual space (see Section 7.10 in Aliprantis and Border (2006) [16]). In addition, the result of Theorem 1 holds even if agent types are multi-dimensional or correlated. Goeree and Kushnir (2016) [28] also extend this result to environments with non-linear utilities and interdependent values.

3.2. Reduced Form Implementation

It is insightful to work out the inequalities in Theorem 1 for single-unit auctions, which fit the social choice framework as follows: alternative $i = 1,\ldots,I$ corresponds to the event when bidder i wins, i.e. $a_i^i = 1$ and $a_k^i = 0$ for $k \ne i$, and alternative $I+1$ corresponds to the event when the seller keeps the object. In this case, the reduced form value $V^i(x^i)$ is equal to a bidder i's interim chance of winning $Q^i(x^i) = E_{\mathbf{x}^{-i}}(q^i(\mathbf{x}))$ and the support function in Theorem 1 simplifies to

$$S^{\mathbf{Q}}(\mathbf{W}) \;=\; E_{\mathbf{x}}\big(\max_{i\,\in\,\mathcal{I}}\,(0, W^i(x^i))\big), \tag{7}$$

where the zero corresponds to the alternative when the seller keeps the object. An exhaustive set of inequalities follows by choosing, for each $i \in \mathcal{I}$, a subset $S^i \subseteq X^i$ and setting $W^i(x^i) = 1$ for $x^i \in S^i$ and 0 otherwise and then varying the set S^i.

[6] For social choice problems, one could be interested in characterizing the set of feasible interim allocation probabilities $Q_k^i(x^i) = \sum_{x^{-i}} f^{-i}(x^{-i}) q_k^i(\mathbf{x})$, $k \in \mathcal{K}, i \in \mathcal{I}$. The support function for this set could be derived similarly and equals

$$S^{\mathbf{Q}}(\widehat{\mathbf{W}}) \;=\; E_{\mathbf{x}}\,\big(\max_{k\,\in\,\mathcal{K}} \sum_{i\,\in\,\mathcal{I}} \widehat{W}_k^i(x^i)\big),$$

for all $\widehat{\mathbf{W}} \in \mathbb{R}^{K\sum_i |X^i|}$.

Corollary 1. *For single-unit auctions, the set of feasible interim allocations is determined by*

$$\sum_{i \in \mathcal{I}} \sum_{x^i \in S^i} f^i(x^i) Q^i(x^i) \;\leq\; 1 - \prod_{i \in \mathcal{I}} \sum_{x^i \notin S^i} f^i(x^i) \tag{8}$$

for any subset $S^i \subseteq X^i$, $i = 1, \ldots, I$.

The inequalities in Corollary 1 are known as the Maskin-Riley-Matthews conditions for reduced form auctions. They were conjectured to be necessary and sufficient by Matthews (1984) [49] based on the following intuition: the probability that a certain bidder with a certain type wins (left side) can be no higher than the probability that such a bidder exists (right side). The conjecture was subsequently proven and generalized by Border (1991, 2007) [11, 12]. Since then an extensive literature has been developed to extend the characterization to multi-unit auctions (Alaei et al., 2019 [3]), auctions with heterogenous objects (Cai et al., 2012ab [14, 15]), and settings with various capacity constraints (Che et al., 2013 [17]).[7] Our *geometric approach* (Theorem 1) provides a simple treatment of auctions settings and further extends them to *social choice environments* with multi-dimensional and possibly correltated types.

3.3. Incentive Compatibility

A mechanism $(\mathbf{q}, \mathbf{t})$ is *Bayesian incentive compatible* (BIC) if truthful reporting is a Bayes-Nash equilibrium. We also say that an allocation is BIC implementable if there exist transfers that form a BIC mechanism when coupled with the allocation. Myerson (1981) [52] showed that an allocation $\mathbf{q}$ is BIC implementable if and only if for each $i = 1, \ldots, I$ the interim values are increasing: $V^i(x^i_{j-1}) \leq V^i(x^i_j)$ for $j = 2, \ldots, N^i$.[8] Let $\mathbf{e}(x^i_j)$ denote the unit vector of $\mathbb{R}^{\sum_i |X^i|}$ in the direction x^i_j for $i = 1, \ldots, I$ and $j = 1, \ldots, N^i$. Using the definition of the probability-weighted inner product (5) the Bayesian incentive constraints can be written as

$$\big(\mathbf{e}(x^i_{j-1})/f^i(x^i_{j-1}) - \mathbf{e}(x^i_j)/f^i(x^i_j)\big) \cdot \mathbf{V} \;\leq\; 0$$

[7] See also Maskin and Riley (1984) [48], Belloni et al. (2010) [9], Hart and Reny (2015) [30], Gershkov et al. (2019) [26], Kleiner et al. (2021) [34], Lang and Yang (2021) [40], and Zheng (2021) [58].

[8] An *increasing* sequence refers to a *weakly increasing* sequence throughout the paper.

for $i = 1, \ldots, I$ and $j = 2, \ldots, N^i$. These constraints define half spaces and their intersection with the set of feasible values defines the set of feasible BIC values. The support function for this intersection is (e.g. Rockafellar, 1997 [54])

$$S^{BIC}(\mathbf{W}) \;=\; \inf_{\mathbf{\Lambda} \geq 0} S^{\mathbf{V}}(\mathbf{W} - \sum_{i=1}^{I} \sum_{j=2}^{N^i} \Lambda^i(x^i_{j-1})(\frac{\mathbf{e}(x^i_{j-1})}{f^i(x^i_{j-1})} - \frac{\mathbf{e}(x^i_j)}{f^i(x^i_j)})) \;=\; \inf_{\mathbf{\Lambda} \geq 0} S^{\mathbf{V}}(\widehat{\mathbf{W}}) \qquad (9)$$

where $\widehat{W}^i(x^i_j) = W^i(x^i_j) - (\Lambda^i(x^i_j) - \Lambda^i(x^i_{j-1}))/f^i(x^i_j)$ for $i = 1, \ldots, I$ and $j = 1, \ldots, N^i$ with $\Lambda^i(x^i_0) = \Lambda^i(x^i_{N^i}) = 0$. Since the Λ's are non-negative it is readily verified that

$$\sum_{j=1}^{l} f^i(x^i_j)\widehat{W}^i(x^i_j) \;\leq\; \sum_{j=1}^{l} f^i(x^i_j)W^i(x^i_j)$$

for $l = 1, \ldots, N^i$ with equality for $l = N^i$, which we abbreviate as $\widehat{\mathbf{W}}^i \preceq_{f^i} \mathbf{W}^i$. For two increasing sequences, $\mathbf{W}$ and $\mathbf{W}'$, we say that $\mathbf{W}'$ f^i*-majorizes* $\mathbf{W}$ if $\mathbf{W} \preceq_{f^i} \mathbf{W}'$.

The minimization problem in (9) can thus be written as $\inf_{\widehat{\mathbf{W}}^i \preceq_{f^i} \mathbf{W}^i} S^{\mathbf{V}}(\widehat{\mathbf{W}})$. Using the seminal Hardy, Littlewood, and Pólya's (1929) [29] result we show that its solution $\mathbf{W}^i_+$ is the "largest" increasing sequence such that $\mathbf{W}^i_+ \preceq_{f^i} \mathbf{W}^i$, i.e. $\mathbf{W}^i_+$ f^i-majorizes any other increasing sequence of weights $\widehat{\mathbf{W}}^i$ that satisfy $\widehat{\mathbf{W}}^i \preceq_{f^i} \mathbf{W}^i$. We note that sequence $\mathbf{W}^i_+$ has to be increasing and generally depends on the distribution of agent i's types.

Theorem 2. *The support function for the set of feasible and Bayesian incentive compatible interim values is given by*

$$S^{BIC}(\mathbf{W}) \;=\; E_{\mathbf{x}}\left(\max_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} a^i_k W^i_+(x^i)\right)$$

for any $\mathbf{W} \in \mathbb{R}^{\sum_i |X^i|}$.

The discrete majorization procedure thus parallels the "ironing" technique introduced by Mussa and Rosen (1978) [51] and Myerson (1981) [52] for continuous types.[9] This par-

[9]See Weymark (1986) [57] for the ironing for the discrete case in optimal income taxation setting.

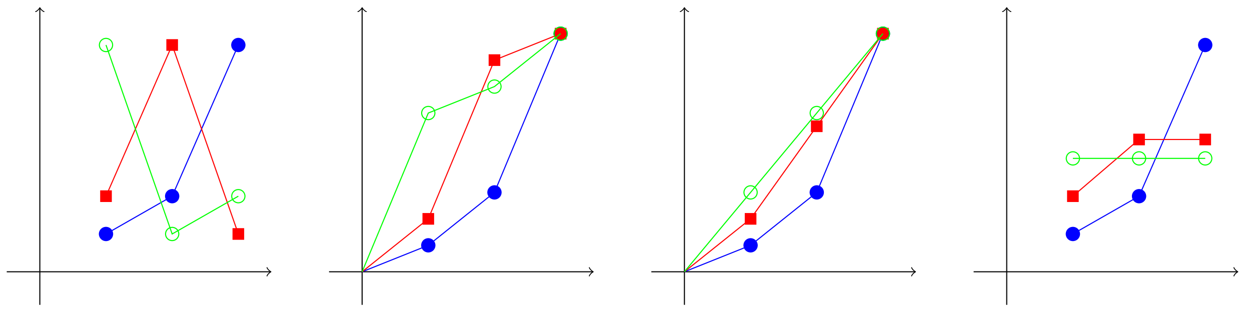

Figure 3. The three sequences in the leftmost panel are $\mathbf{W}^1 = (1, 2, 6)$ (solid blue circles), $\mathbf{W}^2 = (2, 6, 1)$ (red squares), and $\mathbf{W}^3 = (6, 1, 2)$ (open green circles). The rightmost panel shows the majorized sequences: $\mathbf{W}^1_+ = (1, 2, 6)$, $\mathbf{W}^2_+ = (2, \frac{7}{2}, \frac{7}{2})$, and $\mathbf{W}^3_+ = (3, 3, 3)$. The two middle panels (with rescaled y-axis) show the cumulative sequences for $\mathbf{W}$ (middle-left) and $\mathbf{W}_+$ (middle-right). The cumulative of $\mathbf{W}_+$ is the largest convex function below the cumulative of $\mathbf{W}$.

allel establishes a convenient way to derive the ironed values. For any sequence $\mathbf{W} = (W_1, ..., W_N)$, to compute $\mathbf{W}_+$ one could follow the following simple procedure. First, consider sequence $\{\ell_k\}_{k=1,...,N}$ of cumulative sums $\ell_k = \sum_{n=1}^{k} W_n$. Second, find the largest convex function $\{\hat{\ell}_k\}_{k=1,...,N}$ that lies below $\{\ell_k\}_{k=1,...,N}$. The elements of the majorized sequence are then computed as $(W_+)_k = \ell_k - \ell_{k-1}$, $k = 1, ..., N$ with $\ell_0 = 0$. Generally, majorized sequences are characterized by a collection of intervals. Outside these intervals $\mathbf{W}_+$ coincides with original sequence $\mathbf{W}$, and insider each interval the majorized sequence is constant (see Kleiner et al., 2021 [34]).

Figure 3 illustrates majorization for three equally likely types. The leftmost panel shows sequences $\mathbf{W}^1 = (1, 2, 6)$, $\mathbf{W}^2 = (2, 6, 1)$, and $\mathbf{W}^3 = (6, 1, 2)$. The rightmost panel shows the corresponding majorized sequences $\mathbf{W}^1_+ = (1, 2, 6)$, $\mathbf{W}^2_+ = (2, \frac{7}{2}, \frac{7}{2})$, and $\mathbf{W}^3_+ = (3, 3, 3)$. Note that $\mathbf{W} = \mathbf{W}_+$ if and only if original sequence $\mathbf{W}$ is increasing. The middle panels show the cumulative sequences for $\mathbf{W}$ (left) and $\mathbf{W}_+$ (right) and demonstrates that the cumulative of $\mathbf{W}_+$ is the largest convex function below the cumulative of sequence $\mathbf{W}$.

Alternatively, one could obtain a majorized sequence using a minimization procedure. As we show in Appendix (Lemma A2), the majorized sequence delivers the minimum to the sum of functions defined over sequences satisfying the majorized constraints for any increasing convex function. In particular, one could consider a quadratic function and minimization problem $\mathbf{W}_+ = \arg\min_{\widehat{\mathbf{W}} \preceq \mathbf{W}.} \sum_j \widehat{W}_j^2$. This minimization problem could be simply used to derive the majorized or ironed sequence for any given sequence.

3.4. BIC-DIC Equivalence

Similar to BIC we can incorporate dominant strategy incentive compatibility (DIC) into the support function. Surprisingly, this yields the same support function for the interim values.

Example 1. Consider again the auction example of Section 2 but without the symmetry assumption. The support function for the allocation rules $\mathbf{q}^i = (q^i_{ll}, q^i_{lh}, q^i_{hl}, q^i_{hh})$ for $i = 1, 2$ is $\max(0, w^1_{ll}, w^2_{ll}) + \max(0, w^1_{hl}, w^2_{lh}) + \max(0, w^1_{lh}, w^2_{hl}) + \max(0, w^1_{hh}, w^2_{hh})$. Imposing the DIC constraints, $q^i_{ll} - q^i_{hl} \leq 0$ and $q^i_{lh} - q^i_{hh} \leq 0$ for $i = 1, 2$, and applying the interim mapping to derive the support function for interim allocations yields

$$
\begin{aligned}
S^{DIC}(\mathbf{W}) \;=\; \inf_{0 \leq \lambda^i_l, \lambda^i_h} & \frac{1}{4}\max(0, W^1_l - \lambda^1_l, W^2_l - \lambda^2_l) + \frac{1}{4}\max(0, W^1_h + \lambda^1_l, W^2_l - \lambda^2_h) \\
& + \frac{1}{4}\max(0, W^1_l - \lambda^1_h, W^2_h + \lambda^2_l) + \frac{1}{4}\max(0, W^1_h + \lambda^1_h, W^2_h + \lambda^2_h)
\end{aligned}
$$

For agent $i = 1, 2$ there are two minimization parameters, λ^i_l and λ^i_h, while in the BIC case there is only one, Λ^i. However, the above minimization problem has a solution that sets $\lambda^i_l = \lambda^i_h = \frac{1}{2}\max(0, W^i_l - W^i_h)$, which is also the solution for Λ^i so the BIC and DIC support functions coincide. This solution is apparent when considering the minimization problem over one agent's parameters ignoring the dependence on the other's weights. The reason we can consider each agent's DIC constraints separately stems from their geometric interpretation: each represents the intersection of the feasible set with a half space. ■

The next result shows that the BIC and DIC support functions coincide more generally.

Proposition 1. *The support functions for the set of feasible interim values satisfying BIC or DIC constraints coincide:* $S^{DIC}(\mathbf{W}) = S^{BIC}(\mathbf{W})$ *for any* $\mathbf{W} \in \mathbb{R}^{\sum_i |X^i|}$.

This result implies that for any Bayesian incentive compatible mechanism there exists an equivalent dominant strategy incentive compatible mechanism, a result first shown for the auction case by Manelli and Vincent (2010) [44] and generalized to social choice settings by Gershkov et al. (2013) [24] (see also Kushnir (2015) [37], Kushnir and Liu (2019, 2020) [38] [39], and Manelli and Vincent (2019) [45]).

3.5. Optimal Mechanisms for Linear Objectives

Consider maximization of the linear objective $\boldsymbol{\alpha}\cdot\mathbf{V}$ over the set of feasible, incentive compatible interim values. Then $S^{BIC}(\boldsymbol{\alpha})$ is the optimal value and the optimal mechanism follows from Hotelling's lemma, i.e. $\mathbf{V}(\boldsymbol{\alpha}) = \nabla S^{BIC}(\boldsymbol{\alpha})$, see Rockafellar (1997) [54]. Proposition 1 ensures this mechanism can be written as a dominant strategy incentive compatible mechanism.

Proposition 2. *For any social choice problem and any linear objective, $\boldsymbol{\alpha}\cdot\mathbf{V}$, an optimal dominant strategy incentive compatible mechanism is given by the allocation rule*

$$q_k(\mathbf{x}) \;=\; \begin{cases} 1/|M| & \text{if } k \in M \equiv \operatorname{argmax}_{k\in\mathcal{K}} \sum_{i\in\mathcal{I}} a_k^i \alpha_+^i(x^i) \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

and corresponding payment rule[10]

$$t^i(\mathbf{x}) \;=\; \sum_{k\in\mathcal{K}} a_k^i\Big(x^i q_k(\mathbf{x}) - \sum_{x_j^i < x^i} (x_{j+1}^i - x_j^i) q_k(x_j^i, \mathbf{x}^{-i})\Big) \tag{11}$$

Typical examples of linear objectives are expected surplus, $E_{\mathbf{x}}(\sum_{i\in\mathcal{I}} x^i V^i(x^i)) = \mathbf{x}\cdot\mathbf{V}$, and expected revenue, $E_{\mathbf{x}}(\sum_i t^i(\mathbf{x})) = \mathbf{MR}\cdot\mathbf{V}$, where marginal revenues are defined as

$$MR^i(x_j^i) \;=\; x_j^i - \big(x_{j+1}^i - x_j^i\big)\frac{1-F^i(x_j^i)}{f^i(x_j^i)}, \quad i = 1,\ldots,I, \;\; j = 1,\ldots,N^i, \tag{12}$$

with $x_{N^i+1}^i = x_{N^i}^i$ and $F^i(x_j^i) = \sum_{l=1}^j f^i(x_l^i)$. These marginal values are the discrete analogues of Myerson's (1981) [52] "virtual values" for the continuous case.

4. General Concave Objectives

In many applied design problems there are distributional goals besides surplus and revenue maximization. Federal procurement in the US, for instance, awards at least 23% of its $500 billion annual budget to small businesses, with lower targets for businesses owned by women, disabled veterans, and the economically disadvantaged (Athey, Coey, and Levin,

[10] The specified payment rule (which is not unique) ensures that the optimal mechanism $(\mathbf{q},\mathbf{t})$ is also ex post individually rational. See Section 4 for more details.

2013 [4]). One way such preferential treatment can be achieved is by using "set asides," which constrain the allocation rule. For example, in the US, procurement contracts under $100,000 are reserved for small businesses and around $30 billion in contracts is awarded via set-aside programs. An alternative way is to adapt the payment rule to reflect subsidies to favored firms. The US Federal Communications Commission, for instance, has applied bidding credits to minority-owned firms in some of their spectrum auctions. To incorporate set asides and subsidies we consider objectives that depend on both allocations and payments. We drop the restriction the objective is linear and instead assume it is concave. We show that the optimal mechanism can still be derived using Hotelling's lemma, which now results in a fixed-point equation.

We first derive the support function for the set of interim *values and payments*, $\mathbf{V}$ and $\mathbf{T}$, that satisfy, for each $i = 1, \ldots, I$, the Bayesian incentive compatibility (BIC) constraints[11]

$$(V^i(x^i_j) - V^i(x^i_{j-1}))x^i_{j-1} \,\leq\, T^i(x^i_j) - T^i(x^i_{j-1}) \,\leq\, (V^i(x^i_j) - V^i(x^i_{j-1}))x^i_j \tag{13}$$

for $j = 2, \ldots, N^i$, and the interim individually rationality (INIR) constraints: $U^i(x^i) = V^i(x^i)x^i - T^i(x^i) \geq 0$ for $x^i \in X^i$. Dominant strategy incentive compatibility (DIC) and ex post individual rationality (EXIR) are defined similarly. To include interim payments $T^i(x^i)$ into the support function we introduce weights $Z^i(x^i)$ for $x^i \in X^i$, $i = 1, ..., I$ and generalize the marginal revenues in (12) to allow for different weights for each of the payments:

$$MR_{Z^i}(x^i_j) \,=\, x^i_j Z^i(x^i_j) - \frac{x^i_{j+1} - x^i_j}{f^i(x^i_j)} \sum_{l>j} f^i(x^i_l) Z^i(x^i_l) \tag{14}$$

for $j = 1, ..., N^i$ with $x^i_{N^i+1} = x^i_{N^i}$. This expression reduces to (12) when $Z^i(x^i_j) \equiv 1$.

Theorem 3. *The support function for the set of feasible interim values and payments that satisfy BIC (DIC) and INIR (EXIR) is given by*

$$S^{DIC}(\mathbf{W}, \mathbf{Z}) \,=\, S^{BIC}(\mathbf{W}, \mathbf{Z}) \,=\, E_{\mathbf{x}} \Big(\max_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} a^i_k (\mathbf{W}^i + \mathbf{MR}_{Z^i})_+(x^i) \Big) \tag{15}$$

for any $\mathbf{W} \in \mathbb{R}^{\sum_i |X^i|}$ *and* $\mathbf{Z} \in \mathbb{R}_+^{\sum_i |X^i|}$.

[11] We consider only adjacent incentive constraints because utilities satisfy the single-crossing condition.

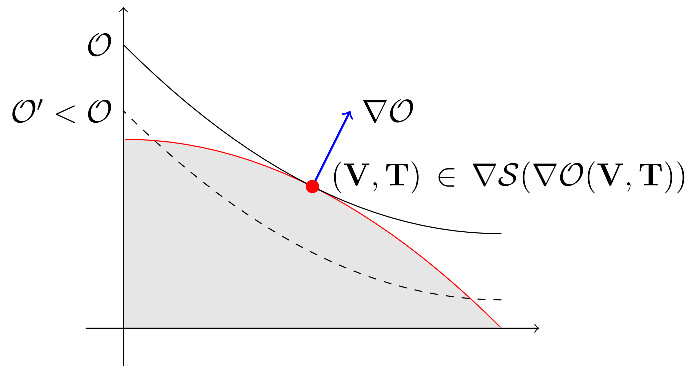

Figure 4. The *optimal* interim expected values and payments belong to the subdifferential of the support function evaluated at the vector of weights that is equal to the gradient of the objective function evaluated at the optimal interim expected values and payments.

Now consider a differentiable concave objective function $\mathcal{O}(\mathbf{V},\mathbf{T})$ that is increasing in interim payments. Concave objectives have convex indifference curves and maximization requires that, at the optimal point, the gradient $\nabla\mathcal{O}$ is normal to the surface of the feasible, incentive compatible set, see Figure 4. Moreover, the subdifferential of the support function evaluated at this normal vector should contain the optimal point.

Proposition 3. *For any social choice problem and any concave differentiable objective $\mathcal{O}(\mathbf{V},\mathbf{T})$ increasing in interim payments the interim values and payments corresponding to an optimal DIC and EXIR mechanism satisfy*[12]

$$(\mathbf{V},\mathbf{T}) \in \nabla\mathcal{S}^{DIC}(\nabla\mathcal{O}(\mathbf{V},\mathbf{T})). \tag{16}$$

Fixed point equation (16) provides a characterization of optimal mechanisms maximizing any given concave objective. This novel result follows naturally from the geometric approach developed in this paper, but would be quite hard to derive using classical techniques. Fixed-point condition (16) holds even if objective $\mathcal{O}$ is not increasing in interim payments. However, the explicit expression (15) for $\mathcal{S}^{DIC}(\mathbf{W},\mathbf{Z})$ was derived in Theorem 3 under assumption $\mathbf{Z} \in \mathbb{R}_+^{\sum_i |X^i|}$ and it can be used only if objective $\mathcal{O}$ is increasing in interim payments.

In contrast to the linear case of Proposition 2, generally it is not possible to provide

[12]Note that the statement of the proposition immediately extends to any differentiable quasi-concave objective increasing in interim payments. We thank Jacques-François Thisse for pointing this out to us.

explicit solutions for the ex post allocation and payment rules, or their interim equivalents for that matter. Note, however, that the weights for the values and payments enter the support function (15) as linear combinations, which implies that their subgradients are closely related. This observation can be used to express the optimal interim payments in terms of the optimal interim values

$$T^i(x^i) \;=\; V^i(x^i)x^i - \sum\nolimits_{x^i_j < x^i} V^i(x^i_j)(x^i_{j+1} - x^i_j)$$

which is the interim version of (11).[13]

To illustrate Proposition 3, consider a single-unit auction with two ex ante symmetric bidders whose private values are equally likely to be x_1 or x_2 with $x_1 < x_2 < 2x_1$. Suppose the seller wants to maximize the following objective: $\mathcal{O} = T(x_1) + T(x_2) + \beta(x_2 - x_1)Q(x_1)Q(x_2)$, where $\beta \geq 0$ and $U(x_j) = x_jQ(x_j) - T(x_j)$ denotes agent's interim expected utility when her type is x_j (note that $V(x_j) = Q(x_j)$). The objective reflects the desire of the seller to maximize revenue (the first two terms) while providing different types of bidders with high and similar chances of winning (the last term). The gradient of the objective is

$$\nabla\mathcal{O} \;=\; \begin{pmatrix} \beta(x_2 - x_1)Q(x_2) \\ \beta(x_2 - x_1)Q(x_1) \\ 1 \\ 1 \end{pmatrix}.$$

The support function for the single unit auction can be written as

$$S^{DIC}(\mathbf{W}, \mathbf{Z}) \;=\; \frac{1}{4}\sum_{j,k=1}^{2} \max(0, (W + MR_Z)_+(x_j), (W + MR_Z)_+(x_k)).$$

Using (16), we determine when the efficient allocation, in the sense that the higher type is awarded the object, is optimal. From the above expression for $\nabla\mathcal{O}$ we have $(W + MR_Z)(x_1) = 2x_1 - x_2 + \beta(x_2 - x_1)Q(x_2)$ and $(W + MR_Z)(x_2) = x_2 + \beta(x_2 - x_1)Q(x_1)$. These weights should be non-negative and non-decreasing in type for the efficient allocation to be optimal, which requires $\beta \leq 4$ (since $Q(x_1) = \frac{1}{4}$ and $Q(x_2) = \frac{3}{4}$ in an efficient allocation).

[13] This formula is analogous to the envelope theorem for continuous case (Milgrom and Segal, 2002 [50]).

When $\beta > 4$, the ironed weights are equal, which implies $Q(x_1) = \frac{1}{2} - \frac{1}{\beta}$ and $Q(x_2) = \frac{1}{2} + \frac{1}{\beta}$, i.e. the allocation is distorted in the direction of a lottery to raise the lower type's chance of winning.

5. Geometric vs. Other Approaches

In this section, we discuss how the geometric approach is compared to the classical Myersonian approach and some recent advances in the mechanism design and computer science literatures.

In a seminal paper, Myerson (1981) [52] analyzes the problem of designing a revenue-maximizing auction when bidders have independently distributed values. Myerson shows that if the auctioneer maximizes revenue, the auctioneer does not always want to allocate the object (there is a reserve price). Moreover, the auctioneer does not necessarily allocate the object to a bidder with the highest value. Instead, the object is allocated to a bidder with so called highest "virtual value" in the case of regular distributed types. For generally distributed types, the virtual values first need to be "ironed."[14] The geometric approach also considers a linear maximization problem over the set of feasible and incentive compatible interim values. In contrast to the classical approach, which uses the standard optimization techniques, the geometric approach utilizes the algebraic properties of support functions, and, in particular, how they change under Cartesian product, linear transformation, and intersection. Using these properties, we derive an analytical expression of the support function for the set of feasible and Bayesian incentive compatible interim values.[15] For *any* given linear objective, the subdifferential of the support function then determines the optimal mechanisms including the allocation rule and transfer payment. For concave objectives, the geometric approach provides a fixed-point condition characterizing the optimal mechanisms.

Though we first to point out the connection between the ironing procedure and the seminal concept of majorization, a more fundamental study of the majorization has been

[14] Musa and Rosen (1978) [51] are the first to use ironing procedure in mechanism design literature.

[15] Using classical Lagrangian methods, Ledyard and Palfrey (1999, 2007) [41, 42] obtain related results characterizing incentive compatible mechanisms. Kucuksenel (2012) [36] studies interim efficient mechanisms in settings with interdependent values.

recently done by Kleiner et al. (2021) [34].[16] They characterize the set of extreme points of monotonic functions that are either majorized by or themselves majorize a given function. For mechanism design applications, they consider an environment of ranked-item auctions with *one-dimensional independently distributed types.* They provide a relation between the implementability of symmetric and monotonic interim allocation rules and the efficient allocation using majorization.[17] Kleiner et al. (2021) [34] also show how the equivalence between Bayesian and dominant strategy implementation for *symmetric* allocation rules in ranked-items auctions follows from their main results. In contrast, the results of this paper do not require the symmetry assumption and extend to *social choice environments.* For the equivalence between Bayesian and dominant strategy implementation, we also assume one-dimensional types, whereas for reduced-form implementation our results apply to general multi-dimensional settings with possibly correlated types. The results of Kleiner et al. (2021) [34] could also be extended to non-linear maximization problems with super-modular objectives due to Fan and Lorents (1954) [22].[18] The geometric approach is also applicable to the maximization of concave objectives as Section 4 shows. At the same time, the applications of Kleiner et al. (2021) [34] extend beyond mechanism design settings and consider matching contests, Bayesian persuasion, optimal delegation, and decision making under uncertainty.

There have recently appeared several analytical and computational approaches to reduced form implementation. Importantly, Che et al. (2013) [17] convert the problem of identifying whether an interim allocation rule is implementable into the problem of whether there exists a feasible flow in a certain network. Using this network-flow approach they provide the characterization of reduced-form auctions in a multi-unit setting with bidder *capacity constraints.* Zheng (2021) [58] extended their analysis to multiple heterogeneous objects with multiple units.[19] In contrast, the characterization of implementable interim allocation rules in the current paper applies to *social-choice environments* beyond multi-

[16]The working version of this paper Goeree and Kushnir (2011) [27] preceded their analysis. See also Gershkov and Moldovanu (2010) [25] for an earlier use of majorization in the study of dynamic assignment of heterogeneous objects.

[17]The connection between reduced form implementation and majorization was originally identified by Hart and Reny (2015) [30] for single-object auctions. See also Gershkov et al. (2019) [26] for multi-unit auctions.

[18]In the paper, Kleiner et al. (2021) [34] restrict their applications only to linear objectives.

[19]See also Lang and Yang (2021) [40] for an alternative approach to reduced-form auctions with multiple heterogeneous objects and capacity constraints.

unit auctions. Our analysis also extends to settings with value-interdependence (see Goeree and Kushnir, 2016 [28]). At the same time, we do not attempt to extend our analysis to incorporate capacity constraints.[20]

For computational approaches to reduced form auctions one should first highlight Belloni et al. (2010) [9], who developed a tractable computational way to find a revenue maximizing auction in multi-dimensional single-item auction. Their main insight is to develop an efficient algorithm that accounts for the reduced-form implementation constraints. Alaei et al. (2019) [3] extended their results to multi-unit auction settings and to the settings with general matroid feasibility constraints.[21] In addition, they provide an efficient procedure how to provide a description of a mechanism for a given feasible interim allocation. Cai et al. (2012a) [14] also provide an alternative algorithm to characterize reduced-form allocations for auctions with multiple heterogeneous objects. They also provide an efficient algorithm to calculate optimal revenue-maximizing auction in the setting. Cai et al. (2012b) [15] extended their results by showing that the vertices of the interim feasible polytope can be implemented by an ex post allocation that optimizes an appropriately defined virtual welfare. In contrast, we pursue only analytical characterization of reduced-form allocations, and our characterization applies to general social choice settings.

6. Conclusion

Mechanism design has been successfully applied to a variety of societal issues including the matching of students to schools, interns to hospitals, and organ donors to patients as well as the design of high-stakes auctions to allocate public assets. This paper provides a new powerful perspective on some of these and many other applications by introducing a novel approach to the analysis of optimal mechanisms maximizing given objectives.[22] Our approach is based on the one-to-one relation between a convex set and its support function.

[20] The main results of this paper were developed independently and contemporaneously with Che et al. (2013) [17] (see Goeree and Kushnir, 2011 [27]).

[21] Vohra (2011) [56] was first to show that interim feasible allocation rules form a polymatroid.

[22] Hence, our paper is related, in spirit, to Bulow and Roberts (1989) [13] who gracefully reinterpret the problem of the revenue-maximizing auction through the prism of monopolistic third-degree price discrimination. Similarly, we use basic tools of convex analysis to provide a unified and simple treatment of optimal mechanisms.

While we are the first to exploit this relation in mechanism design, related methods have a long history in economics and finance and are now standardly taught in micro PhD classes (e.g. Mas-Colell, Whinston, and Green, 1995, p.63 [47]).[23]

In conclusion, we present our perspective on the most promising research areas where the geometric approach can be successfully applied. First, the geometric approach extends the recent powerful methods in reduced form auctions (Alaei et al., 2012 [3]; Cai et al., 2012ab [14] [15]; Che et al., 2013 [17]) to social choice environments. Hence, it provides the characterization of interim allocation probabilities to solve for optimal mechanisms in such settings as public good provision, matching, voting, and scheduling.[24]

Second, the geometric approach can be extended to characterize reduced form implementation in environments with value interdependencies (see Goeree and Kushnir, 2016 [28]). Hence, it should be a useful tool to analyze welfare- or reveneue-maximizing mechanism in these often complex settings.

Third, the geometric approach should be helpful in challenging multi-dimensional mechanism design problems. The analysis of revenue-maximizing auction in settings with several objects and several buyers with multi-dimensional valuations is a long-standing open problem (see Sandholm and Likhodedov, 2015 [55]; Daskalakis et al., 2017 [19]; Kleiner and Manelli, 2019 [34]). Incorporating incentive compatibility constraints into the support function approach or understanding the set of extreme points of functions that are majorized by a given function in multi-dimensional settings (see Bedard et al., 2020 [8], Kleiner et al., 2021 [34]) will be an important methodological breakthrough with many useful applications in the areas of auction design, information design, matching contests, contracting, bargaining, etc.

Fourth, the support function approach could also be applied to the analysis of equilibria in dynamic games. The Folk theorem identifies subgame perfect equilibria for discount factors approaching one. Abreu (1986 [1]) and Abreu et al. (1986) [2] provide an approach to analyze the set of subgame perfect equilibria for any given discount factor (see also

[23] The relation between a convex closed set and its support function is an example of the duality in convex analysis that has been previously heavily exploited in economics (see Bardsley, 2012 [7]; Makowski and Ostroy, 2013 [43]; Baldwin and Klemperer, 2019 [5]; Daskalakis et al., 2017 [19]; and Kleiner and Manelli, 2019 [33]).

[24] Theorem 1 provides the characterization of the set of feasible interim values in social choice settings. For how this result extends to the characterization of interim allocation probabilities see footnote 6.

Judd et al., 2003 [31]). One potential application of the geometric approach is to apply the algebra of support functions to characterize the set of subgame perfect equilibria in dynamic games for a given discount factor.[25]

Finally, the properties of support functions have been previously used in decision theory (Dekel et al., 2001 [20]), econometrics (Beresteanu and Molinari, 2008 [10]), and mathematical finance (Ekeland et al., 2012 [21]). We believe that the analysis of this paper will further promote these techniques in the corresponding areas. We leave these exciting opportunities for future research.

[25] In an early paper, Fudenberg and Levine (1994) [23] used related ideas to characterize the equilibrium payoffs in repeated games with long-run and short-run players for discount factors approaching one. Recently, Balseiro et al. (2019) [6] and Chen et al. (2020) [18] used support function techniques in dynamic mechanism design without transfers and principal-agent problem with costly monitoring.

A. Appendix

Proof of Corollary 1. Necessity of the inequalities follows from the definition of the support function. Sufficiency also follows easily from our approach by interpreting (8) in terms of hyperplanes that bound the interim expected probability set. Any boundary point of the interim expected probability set, i.e. any $\mathbf{Q}$ that satisfies $\mathbf{Q} \cdot \mathbf{W} = \mathcal{S}^{\mathbf{Q}}(\mathbf{W})$ for some $\mathbf{W}$, can be written as $\mathbf{Q} = \nabla \mathcal{S}^{\mathbf{Q}}(\mathbf{W})$ at points of differentiability of the support function from the envelope theorem. Furthermore, if $\mathcal{S}^{\mathbf{Q}}(\mathbf{W})$ is not differentiable at $\mathbf{W}$ then the subdifferential $\nabla \mathcal{S}^{\mathbf{Q}}(\mathbf{W})$ produces a face on the boundary: for any $\mathbf{Q}$ belonging to such a face we have

$$(\mathbf{Q} - \mathbf{Q}') \cdot \mathbf{W} \;=\; \mathcal{S}^{\mathbf{Q}}(\mathbf{W}) - \mathcal{S}^{\mathbf{Q}}(\mathbf{W}) \;=\; 0.$$

Each point of non-differentiability, $\mathbf{W}$, therefore defines a normal vector to the face of the polyhedron, formed by $\nabla \mathcal{S}^{\mathbf{Q}}(\mathbf{W})$. For the support function (7) the points of non-differentiability are weight vectors with several equal entries, and those equal entries are the largest entries for some profile of types $\mathbf{x}$. Since non-maximum entries does not change the value of the support function we can consider only weights where these entries are 0. Since the support function is homogeneous of degree one we can restrict ourselves to weights with only 1 and 0 entries. Then considering all non-trivial $\mathbf{W} \in \{0,1\}^{\sum_i X^i}$ exhausts all hyperplanes containing one of the boundary faces of the interim expected probability set. ■

Proof of Theorem 2 and Propositions 1. The statements of the theorem and the proposition follow from more general results established in Theorem 3 incorporating also payments into the support function. ■

Proof of Proposition 2. Using Theorems 1, 2, Proposition 1, and the definition of the interim support function we have

$$\mathcal{S}^{DIC}(\boldsymbol{\alpha}) \;=\; \mathcal{S}^{BIC}(\boldsymbol{\alpha}) \;=\; \mathcal{S}^{\mathbf{V}}(\boldsymbol{\alpha}_+) \;=\; \max\Big\{\, E_{\mathbf{x}}\Big(\sum_{k\in\mathcal{K}} q_k(\mathbf{x}) \sum_{i\in\mathcal{I}} a^i_k \alpha^i_+(x^i)\Big) \,\big|\, \mathbf{q} \text{ is feasible}\Big\}.$$

This establishes the optimality of the allocation rule in (10). To derive the payments consider dominant strategy incentive compatibility constraints[26]

$$(v^i(x^i_j,\mathbf{x}^{-i})-v^i(x^i_{j-1},\mathbf{x}^{-i}))x^i_{j-1} \,\le\, t^i(x^i_j,\mathbf{x}^{-i})-t^i(x^i_{j-1},\mathbf{x}^{-i}) \,\le\, (v^i(x^i_j,\mathbf{x}^{-i})-v^i(x^i_{j-1},\mathbf{x}^{-i}))x^i_j$$

for $j=2,\dots,N^i$. Moreover, ex post individual rationality requires that $v^i(\mathbf{x})x^i - t^i(\mathbf{x}) \ge 0$ for all $\mathbf{x}\in X$. Considering the payments binding the upward incentive constraints and the ex post individually rationality constraint for the lowest type we recursively calculate

$$t^i(\mathbf{x}) = v^i(\mathbf{x}) - \sum_{x^i_j < x^i} (x^i_{j+1}-x^i_j)v^i(x^i_j,\mathbf{x}^{-i})$$

for $\mathbf{x}\in X$ and $i\in\mathcal{I}$. This establishes the claim of the proposition. ■

Proof of Theorem 3. For clarity, we first outline the main steps of the proof. As a first step, we derive the support function for the set of feasible interim expected values and payments (similarly to (6)). As a second step, we state the result from the majorization theory and prove a useful lemma. Using this lemma we then incorporate Bayesian incentive compatibility (13) and interim individual rationality constraints into the support function. Finally, as a third step, we consider the dominant strategy incentive compatibility and ex post individual rationality constraints and show that these constraints result into the same support function.

We begin by deriving the support function for the set of feasible interim expected values and payments. Since we do not restrict payments, the feasible set of payments (not yet taking into account incentive constraints) is the whole space $\mathbb{R}^{\sum_i |X^i|}$. Hence, the support function for the feasible set equals $E_{\mathbf{x}}\big(\delta(Z^i(x^i)=0,\,\forall x^i,\forall i)$, where we use the standard definition of δ-function that equals 0 if its argument is true and $+\infty$ otherwise, and weight $Z^i(x^i)$ corresponds to $T^i(x^i)\in\mathbb{R}$ for $x^i\in X^i$, $i=1,...,I$. Combining this expression with the result of Theorem 1 we obtain that the support function for the set of feasible interim expected values and payments equals

$$\mathcal{S}^{\mathbf{VT}}(\mathbf{W},\mathbf{Z}) \;=\; E_{\mathbf{x}}\Big(\max_{k\in\mathcal{K}}(\sum_{i\in\mathcal{I}} a^i_k W^i(x^i)) + \delta(Z^i(x^i)=0,\,\forall x^i,\forall i)\Big) \tag{A.1}$$

[26]We consider only adjacent incentive constraints because utilities satisfy the single-crossing condition.

where $\mathbf{W} \in \mathbb{R}^{\sum_i |X^i|}$ and $\mathbf{Z} \in \mathbb{R}^{\sum_i |X^i|}$.

We now state an important result from the majorization theory that dates back to Hardy, Littlewood, and Pólya (1929) [29] (see Marshall et al., 2011 [46]).[27] Let $f_1, \dots, f_n$ denote arbitrary non-negative numbers and consider two *increasing* sequences $\boldsymbol{\sigma}$ and $\boldsymbol{\varsigma}$ of length n related with the majorization order $\boldsymbol{\sigma} \succeq_f \boldsymbol{\varsigma}$ (see the definition in Section 3.3). We then say that sequence $\boldsymbol{\sigma}$ f-majorizes $\boldsymbol{\varsigma}$.

Lemma A1. *If $\boldsymbol{\sigma}$ f-majorizes $\boldsymbol{\varsigma}$ we have*

$$\sum_{j=1}^{n} f_j g(\sigma_j) \leq \sum_{j=1}^{n} f_j g(\varsigma_j)$$

for any continuous increasing convex function $g : \mathbb{R} \to \mathbb{R}$.

We use this result to prove the following powerful lemma that will be useful for incorporating the incentive constraints into the support function.

Lemma A2. *For any sequence $\boldsymbol{\sigma}$,*

$$\boldsymbol{\sigma}_+ = \arg\min_{\boldsymbol{\sigma} \succeq_f \boldsymbol{\varsigma}} \sum_{j=1}^{n} f_j g(\varsigma_j) \tag{A.2}$$

for any continuous increasing convex function $g : \mathbb{R} \to \mathbb{R}$.

Proof: Let us first construct $\boldsymbol{\sigma}_+$. For any increasing sequence $\boldsymbol{\varsigma} \in \mathbb{R}^n$, let us define function $h_l(\boldsymbol{\varsigma}) = \sum_{j=1}^{l} f_j \varsigma_j$ and $\alpha_l = \sup_{\boldsymbol{\sigma} \succeq_f \boldsymbol{\varsigma}} h_l(\boldsymbol{\varsigma})$, $l = 1, .., n$, where the supremum is taken only over increasing sequences. Define now sequence $\boldsymbol{\sigma}_+$ as $(\sigma_+)_l = (\alpha_l - \alpha_{l-1})/f_l$, where $\alpha_0 = 0$. Clearly, we have (i) $\boldsymbol{\sigma} \succeq_f \boldsymbol{\sigma}_+$ and (ii) $\boldsymbol{\sigma}_+ \succeq_f \boldsymbol{\varsigma}$ for any *increasing* sequence $\boldsymbol{\varsigma}$ satisfying $\boldsymbol{\sigma} \succeq_f \boldsymbol{\varsigma}$. To prove that $\boldsymbol{\sigma}_+$ is itself increasing we notice that $\frac{h_l(\boldsymbol{\varsigma})}{f_l} + \frac{h_{l-2}(\boldsymbol{\varsigma})}{f_{l-1}} \geq (\frac{1}{f_l} + \frac{1}{f_{l-1}}) h_{l-1}(\boldsymbol{\varsigma})$ for any increasing sequence $\boldsymbol{\varsigma}$ and $l = 2, ..., n$. Therefore,

$$\sup_{\boldsymbol{\sigma} \succeq_f \boldsymbol{\varsigma}} \left(\tfrac{h_l(\boldsymbol{\varsigma})}{f_l} + \tfrac{h_{l-2}(\boldsymbol{\varsigma})}{f_{l-1}} \right) \geq \left(\tfrac{1}{f_l} + \tfrac{1}{f_{l-1}} \right) \sup_{\boldsymbol{\sigma} \succeq_f \boldsymbol{\varsigma}} h_{l-1}(\boldsymbol{\varsigma})$$

[27] This result is also closely related to Karamata's inequality (see Karamata, 1932 [32]).

where the suprema are taken over only increasing sequences. Notice that the supremum of a sum is smaller than the sum of the suprema. After a rearrangement we then obtain $(\alpha_l - \alpha_{l-1})/f_l \geq (\alpha_{l-1} - \alpha_{l-2})/f_{l-1}$, which proves that $\boldsymbol{\sigma}_+$ is increasing.

We now consider minimization problem (A.2). We show that, without loss of generality, we can restrict attention to increasing sequences $\boldsymbol{\varsigma}$. Consider some $\boldsymbol{\varsigma}$ with $\varsigma_l > \varsigma_k$ for some $l < k$. Then define the sequence $\tilde{\boldsymbol{\varsigma}}$ with elements $\tilde{\varsigma}_l = \varsigma_l - \varepsilon(\varsigma_l - \varsigma_k)/f_l$ and $\tilde{\varsigma}_k = \varsigma_k + \varepsilon(\varsigma_l - \varsigma_k)/f_k$ while $\tilde{\varsigma}_j = \varsigma_j$ for $j \neq l, k$. The sequence $\tilde{\boldsymbol{\varsigma}}$ also satisfies $\boldsymbol{\sigma} \succeq_f \tilde{\boldsymbol{\varsigma}}$. Since $g(\cdot)$ is convex we have

$$f_l g(\tilde{\varsigma}_l) + f_k g(\tilde{\varsigma}_k) \leq f_l g(\varsigma_l) + f_k g(\varsigma_k)$$

and, hence, $\sum_{j=1}^n f_j g(\tilde{\varsigma}_j) \leq \sum_{j=1}^n f_j g(\varsigma_j)$. Repeatedly applying this procedure results in a increasing sequence $\tilde{\boldsymbol{\varsigma}}$ that satisfies $\boldsymbol{\sigma} \succeq_f \tilde{\boldsymbol{\varsigma}}$. But any such sequence is f-majorized by $\boldsymbol{\sigma}_+$. Hence, the statement of the lemma follows from Lemma A1. ■

Using the above result we now incorporate the Bayesian incentive compatibility and interim individual rationality constraints into the support function. For convenience we rewrite these constraints as follows.

$$T^i(x^i_j) - T^i(x^i_{j-1}) \;\; \geq \;\; x^i_{j-1}\big(V^i(x^i_j) - V^i(x^i_{j-1})\big) \tag{A.3}$$

$$T^i(x^i_j) - T^i(x^i_{j-1}) \;\; \leq \;\; x^i_j\big(V^i(x^i_j) - V^i(x^i_{j-1})\big) \tag{A.4}$$

$$T^i(x^i_j) \;\; \leq \;\; x^i_j V^i(x^i_j) \tag{A.5}$$

The support function of the intersection of non-empty closed convex sets is the convolution of the support functions of these sets. When some sets are half spaces $\mathbf{B}^m \cdot \mathbf{V} \leq 0$ for $m = 1, \ldots, M$ the operation of convolution reduces to $\inf_{\Lambda^m \geq 0} S^{\mathbf{VT}}(\mathbf{W} - \sum_m \Lambda^m \mathbf{B}^m)$ (see Rockafellar, 1997 [54]).

Let us denote parameters corresponding to constraints (A.3), (A.4), and (A.5) as $\Lambda^i(x^i_{j-1})$, $\gamma^i(x^i_j)$, and $\mu^i(x^i_j)$ respectively. The support function for feasible values and payments satisfying these constraints can be calculated as

$$\mathcal{S}^{BIC}(\mathbf{W}, \mathbf{Z}) \; = \; \inf_{\Lambda,\gamma,\mu \geq 0} E_{\mathbf{x}} \left(\max_{k \in \mathcal{K}} (\sum_i a^i_k \widehat{W}^i(x^i)) + \delta(\widehat{Z}^i(x^i) = 0, \, \forall x^i, \forall i) \right) \tag{A.6}$$

where we denote

$$\begin{aligned}
\widehat{W}^i(x^i_j) &= W^i(x^i_j) + \tfrac{1}{f^i(x^i_j)}(-x^i_{j-1}\Lambda^i(x^i_{j-1}) + x^i_j\Lambda^i(x^i_j) + x^i_j\gamma^i(x^i_j) - x^i_{j+1}\gamma^i(x^i_{j+1}) + x^i_j\mu^i(x^i_j)) \\
\widehat{Z}^i(x^i_j) &= Z^i(x^i_j) + \tfrac{1}{f^i(x^i_j)}(\Lambda^i(x^i_{j-1}) - \Lambda^i(x^i_j) - \gamma^i(x^i_j) + \gamma^i(x^i_{j+1}) - \mu^i(x^i_j))
\end{aligned}$$

Note that we use convention that $\Lambda^i(x^i_0) = \Lambda^i(x^i_{N^i}) = 0$ and $\gamma^i(x^i_1) = \gamma^i(x^i_{N^i+1}) = 0$. Since agents' utilities satisfy the single crossing condition the interim individual rationality constraints are binding only for the lowest type, i.e. $\mu^i(x^i_j) = 0$ for $j = 2, ..., N^i$. Summing up constraints $\widehat{Z}^i(x^i) = 0$ of formula (A.6) over all types $x^i \in X^i$ we then obtain

$$\mu^i(x^i_1) = \sum_{l=1}^{N^i} Z^i(x^i_l) f^i(x^i_l)$$

Similarly, summing up constraints $\widehat{Z}^i(x^i) = 0$ starting from x^i_j, $j = 2, ..., N^i$ we obtain

$$\gamma^i(x^i_j) = \sum_{l=j}^{N^i} Z^i(x^i_l) f^i(x^i_l) + \Lambda^i(x^i_{j-1})$$

Note that for non-negative weights $\mathbf{Z} \in \mathbb{R}_+^{\sum^i |X^i|}$ inequalities $\mu^i(x^i_1) \geq 0$ and $\gamma^i(x^i_j) \geq 0$ are automatically satisfied. With some abuse of notation we replace $(x^i_{j+1} - x^i_j)\Lambda^i(x^i_j)$ with $\Lambda^i(x^i_j)$. Substituting the above expressions into formula (A.6) we obtain

$$\mathcal{S}^{BIC}(\mathbf{W}, \mathbf{Z}) = \inf_{0 \leq \Lambda^i(x^i)} E_{\mathbf{x}} \Big(\max_{k \in \mathcal{K}} \sum_{i \in \mathcal{I}} a^i_k \big(W^i(x^i) + MR_{Z^i}(x^i) - \frac{\Delta\Lambda^i(x^i)}{f^i(x^i)} \big) \Big)$$

Let us now define shifted weights $\widehat{W}^i(x^i) = W^i(x^i) + MR_{Z^i}(x^i) - \Delta\Lambda^i(x^i)/f^i(x^i)$. It is straightforward to verify that $\mathbf{W}^i + \mathbf{MR}_{\mathbf{Z}^i} \succeq_{f^i} \widehat{\mathbf{W}}^i$ for all $i \in \mathcal{I}$.[28] Therefore, Lemma A2 implies that $(\mathbf{W}^i + \mathbf{MR}_{\mathbf{Z}^i})_+$ delivers the minimum to the above expression, which establishes the claim of the theorem for support function $\mathcal{S}^{BIC}(\mathbf{W}, \mathbf{Z})$.

As the last step of the proof, we show that the introduction of the dominant strategy

[28] Note that $\sum_{j=1}^{l} \Delta\Lambda^i(x^i_j) = \Lambda^i(x^i_l) - \Lambda^i(x^i_0) \geq 0$ for $l = 1, \ldots, N^i$ with equality for $l = N^i$.

incentive compatibility constraints

$$t^i(x^i_j,\mathbf{x}^{-i}) - t^i(x^i_{j-1},\mathbf{x}^{-i}) \;\;\geq\;\; x^i_{j-1}\big(v^i(x^i_j,\mathbf{x}^{-i}) - v^i(x^i_{j-1},\mathbf{x}^{-i})\big) \tag{A.7}$$

$$t^i(x^i_j,\mathbf{x}^{-i}) - t^i(x^i_{j-1},\mathbf{x}^{-i}) \;\;\leq\;\; x^i_{j}\big(v^i(x^i_j,\mathbf{x}^{-i}) - v^i(x^i_{j-1},\mathbf{x}^{-i})\big), \tag{A.8}$$

and ex post individual rationality constraints

$$t^i(x^i_j,\mathbf{x}^{-i}) \;\;\leq\;\; x^i_j v^i(x^i_j,\mathbf{x}^{-i}) \tag{A.9}$$

lead to the same support function. To accomplish this we use the geometric interpretation of incentive constraints: the support function minimization problem corresponds to the intersection of the feasible set with the corresponding incentive constraint. Hence, we can include the constraints to support function (A.1) for one agent at a time.

We first include only agent 1's constraints to the support function using arguments similar to ones used in the derivation of support function $\mathcal{S}^{BIC}(\mathbf{W},\mathbf{Z})$. Let us denote parameters corresponding to constraints (A.7) as $\lambda^1(x^1_{j-1},\mathbf{x}^{-1})$ with $\lambda^1(x^1_0,\mathbf{x}^{-1}) = \lambda^1(x^1_{N^1},\mathbf{x}^{-1}) = 0$, and $\Delta\lambda^1(x^1_j,\mathbf{x}^{-1}) = \lambda^1(x^1_j,\mathbf{x}^{-1}) - \lambda^1(x^1_{j-1},\mathbf{x}^{-1})$. We then obtain

$$\mathcal{S}^{DIC}_{agent_1}(\mathbf{W},\mathbf{Z}) = \inf_{0\leq\lambda^1(\mathbf{x})} E_{\mathbf{x}}\Big(\max_{k\in\mathcal{K}} \quad \big(a^1_k(W^1(x^1) + MR_{Z^1}(x^1) - \tfrac{\Delta\lambda^1(\mathbf{x})}{f^1(x^1)}\big) + \sum\nolimits_{i\neq 1} a^i_k\big(W^i(x^i) + MR_{Z^i}(x^i)\big)\Big)$$

We again consider the shifted weights $\widehat{W}^1(\mathbf{x}) = W^1(x^1) + MR_{Z^1}(x^1) - \frac{\Delta\lambda^1(\mathbf{x})}{f^1(x^1)}$. For each $\mathbf{x}^{-1}$ vector $\widehat{\mathbf{W}}^1(\cdot,\mathbf{x}^{-1})$ satisfies $\mathbf{W}^1 + \mathbf{MR}_{\mathbf{Z}^1} \succeq_{f^1} \widehat{\mathbf{W}}^1(\cdot,\mathbf{x}^{-1})$ and the above minimization problem can be rewritten as

$$\sum_{\mathbf{x}^{-1}} \inf_{\mathbf{W}^1+\mathbf{MR}_{\mathbf{Z}^1}\succeq_{f^1}\widehat{\mathbf{W}}^1(\cdot,\mathbf{x}^{-1})} \sum_{x^1} f^1(x^1) g^1(\widehat{W}^1(x^1,\mathbf{x}^{-1}))$$

where $g^1(y) = f^{-1}(\mathbf{x}^{-1})\max_{k\in\mathcal{K}}\big(a^1_k y + \sum_{j\neq 1} a^j_k\big(W^j(x^j) + MR_{Z^j}(x^j)\big)$ is a convex function of y. Lemma A2 asserts that $\widehat{\mathbf{W}}^1(\cdot,\mathbf{x}^{-1}) = (\mathbf{W}^1 + \mathbf{MR}_{\mathbf{Z}^1})_+$ for each $\mathbf{x}^{-1}$ solves the above minimization problem.

Let us now assume that we have introduced the constraints of $i-1$ agents. The minimization problem that corresponds to the introduction of the constraints of agent i is

$$\sum_{\mathbf{x}^{-i}} \inf_{\mathbf{W}^i + \mathbf{MR}_{\mathbf{Z}^i} \succeq_{f^i} \widehat{\mathbf{W}}^i(\cdot, \mathbf{x}^{-i})} \sum_{x^i} f^i(x^i) g^i(\widehat{W}^i(x^i, \mathbf{x}^{-i}))$$

where shifted weights equal $\widehat{W}^i(\mathbf{x}) = W^i(x^i) + MR_{Z^i}(x^i) - \frac{\Delta\lambda^i(\mathbf{x})}{f^i(x^i)}$ and function

$$g^i(y) = f^{-i}(\mathbf{x}^{-i}) \max_{k \in \mathcal{K}} \Big(\sum_{j<i} a_k^j \big(W^j(x^j) + MR_{Z^j}(x^j) \big)_+ + a_k^i y + \sum_{j>i} a_k^j \big(W^j(x^j) + MR_{Z^j}(x^j) \big) \Big)$$

is a convex function of y. Lemma A2 again asserts that $\widehat{\mathbf{W}}^i(\cdot, \mathbf{x}^{-i}) = (\mathbf{W}^i + \mathbf{MR}_{\mathbf{Z}^i})_+$ for each $\mathbf{x}^{-i}$ solves the above minimization problem. Proceeding in this way for all agents, we finally obtain that the support function for the feasible interim expected values and payments that satisfies constraints (A.7-A.9) coincides with $\mathcal{S}^{BIC}(\mathbf{W}, \mathbf{Z})$. ■

Proof of Proposition 3. Vector $(\mathbf{V}^*, \mathbf{T}^*)$ belongs to $\nabla \mathcal{S}^{DIC}(\nabla \mathcal{O}(\mathbf{V}^*, \mathbf{T}^*))$ if and only if (see Theorem 23.5 in Rockafellar, 1997)

$$(\mathbf{V}^*, \mathbf{T}^*) \in \operatorname{argmax}((\mathbf{V}, \mathbf{T}) \cdot \nabla \mathcal{O}(\mathbf{V}^*, \mathbf{T}^*) \,|\, (\mathbf{V}, \mathbf{T}) \in C)$$

where C is the set of dominant strategy incentive compatible and ex post individually rational agent interim values and payments. This is equivalent to $\nabla \mathcal{O}(\mathbf{V}^*, \mathbf{T}^*)$ be tangent to set C at $(\mathbf{V}^*, \mathbf{T}^*)$ (see p. 15, Rockafellar, 1997 [54]). Finally, Theorem 27.4 in Rockafellar (1997) [54] establishes that this is equivalent to $(\mathbf{V}^*, \mathbf{T}^*)$ be a vector where maximum of $\mathcal{O}(\mathbf{V}, \mathbf{T})$ relative to C is attained. ■

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