On the Equivalence of Bayesian and Dominant Strategy Implementation for Environments with Non-Linear Utilities*

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Abstract

We extend the equivalence between Bayesian and dominant strategy implementation (Manelli and Vincent, *Econometrica*, 2010; Gershkov et al. *Econometrica*, 2013) to environments with non-linear utilities satisfying the property of increasing differences over distributions and the convex-valued assumption. The new equivalence result produces novel implications to the literature on the principal-agent problem with allocative externalities, environmental mechanism design, and public good provision.

Keywords: Bayesian implementation, dominant strategy implementation, mechanism design, non-linear utilities, increasing differences over distributions
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1 Introduction

Fundamental advances in mechanism design have found vast practical applications including auctions for radio spectrum licenses, carbon emission permits, and online advertising (Siegfried, 2010). One of the most important practical challenges facing mechanism designers is to ensure that they propose *robust* mechanisms, i.e., mechanisms that are not sensitive to the fine details of the environment such as the beliefs of the agents. As argued by Bergemann and Morris (2005), robustness in private value settings is equivalent to dominant strategy incentive compatibility. Indeed, dominant strategy implementation has the significant advantage that it does not rely on the strong assumption that agents share a common prior, and it is resistant to deviations from rationality that are often observed in practice.

A natural concern for insisting on mechanism robustness is that it may undermine the attainability of the mechanism designer's objectives as the implementation concept required is more restrictive (compared to, for example, Bayesian implementation). Nevertheless, recent seminal work by Manelli and Vincent (2010) and Gershkov et al. (2013) has shown that the trade-off between implementability and robustness can be immaterial: For a large class of social choice problems, the mechanism designer can restrict herself to robust dominant strategy incentive compatible mechanisms and gain nothing from designingpossibly more complex mechanisms with Bayes-Nash equilibria. In particular, when agents have linear utilities and independent, one-dimensional private types, they establish that for any Bayesian incentive compatible (BIC) mechanism there exists an equivalent dominant strategy incentive compatible (DIC) mechanism that yields the same interim expected utilities to all agents and generates the same expected social surplus.[1]

The main contribution of this paper is to extend the BIC-DIC equivalence result to environments with *non-linear utilities* satisfying two assumptions. The first one demands that each agent's utility satisfy the *increasing differences over distributions* property, which is a natural extension of the standard increasing differences (or supermodularity) property to the space of lotteries. This novel property delineates the settings where all BIC mechanisms can be conveniently described by a monotonicity condition and an envelope formula. We also fully characterize the set of functions satisfying the increasing differences over distributions property.

The second assumption demands that the mapping of all agents' utilities, as a mapping from the set of feasible allocations to the space of utilities, has *a convex image* for each profile of types.[2] Though this condition might be restrictive in general environments, it is trivially satisfied for linear

[1]Goeree and Kushnir (2017) provides an alternative proof of this equivalence result using a novel geometric approach to mechanism design. Kushnir (2015) extends the result to environments with correlated types. Kushnir and Liu (2017) explain how the BIC-DIC equivalence problem reduces to a purely mathematical question of when a linear transformation of intersection of two closed convex sets coincides with the intersection of their images.

[2]Similar convexity assumptions on the utility possibility set are also made in many seminal papers in the literature of bargaining theory (e.g. Nash, 1950; Kalai and Smorodinsky, 1975; Crawford, 1982).

utilities defined on a convex set (as in Gershkov et al., 2013) and for any symmetric settings.

Assuming the *increasing differences over distributions* property and the mapping of all agents' utilities being *convex-valued*, we establish the BIC-DIC equivalence for non-linear environments. For settings where our main equivalence theorem does not apply, we provide further conditions on agents' utilities when for any given BIC mechanism one could find a DIC mechanism that yields the same interim expected utilities to all agents and generates *at least as large* expected social surplus. The latter requirement captures the economic intuition that one does not need to insert additional money to achieve a more robust solution concept.

Finally, we demonstrate the usefulness of our results by revisiting several important applications, for which the previous works have little bite (e.g. Manelli and Vincent, 2010; Gershkov et al., 2013). We first consider the principal-agent problem in a procurement context and illustrate that many influential papers satisfy our main assumptions (e.g. Laffont and Martimort, 1997; Mookherjee and Tsumagari, 2004). In the same context, we study settings with allocative externalities, when agents care not only about their own contracts, but also about contracts received by other agents (e.g. Jehiel et al., 1996; Segal, 1999). If agents face non-decreasing convex (concave) contracting costs and positive (negative) concave externalities, then for any BIC mechanism one could find a DIC mechanism yielding the same interim expected utilities to all agents and generating *at least as large* social surplus. We also establish that the above result holds for environmental mechanism design problems (Martimort and Sand-Zantman, 2013, 2015; Baliga and Maskin, 2003) when agents have linear (concave) benefits and concave (linear) costs of pollution reduction. We finally consider the evergreen problem of public good provision, where in addition to incentive compatibility and individual rationality constraints the budget-balance constraint is of huge importance (e.g. Mailath and Postlewaite, 1990; Ledyard and Palfrey, 1999; Hellwig, 2003; Norman, 2004). When agents have concave utilities and the cost of public good provision is convex, we show that for any BIC mechanism that is *ex ante budget balanced* there exists an equivalent DIC mechanism that satisfies the same requirement.

The paper is organized as follows. Section 2 presents the model. We introduce the increasing differences over distributions property in Section 3. We prove our main equivalence results in Section 4. Section 5 presents applications and Section 6 concludes. The Appendix contains omitted proofs.

2 Model

We consider environments with a finite set of agents $\mathcal{I} = \{1, 2, .., I\}$ and a compact set of available alternatives $A \subset \mathbb{R}^k$ for some natural k. Agent i's utility when alternative $a \in A$ is chosen equals $v_i(a, x_i) + t_i$, where x_i is agent i's type that is independently distributed according to some probability distribution λ_i with one-dimensional connected support $X_i = [\underline{x}_i, \overline{x}_i] \subset \mathbb{R}$, function $v_i : A \times X_i \to \mathbb{R}$ is continuous in a, is absolutely continuous in x_i, and has a bounded derivative with respect to x_i (i.e., $|v_{ix}(a, x_i)| \leq K_i, \forall a \in A, x_i \in X_i, i \in I$), and $t_i \in \mathbb{R}$ is a monetary transfer. We denote $\mathbf{x} = (x_1, ..., x_I)$, $X = \prod_{i \in \mathcal{I}} X_i$, and $\lambda = \prod_{i \in \mathcal{I}} \lambda_i$.[3]

We consider only direct mechanisms (q, t), where $q : X \to A$ defines an allocation rule and $t = \{t_i\}_{i \in \mathcal{I}}$, with $t_i : X \to \mathbb{R}$ defines monetary transfers to agents. A mechanism (q, t) is Bayesian incentive compatible or BIC (dominant strategy incentive compatible or DIC) if truthful reporting by all agents constitutes a Bayes-Nash equilibrium (a dominant strategy equilibrium). We also say that an allocation rule q is BIC (DIC) if there exists a payment rule t such that mechanism (q, t) is Bayesian incentive compatible (dominant strategy incentive compatible).

When all agents report their types truthfully and agent i's type is x_i, we denote his utility by $u_i(\mathbf{x}) = v_i(q(\mathbf{x}), x_i) + t_i(\mathbf{x})$ and his interim expected utility by $U_i(x_i) = E_{\mathbf{x}_{-i}}(v_i(q(\mathbf{x}), x_i) + t_i(\mathbf{x}))$. The expected social surplus is defined as $E_{\mathbf{x}}\left(\sum_{i \in \mathcal{I}} v_i(q(\mathbf{x}), x_i)\right)$ or, equivalently, as the sum of agents' ex ante expected utilities minus the sum of agents' ex ante expected transfers. As in Gershkov et al. (2013), we employ the following notion of equivalence.

DEFINITION 1. *Two mechanisms (q, t) and $(\tilde{q}, \tilde{t})$ are equivalent if and only if they yield the same interim expected utilities to all agents and generate the same expected social surplus.*

3 The Increasing Difference over Distributions

In this section, we introduce and characterize the *increasing differences over distributions* property. We use this novel property to characterize Bayesian incentive compatible mechanisms in terms of a monotonicity condition and an envelope formula, which is similar to how the standard increasing differences property is used to characterize dominant strategy incentive compatible mechanisms.

To motivate the analysis of our novel property, let us first consider the standard property of increasing differences or supermodularity (see Topkis, 1998).

DEFINITION 2. *Function v_i satisfies the increasing differences property if for any pair of alternatives $a, a' \in A$ the difference $v_i(a, x) - v_i(a', x)$ is either increasing, decreasing, or constant in x.*[4]

Assuming that v_i satisfies the increasing differences property for each $i \in \mathcal{I}$, Mookherjee and Reichelstein (1992) showed that dominant strategy incentive compatibility can be characterized by a monotone-marginal condition and an envelope formula.[5]

[3]Our main results Theorems 1 and 2 can also be extended to discrete types similar to Gershkov et al. (2013).

[4]Throughout the paper, "increasing" ("decreasing") refers to a strictly increasing (decreasing) function.

[5]More precisely, the result below follows from Propositions 1, 2, and 3 of Mookherjee and Reichelstein (1992), because when v_i is differentiable with respect to its second argument the increasing differences property is equivalent

Proposition 1 (Mookherjee and Reichelstein, 1992). *Suppose v_i satisfies the increasing differences property for each $i \in \mathcal{I}$. A mechanism (q,t) is DIC if and only if for each $i \in \mathcal{I}$ and $\mathbf{x} \in X$ (i) $v_{ix}(q(s,\mathbf{x}_{-i}),x_i)$ is non-decreasing in s and (ii) agent i's utility can be expressed as*

$$u_i(x_i,\mathbf{x}_{-i}) \;=\; u_i(\underline{x}_i,\mathbf{x}_{-i}) + \int_{\underline{x}_i}^{x_i} v_{ix}(q(s,\mathbf{x}_{-i}),s)ds.^6 \tag{1}$$

Proposition 1 is a powerful result as it provides a tractable analysis of incentive compatibility constraints in many important applications (e.g., Laffont and Martimort, 1997; Segal, 2003). In the Appendix, we further show that the increasing differences property is a necessary condition for the characterization of Proposition 1. In particular, if some agent's function v_i does not satisfy the increasing differences property then one can always construct a DIC mechanism that does not have non-decreasing marginals (see Proposition A1).

To obtain a similar characterization for BIC mechanisms, we first need an appropriate extension of the increasing differences property to Bayesian settings. Note that from the perspective of each agent, who knows only the distribution of the types of other agents, every allocation rule induces a probability distribution over possible outcomes. This logically leads to the following definition.

Definition 3. *Function v_i satisfies the increasing differences over distributions property if for any pair of probability distributions $G, F \in \Delta(A)$, the difference $\int v_i(a,x)dG(a) - \int v_i(a,x)dF(a)$ is either increasing, decreasing, or constant in x.*

The following proposition shows that, given the increasing differences over distributions property, BIC mechanisms can be indeed characterized by a monotone-expected-marginal condition and an envelope formula.

Proposition 2. *Suppose v_i satisfies the increasing differences over distributions property for each $i \in \mathcal{I}$. A mechanism (q,t) is BIC if and only if for each $i \in \mathcal{I}$ and $x_i \in X_i$ (i) $E_{\mathbf{x}_{-i}} v_{ix}(q(s,\mathbf{x}_{-i}),x_i)$ is non-decreasing in s and (ii) agent i's interim expected utility can be expressed as*

$$U_i(x_i) \;=\; U_i(\underline{x}_i) + \int_{\underline{x}_i}^{x_i} E_{\mathbf{x}_{-i}} v_{ix}(q(s,\mathbf{x}_{-i}),s)ds.^7 \tag{2}$$

Parallel to the result of Proposition 1, the increasing differences over distributions is a necessary condition for the characterization of Proposition 2 (see Proposition A2 in Appendix).

Propositions 1 and 2 are connected to the literature on monotonicity and incentive compatibility. For general quasi-linear environments, Rochet (1987) showed that incentive compatibility

to the weak single-crossing property used in their paper.

[6] See also Milgrom and Segal (2002). The sufficiency part holds even without imposing the increasing differences.

[7] As in Proposition 1, the sufficiency part holds even without imposing increasing differences over distributions.

constraint can be characterized by a condition called cycle-monotonicity. For convex type-spaces, Saks and Yu (2005) advanced Rochet (1987)'s result by establishing that it is sufficient to consider only two-cycle monotonicity.[8] The two-cycle monotonicity condition reduces to the standard monotonicity of the allocation rule when agents have dot product valuations (e.g., $A \subset \mathbb{R}$ and $v_i(a, x_i) = a \cdot x_i$). When agents have non-linear differentiable valuations (and one-dimensional types), the two-cycle monotonicity is equivalent to the monotone-marginal condition (see Proposition 1). Thus, Propositions 1 and 2, together with Propositions A1 and A2, determine the largest set of differentiable quasi-linear utility functions that permit the characterization of incentive compatibility with the two-cycle monotonicty condition for one-dimensional types.

The increasing differences over distributions property gives us a readily workable characterization of Bayesian incentive compatibility. This property is, however, novel, and we want to understand how it restricts agents' utilities before proceeding with further analysis. First of all, if the feasible set A is the set of all possible lotteries over some set of alternatives, increasing differences over distributions and increasing differences properties are equivalent, because in this case any probability distribution over A simply defines a compound lottery over the underlying set of alternatives. In general, however, the increasing differences over distributions property only implies the increasing differences property. To see this, simply note that one can always consider a pair of deterministic distributions in the definition of the increasing differences over distributions. Finally, we provide a full characterization of utility functions that satisfy the increasing differences over distributions property.

Proposition 3. *Function $v_i(a, x_i)$ satisfies increasing differences over distributions if and only if*

$$v_i(a, x_i) = f_i(a)M_i(x_i) + m_i(x_i) + g_i(a), \tag{3}$$

where $f_i, g_i : A \to \mathbb{R}$ are continuous, $M_i, m_i : X_i \to \mathbb{R}$, and M_i is increasing.

In a concurrent paper, Kartik, Lee, and Rappoport (2017) study a less demanding property of the single-crossing expectational differences, which extends the standard single-crossing differences property to the space of lotteries. They show that their novel property admits the characterization $v_i(a, x_i) = f_i(a)M_i(x_i) + g_i(a)\hat{M}_i(x) + m_i(x_i)$, where $f_i, g_i : A \to \mathbb{R}$ and $M_i, \hat{M}_i, m_i : X_i \to \mathbb{R}$ with M_i and $\hat{M}_i$ being each single crossing and ratio ordered. The ratio-ordered requirement reduces to M_i being increasing function when $\hat{M}_i \equiv 1$. Celik (2015) also employs a weaker version of increasing differences over distributions condition to analyze the implementation with gradual-revelation. These weaker properties, however, do not allow a convenient characterization

[8]See also Bikhchandani et al. (2006) and Ashlagi et al. (2010) for the analysis of incentive compatibility in convex domains. Mishra, Pramanik and Roy (2014) and Kushnir and Galichon (2017) analyze two-cycle monotonicity condition in important non-convex domains.

of Bayesian incentive compatibility as Proposition A2 in Appendix highlights.

Liu and Pei (2017) also consider a related but more demanding property of the increasing absolute differences over distributions. They show that this property together with monotone-supermodularity are sufficient to guarantee the monotonicity of sender's equilibrium strategy with respect to her type in signalling games.

The increasing differences over distributions property is also related to the aggregation of the single-crossing property analyzed by Quah and Strulovici (2012). They consider function $v(a,x,t)$ that satisfies the single-crossing differences property in (a,x) for each t. They ask under what conditions the aggregate function $\int v(a,x,t)dF(t)$ will also satisfy the single-crossing differences property for all distributions F. While this question is not trivial, the answer to the parallel question for the increasing differences property is rather straightforward if one fixes the direction of monotonicity: If for given a and a' the difference $v(a',x,t)-v(a,x,t)$ is increasing in x for each t, the aggregate difference has to be increasing.[9] However, requiring the increasing differences property to hold in the space of lotteries is different from requiring it to be preserved under aggregation, as Proposition 3 shows.

Given the result of Proposition 3, we assume in the rest of the paper that agent i's value function v_i takes the form of (3). With this specification, for each $i \in \mathcal{I}$, the monotonicity conditions in the characterizations of DIC and BIC mechanisms are now equivalent to $f_i(q(s,\mathbf{x}_{-i}))$ being non-decreasing in s for $\mathbf{x}_{-i} \in X_{-i}$ and $E_{\mathbf{x}_{-i}} f_i(q(s,\mathbf{x}_{-i}))$ being non-decreasing in s, respectively.[10]

4 The BIC-DIC Equivalence

We use the following logic to prove the equivalence between Bayesian and dominant strategy implementation. The characterizations of DIC and BIC mechanisms (Propositions 1 and 2) imply that the interim expected utilities of agents are determined by the allocation rule (up to a constant). Therefore, to match agents' interim expected utilities, we need to match $E_{\mathbf{x}_{-i}} f_i(q(x_i,\mathbf{x}_{-i}))$ for each $x_i \in X_i$ and $i \in \mathcal{I}$. To respect the incentive compatibility, we need to satisfy the monotone-marginal condition, i.e. $f_i(q(\cdot,\mathbf{x}_{-i}))$ is non-decreasing for each $\mathbf{x}_{-i} \in X_{-i}$ and $i \in \mathcal{I}$. Finally, we need to make sure that the equivalent mechanisms generate the same expected social surplus.

To state our main result, we introduce first the notion of convex-valued mappings. A mapping $\mathbf{f} : A \to \mathbb{R}^I$ with $\mathbf{f}(a) = (f_1(a),...,f_I(a))$ is *convex-valued* if its image is convex, i.e., for any $a,b \in A$ and $\alpha \in [0,1]$ there exists $c \in A$ such that $\mathbf{f}(c) = \alpha\mathbf{f}(a)+(1-\alpha)\mathbf{f}(b)$. We also note a useful property of mappings $\mathbf{g} = (g_1,...,g_I)$ and $\mathbf{f} = (f_1,...,f_I)$ in (3): If $\mathbf{g}$ is a linear transformation of $\mathbf{f}$, i.e.,

[9] We thank Navin Kartik, SangMok Lee, and Daniel Rappoport for pointing out this connection to us.

[10] In specification (3) we could redefine types $\tilde{x}_i \sim M_i(x_i)$ and drop function $m_i(x_i)$ because it does not interact with allocation. We cannot, however, modify g_i and f_i as it becomes clear from applications of Section 5.

$\mathbf{g} \equiv M\mathbf{f}$ for some $I \times I$ matrix M, then $\mathbf{f}$ is convex-valued if and only if the mapping of all agents utilities $(v_1(\cdot, x_1) + t_1, ..., v_I(\cdot, x_I) + t_I)$ is convex-valued for each $(x_1, ..., x_I) \in X$.[11]

Theorem 1. *Assume mapping* $\mathbf{f}$ *is convex-valued, and* $\mathbf{g}$ *is a linear transformation of* $\mathbf{f}$. *Then for any BIC mechanism* $(\tilde{q}, \tilde{t})$ *there exists an equivalent DIC mechanism* (q, t).

The main part of the argument proving the theorem establishes that for a given BIC allocation rule $\tilde{q}$ there exists a feasible allocation q that satisfies

$$E_{\mathbf{x}_{-i}} f_i(q(x_i, \mathbf{x}_{-i})) = E_{\mathbf{x}_{-i}} f_i(\tilde{q}(x_i, \mathbf{x}_{-i})), \forall x_i \in X_i, \forall i \in \mathcal{I}, \tag{4}$$

and that has non-decreasing marginals $f_i(q(\cdot, \mathbf{x}_{-i}))$ for all $\mathbf{x}_{-i} \in X_{-i}$ and $i \in \mathcal{I}$. We establish this statement for discrete and uniformly distributed types in Lemma 1 below. In particular, we develop an algorithm that finds a feasible allocation that satisfies (4) and that has non-decreasing marginals.[12] We then extend the proof to continuous types and arbitrary distributions (see Lemmas A1 and A2). Finally, we construct transfers that lead to the same interim expected utilities and generate the same expected social surplus using the envelope formula (see Proposition 1).

Lemma 1. *Suppose, for all* $i \in \mathcal{I}$, X_i *is a finite discrete set and* λ_i *is the uniform distribution on* X_i. *For any BIC allocation* $\tilde{q}$ *there exists a feasible allocation* q *satisfying (4) and* $f_i(q(\cdot, \mathbf{x}_{-i}))$ *being non-decreasing for all* $\mathbf{x}_{-i} \in X_{-i}$ *and* $i \in \mathcal{I}$.

Proof. Consider an arbitrary BIC allocation $\tilde{q}$, and let us assume $f_j(\tilde{q}(\cdot, \mathbf{x}_{-j}))$ is not non-decreasing for some j and $\mathbf{x}_{-j}$; otherwise the statement is trivial. Then, there exists some $x'_j > x_j$ such that $f_j(\tilde{q}(x'_j, \mathbf{x}_{-j})) < f_j(\tilde{q}(x_j, \mathbf{x}_{-j}))$. Since agent j's expected marginal $E_{\mathbf{x}_{-j}} f_j(\tilde{q}(\cdot, \mathbf{x}_{-j}))$ is non-decreasing there also exists a set of other agents' types X'_{-j} such that $f_j(\tilde{q}(x'_j, \mathbf{x}'_{-j})) > f_j(\tilde{q}(x_j, \mathbf{x}'_{-j}))$ for all $\mathbf{x}'_{-j} \in X'_{-j}$. Now consider a new allocation $\hat{q} \neq \tilde{q}$ such that

$$\mathbf{f}(\hat{q}(x_j, \mathbf{x}_{-j})) = \frac{1}{2}\mathbf{f}(\tilde{q}(x_j, \mathbf{x}_{-j})) + \frac{1}{2}\mathbf{f}(\tilde{q}(x'_j, \mathbf{x}_{-j})),\ \mathbf{f}(\hat{q}(x_j, \mathbf{x}'_{-j})) = (1-\delta)\mathbf{f}(\tilde{q}(x_j, \mathbf{x}'_{-j})) + \delta\mathbf{f}(\tilde{q}(x'_j, \mathbf{x}'_{-j})),$$

$$\mathbf{f}(\hat{q}(x'_j, \mathbf{x}_{-j})) = \frac{1}{2}\mathbf{f}(\tilde{q}(x'_j, \mathbf{x}_{-j})) + \frac{1}{2}\mathbf{f}(\tilde{q}(x_j, \mathbf{x}_{-j})),\ \mathbf{f}(\hat{q}(x'_j, \mathbf{x}'_{-j})) = (1-\delta)\mathbf{f}(\tilde{q}(x'_j, \mathbf{x}'_{-j})) + \delta\mathbf{f}(\tilde{q}(x_j, \mathbf{x}'_{-j})),$$

for all $\mathbf{x}'_{-j} \in X'_{-j}$ and $\hat{q}(\mathbf{x}) = \tilde{q}(\mathbf{x})$ for all other $\mathbf{x} \in \mathbf{X}$, where

[11]The necessity part actually holds only under an additional mild condition. If we denote the matrix transforming $\mathbf{f}$ to $\mathbf{g}$ as A and the diagonal matrix with elements $M_i(x_i)$ as $M(\mathbf{x})$ with $\mathbf{x} = (x_1, ..., x_I)$, the additional condition states that the sum of matrices $M(\mathbf{x}) + A$ has a full rank.

[12]Gershkov et al. (2013) use a minimization problem to find a feasible allocation that satisfies (4) and that has non-decreasing marginals. Their approach could also be adapted to our settings. We use an algorithmic proof because of its convenience in the proofs of our Theorem 2 and the applications presented in Section 5.

$$\delta = \frac{1}{2}(f_j(\tilde{q}(x_j, \mathbf{x}_{-j})) - f_j(\tilde{q}(x'_j, \mathbf{x}_{-j})))/\sum\nolimits_{\mathbf{x}'_{-j} \in X'_{-j}} (f_j(\tilde{q}(x'_j, \mathbf{x}'_{-j})) - f_j(\tilde{q}(x_j, \mathbf{x}'_{-j}))). \qquad (5)$$

Since $E_{\mathbf{x}_{-j}} f_j(\tilde{q}(\cdot, \mathbf{x}_{-j}))$ is non-decreasing we have $0 \leq \delta \leq \frac{1}{2}$. In addition, a feasible allocation $\hat{q}$ with $\hat{q}(\mathbf{x}) \in A$, for each $\mathbf{x} \in X$, is guaranteed to exist, because mapping $\mathbf{f}$ is convex-valued. Equation (5) guarantees that the equal expected marginal condition (4) is satisfied for agent j having types x_j and x'_j. For agent j having other types, condition (4) follows trivially. For agent i, $i \neq j$, condition (4) follows from $\mathbf{f}(\hat{q}(x_j, \mathbf{x}_{-j})) + \mathbf{f}(\hat{q}(x'_j, \mathbf{x}_{-j})) = \mathbf{f}(\tilde{q}(x_j, \mathbf{x}_{-j})) + \mathbf{f}(\tilde{q}(x'_j, \mathbf{x}_{-j}))$ and $\mathbf{f}(\hat{q}(x_j, \mathbf{x}'_{-j})) + \mathbf{f}(\hat{q}(x'_j, \mathbf{x}'_{-j})) = \mathbf{f}(\tilde{q}(x_j, \mathbf{x}'_{-j})) + \mathbf{f}(\tilde{q}(x'_j, \mathbf{x}'_{-j}))$.

Let us now define $\hat{s} = E_{\mathbf{x}}(||\mathbf{f}(\hat{q}(\mathbf{x})||^2)$ and $\tilde{s} = E_{\mathbf{x}}(||\mathbf{f}(\tilde{q}(\mathbf{x})||^2)$, where $||\cdot||$ denotes the Euclidean norm $||\mathbf{f}(q(\mathbf{x}))||^2 = \sum_{i \in \mathcal{I}} f_i(q(\mathbf{x}))^2$. Taking into account that λ_i is uniformly distributed, we have

$$\hat{s} - \tilde{s} = \left(-1/2||\mathbf{f}(\tilde{q}(x_j, \mathbf{x}_{-j})) - \mathbf{f}(\tilde{q}(x'_j, \mathbf{x}_{-j}))||^2 - 2\delta(1-\delta)\, ||\mathbf{f}(\tilde{q}(x'_j, \mathbf{x}'_{-j})) - \mathbf{f}(\tilde{q}(x_j, \mathbf{x}'_{-j}))||^2\right)/|X| < 0.$$

If $f_j(\hat{q}(\cdot, \mathbf{x}_{-j}))$ is not non-decreasing for some j and $\mathbf{x}_{-j}$, we repeat the above procedure. Iterating the procedure, we finally obtain a sequence of allocations $q^n \in A$ and a sequence of values $s^n \geq 0$ for $n = 1, 2, \ldots$. If for some n we find that $f_j(q^n(\cdot, \mathbf{x}_{-j}))$ is non-decreasing for all j and $\mathbf{x}_{-j}$, we set $q^{n+1} \equiv q^n$ and $s^{n+1} \equiv s^n$. By construction, s^n is a weakly decreasing sequence that is bounded below by 0. Hence, s^n has a limit that we denote as s. Since set A is compact, there also exists a convergent subsequence of q^n with a limit q such that $q(\mathbf{x}) \in A$ for all $\mathbf{x} \in X$. By construction, $s = E_{\mathbf{x}}(||\mathbf{f}(q(\mathbf{x})||^2)$.

We argue for the limit allocation q that $f_j(q(\cdot, \mathbf{x}_{-j}))$ has to be non-decreasing for each j and $\mathbf{x}_{-j}$. Suppose, in contradiction, that for some $j \in \mathcal{I}$ and $\mathbf{x}_{-j} \in \mathbf{X}_{-j}$ $f_j(q(\cdot, \mathbf{x}_{-j}))$ is not non-decreasing. Using the above construction, we can obtain an allocation q' with $s' = E_{\mathbf{x}}(||\mathbf{f}(q'(\mathbf{x})||^2) < s$. This contradicts to the fact that s is a limit of decreasing sequence s^n constructed above. □

Proof of Theorem 1. Lemmas A1 and A2 (postponed to Appendix) extend Lemma 1 to show that, given any set $X_i \subset \mathbb{R}$ and any distribution λ_i, for any BIC allocation $\tilde{q}$ there exists a feasible allocation q satisfying (4) with non-decreasing marginals $f_i(q(\cdot, \mathbf{x}_{-i}))$ for all $i \in \mathcal{I}$ and $\mathbf{x}_{-i} \in X_{-i}$. To complete the construction of an equivalent DIC mechanism we consider transfers t defined by

$$t_i(x_i, \mathbf{x}_{-i}) = t_i(\underline{x}_i, \mathbf{x}_{-i}) + v_i(q(\underline{x}_i, \mathbf{x}_{-i}), \underline{x}_i) - v_i(q(x_i, \mathbf{x}_{-i}), x_i) + \int_{\underline{x}_i}^{x_i} v_{ix}(q(s, \mathbf{x}_{-i}), s)ds, \qquad (6)$$

for all $\mathbf{x} \in X$, $i \in \mathcal{I}$, where $t_i(\underline{x}_i, \mathbf{x}_{-i}) = E_{\mathbf{x}_{-i}}\big(v_i(\tilde{q}(\underline{x}_i, \mathbf{x}_{-i}), \underline{x}_i) + \tilde{t}_i(\underline{x}_i, \mathbf{x}_{-i})\big) - v_i(q(\underline{x}_i, \mathbf{x}_{-i}), \underline{x}_i)$. Proposition 1 guarantees that mechanism (q, t) is DIC. In addition, mechanism (q, t) leads to the same interim expected utilities as in BIC mechanism $(\tilde{q}, \tilde{t})$. In particular,

$$\begin{aligned} U_i(x_i) &= E_{\mathbf{x}_{-i}}(\tilde{t}_i(\underline{x}_i, \mathbf{x}_{-i}) + v_i(\tilde{q}(\underline{x}_i, \mathbf{x}_{-i}), \underline{x}_i)) + \int_{\underline{x}_i}^{x_i} E_{\mathbf{x}_{-i}} v_{ix}(q(s, x_{-i}), s) ds \\ &= E_{\mathbf{x}_{-i}}(\tilde{t}_i(\underline{x}_i, \mathbf{x}_{-i}) + v_i(\tilde{q}(\underline{x}_i, \mathbf{x}_{-i}), \underline{x}_i)) + \int_{\underline{x}_i}^{x_i} E_{\mathbf{x}_{-i}} v_{ix}(\tilde{q}(s, x_{-i}), s) ds \;=\; \tilde{U}_i(x_i), \end{aligned} \quad (7)$$

where the first equality follows from (6), the second one from (4), and the third one from the characterization of BIC mechanisms (Proposition 2). When mapping $\mathbf{g}$ is a linear transformation of $\mathbf{f}$, the equal expected marginal conditions in (4) also imply $E_{\mathbf{x}}[\sum_{i\in\mathcal{I}} g_i(q(\mathbf{x}))] = E_{\mathbf{x}}[\sum_{i\in\mathcal{I}} g_i(\tilde{q}(\mathbf{x}))]$. Hence, both mechanisms also generate the same social surplus. □

Theorem 1 extends the BIC-DIC equivalence result to non-linear environments where each agent's utility satisfies the increasing differences over distributions property and the mapping of all agents' utilities is convex-valued. The convex-valued assumption is generally indispensable for the equivalence result as Example A1 in Appendix shows. In addition, the new proof requires only that the set of feasible allocations A is compact instead of being a simplex as in Gershkov et al. (2013).

The requirement that $\mathbf{g}$ is a linear transformation of $\mathbf{f}$ is satisfied, for example, if $g_i \equiv 0 \,\forall i \in \mathcal{I}$ as in some applications of Section 5. For general $\mathbf{g}$, the constructed DIC mechanism, however, does not necessarily match the expected social surplus of the BIC mechanism.[13] We now analyze the conditions when for any BIC mechanism one could find a DIC mechanism that produces the same interim expected utilities and generates *at least as large* expected social surplus. In addition to being more flexible than the equivalence, this way of comparing the implementation concepts better captures the economic intuition that one does not need to insert additional money to achieve a more robust solution concept.

For this purpose, we consider environments where the set of feasible allocations A is a convex and compact subset of $\mathbb{R}^I$ with $q = (q_1, ..., q_I)$, where $q_i \in \mathbb{R}$ for each $i \in \mathcal{I}$. We also assume that functions f_i depend on different components of allocations, i.e., $f_i(q) = \check{f}_i(q_i)$, for all $i \in \mathcal{I}$, $q \in A$.

Theorem 2. *Assume mapping* $\mathbf{f}$ *is convex-valued. For any BIC mechanism there exists a DIC mechanism that delivers the same interim expected utilities for all agents. In addition, the DIC mechanism generates at least as large expected social surplus, if*

(i) for each $i \in \mathcal{I}$ $\check{f}_i(q_i)$ *is non-decreasing and concave (or non-increasing and convex) and* $g_i(q)$ *is continuous, non-increasing, and concave in each component, or*

(ii) for each $i \in \mathcal{I}$ $\check{f}_i(q_i)$ *is non-increasing and concave (or non-decreasing and convex) and* $g_i(q)$ *is continuous, non-decreasing, and concave in each component.*

[13] For general $\mathbf{g}$ the constructed DIC mechanism still delivers the same interim expected utilities.

Remark. The theorem also extends to settings where the set of feasible allocations A is compact, mapping $\mathbf{f}$ is convex-valued, and the utility of each agent satisfies the following condensation property: Functions f_i and g_i can be written as $f_i(q) = \check{f}_i(h_i(q))$ and $\sum_i g_i(q) = G(h_1(q), ..., h_I(q))$ for all $q \in A$, where $h_i : A \to \mathbb{R}$, $\check{f}_i$ is non-decreasing and concave (or non-increasing and convex), and the aggregate function $G : \mathbb{R}^I \to \mathbb{R}$ is continuous, non-increasing, and concave in each component.[14] The proof of this extension repeats the steps of the proof of Theorem 2 presented in Appendix, and we omit it to avoid repetition. We exploit this observation when we consider the environmental mechanism design applications in Section 5.

The requirement of Theorem 2 that the DIC mechanism produces only at least as large expected social surplus compared to the original BIC mechanism is less demanding than the one of mechanisms equivalence (see Definition 1). Hence, it also has a broader range of meaningful economic applications, which we illustrate in Section 5.

5 Applications

In this section, we demonstrate that Theorems 1 and 2 apply to many important environments where previous works have little bite (e.g. Manelli and Vincent, 2010; Gershkov et al., 2013). In addition, they produce several novel implications that are of independent interest.

5.1 Principal-Agent Problem with Allocative Externalities

Consider a standard contracting setting where a principal needs to procure I goods from I agents. Assume the principal chooses a production plan $q = (q_1, ..., q_I) \in A \equiv \Pi_{i=1}^{I}[\underline{q}_i, \overline{q}_i] \subseteq \mathbb{R}^I$ and a transfer scheme $(t_1, ..., t_I) \in \mathbb{R}^I$. The payoff of agent i is then given by $-c_i(q_i)x_i + t_i$, where $c_i : [\underline{q}_i, \overline{q}_i] \to \mathbb{R}$ is some continuous non-decreasing function with an interpretation of $c_i(q_i)x_i$ being agent i's cost of supplying q_i units of good i. Many influential papers analyzing the optimal procurement contracts fall into this setting (e.g. Laffont and Martimort, 1997; Mookherjee and Tsumagari, 2004; Severinov, 2008; Duenyas, et al., 2013). In this setting, we have $f_i(q) = -c_i(q_i)$ and $g_i(q) = 0$ for each $i \in \mathcal{I}$. Since functions c_i are continuous, the Intermediate Value Theorem implies that mapping $\mathbf{f}(\cdot) = (-c_1(\cdot), ..., -c_I(\cdot))$ is convex-valued. Thus, Theorem 1 leads to the following corollary.

Corollary 1. *Consider the standard procurement setting. If c_i is continuous for $i \in \mathcal{I}$, then for any BIC mechanism there exists an equivalent DIC mechanism.*

In many contracting situations, agents may not only care about their own contracts with the

[14] Similar to condition (ii) in Theorem 2, the result also extends to settings when $\check{f}_i$ is non-increasing and concave (or non-decreasing and convex) and G is continuous, non-decreasing, and concave in each component.

principal, but also have preferences about contracts received by other agents. For instance, a country may prefer its ally rather than its enemy to receive a weapon contract (see Jehiel et al., 1996). Similar concerns arise in the presence of downstream competition among firms (Segal, 1999). Within the current framework, type-independent allocative externalities can be captured by incorporating an additional term into agent's utility function, i.e., $-c_i(q_i)x_i + g_i(q) + t_i$. Assuming that the cost and externality functions satisfy the conditions of Theorem 2, we establish the following result.

COROLLARY 2. *Consider a procurement setting with allocation externalities. If c_i is continuous for each $i \in \mathcal{I}$, then for any BIC mechanism there exists a DIC mechanism that delivers the same interim expected utilities for all agents. If c_i is also non-decreasing and convex (concave) and g_i is continuous, non-decreasing (non-increasing), and concave in each component for each $i \in \mathcal{I}$, then the DIC mechanism generates at least as large expected social surplus as the BIC mechanism.*

Corollary 2 identifies environments with allocative externalities where a mechanism designer can rely on dominant strategy implementation and gains nothing from designing more complex BIC mechanisms. This is in sharp contrast to results pertaining to environments with both allocative and information externalities, where more robust solution concepts appear to be much more restrictive (see Jehiel and Moldovanu (2006) for an excellent survey).

5.2 Environmental Mechanism Design

Let us first consider the environmental mechanism design model of Martimort and Sand-Zantman (2013, 2015), who analyze feasible agreements in reducing the aggregate pollution of I countries. Each country i can exert effort $q_i \in [\underline{q}, \overline{q}] \subseteq \mathbb{R}_+$ that has both local benefits of size αq_i (with $\alpha \in [0, 1)$) and global benefits of size $(1 - \alpha)q_i$, which accrue worldwide. The countries differ in their costs of effort $q_i^2 x_i/2$, with x_i being country i's efficiency parameter. Efficiency parameters are drawn independently from the same cumulative distribution λ with support $[\underline{x}, \overline{x}] \subseteq \mathbb{R}$. Overall, country i's payoff is given by $-q_i^2 x_i/2 + \alpha q_i + (1 - \alpha)Q + t_i$, where $Q = \sum_{i=1}^{I} q_i$ is aggregate global benefits and t_i is a monetary transfer to country i. Taking into account that function $-q_i^2/2$ is non-increasing and concave and the externality function $\alpha q_i + (1 - \alpha)Q$ is non-decreasing and linear (and, hence, concave), the following result directly follows from Theorem 2.

COROLLARY 3. *Consider the setting of Martimort and Sand-Zantman (2013, 2015). Then, for any BIC mechanism there exists a DIC mechanism producing the same interim expected utilities to all agents and generating at least as large expected social surplus.*

Baliga and Maskin (2003) also study feasible agreements to efficiently reduce the aggregate pollution level, but consider a slightly different model. Although they assume that agents' costs are type-independent, agents have private information about their value of the pollution reduction. More specifically, agent i's utility equals $x_i Q^{1/2} - q_i + t_i$, where $x_i Q^{1/2}$ is the gross benefits to agent i from aggregate reduction Q. Though Theorem 2 does not formally apply to this environment, each agent i's benefits and costs from pollution reduction satisfy the condensation property defined in the remark after Theorem 2. In particular, agent i's benefits equal $\check{f}_i(h_i(q)) = h_i(q)^{\frac{1}{2}}$ and the aggregate costs equal $\sum_{i \in \mathcal{I}} q_i = \sum_{i \in \mathcal{I}} q_i = -G(h_1(q), ..., h_I(q))$, where the condensation function $h_i(q) = \sum_{i \in \mathcal{I}} q_i$, $i \in \mathcal{I}$, is the same for all agents. The mapping $\mathbf{f} = (f_1(\cdot), ..., f_I(\cdot))$ is symmetric and, hence, the Intermediate Value Theorem implies that it is convex-valued. In addition, $\check{f}_i$ is non-decreasing and concave, and function G is non-increasing and linear (and, hence, concave). Hence, the following result is implied by the extension discussed in the remark after Theorem 2.

Corollary 4. *Consider the environmental mechanism design setting of Baliga and Maskin (2003). Then, for any BIC mechanism there exists a DIC mechanism producing the same interim expected utilities to all agents and generating at least as large expected social surplus.*

Corollaries 3 and 4 imply that the mechanism designer would lose nothing by restricting himself/herself to DIC mechanisms for environmental design problems, if he/she wanted to maintain the same level of agents' interim expected utility without the influx of additional money into the system. This result, however, may no longer hold when additional constraints - such as ex post budget balance - are imposed, as thoroughly discussed in Baliga and Maskin (2003). Though Bayesian implementation is more permissive when ex post budget balance is imposed, the mechanism designer can still rely only on DIC mechanisms if the budget balance constraint needs to be satisfied in expectations. We show this result in the next application.

5.3 Public Good Provision

Consider a standard setting of public good provision with $I \geq 2$ agents. If $q \in A = [\underline{q}, \overline{q}]$ units of public good are provided, agent i's utility is given by $f(q)x_i + t_i$, where $f(q)x_i$ is agent i's valuation of the public good and $t_i \in \mathbb{R}$ is the units of private good that he receives. Many influential papers on public good provision fall into this setting (e.g., Mailath and Postlewaite (1990), Ledyard and Palfrey (1999), Hellwig (2003), Norman (2004)). If $f : A \to \mathbb{R}$ is continuous in q, it again follows from the Intermediate Value Theorem that the mapping $\mathbf{f}(\cdot) = (f(\cdot), ..., f(\cdot))$ is convex-valued and, hence, Theorem 1 can be applied here.

Corollary 5. *Consider the public good provision setting. If f is continuous, then for any BIC mechanism there exists an equivalent DIC mechanism.*

While the equivalent DIC mechanism, constructed in Theorem 1, inherits interim individual rationality from the BIC mechanism,[15] there is no guarantee that other constraints imposed on the BIC mechanism will remain satisfied as well. For example, when designing a mechanism for public good provision, it is typical to require that the private goods raised from the agents are enough to cover the cost of the public good. Formally, a direct mechanism (q, t) is *ex ante budget balanced* if

$$\int_{\mathbf{x}\in X} \left[K(q(\mathbf{x})) + \sum_{i=1}^{I} t_i(\mathbf{x}) \right] d\lambda(\mathbf{x}) \leq 0, \tag{8}$$

where $K : A \to \mathbb{R}$ is the cost function of producing the public good. The following corollary of Theorem 2 provides a sufficient condition under which the equivalent DIC mechanism constructed in Theorem 1 also inherits ex ante budget balance from the original BIC mechanism.[16]

Corollary 6. *Suppose f is continuous, non-decreasing, and concave and K is continuous, non-decreasing, and convex. For any BIC mechanism that is ex ante budget balanced, the equivalent DIC mechanism, constructed in Theorems 1, is also ex ante budget balanced.*

Intuitively, the monotonicity and concavity of utility functions imply that the provision of public good is more balanced across states in the equivalent DIC mechanism than that in the BIC mechanism. Consequently, the expected cost of providing the public good is lower. Since the expected transfers remain unchanged in the equivalent DIC mechanism constructed in Theorem 1, the property of ex ante budget balance is preserved.

Our result thus suggests that for a quite general class of public good provision problems it is without loss of generality to insist on dominant-strategy incentive compatibility, even when the additional ex ante budget balance constraint is imposed.[17] For example, the second-best allocation rule in Hellwig (2003) can be equivalently implemented in dominant strategies without violating the ex ante budget balance condition if functions f and K are concave and convex respectively.

6 Conclusion

This paper extends the equivalence between Bayesian and dominant strategy implementation to environments where each agent's utility satisfies *the increasing differences over distributions property* and the mapping of all agents' utilities is *convex-valued*. These assumptions are satisfied by many important models that are studied in the literature on principal-agent problems with alloca-

[15]The constructed DIC mechanism satisfies even a stronger notion of ex post individual rationality.

[16]The result of Corollary 6 extends without any change to non-symmetric settings with mapping $\mathbf{f} = (f_1(\cdot), ..., f_I(\cdot))$ being convex-valued and functions f_i, $i \in \mathcal{I}$, being continuous, non-decreasing, and concave.

[17]For some applications, it is natural to require mechanisms to be *ex post budget balanced*, i.e., inequality (8) holds for each $\mathbf{x} \in X$. Börgers and Norman (2009) show that for every ex ante budget balanced DIC mechanism (q, t) there exist transfers t' such that (q, t') is (i) BIC for all agents and DIC for all but one agent and (ii) ex post budget balanced. Agents also have the same interim expected payments in both mechanisms (see also Börgers, 2015, Ch. 3).

tive externalities, environmental mechanism design, and public good provision. Since the results of the previous papers (Manelli and Vincent, 2010; Gershkov et al., 2013) do not apply to these environments, the current paper significantly enlarges the set of settings where the mechanism designer can rely on a more robust solution concept of dominant strategy implementation.

In this paper, we also provide sufficient conditions when for a given BIC mechanism there exists a DIC mechanism that yields the same interim expected utilities to all agents and generates *at least as large* social surplus (see also Kushnir, 2015). Using this result, we provide several novel implications for the above-mentioned environments. In addition, being less demanding than the notion of equivalence due to Gershkov et al. (2013), this way of comparing two implementation concepts broadens the set of environments when the mechanism designer could insist on a more robust notion of implementation without sacrificing his/her objectives. Hence, we believe this notion will be useful for future studies.

Our proof of the BIC-DIC equivalence result relies heavily on the characterization of incentive compatibility using the novel *increasing differences over distributions property.* We show that increasing differences over distributions is necessary and sufficient for Bayesian incentive compatibility to be conveniently characterized in terms of a monotone-expected-marginal condition and an envelope formula.[18] The equivalence result could potentially hold in environments where the above properties are not satisfied. The proof should then employ quite different techniques.

One possible approach has been discussed in our recent work. In Kushnir and Liu (2017), we explain how the BIC-DIC equivalence reduces to a purely mathematical question when a linear transformation of intersection of two closed convex sets coincides with the intersection of their images. Another possible approach has been proposed by Goeree and Kushnir (2017) who develop a novel geometric approach to mechanism design using basic tools from convex analysis. Applying these techniques to the question of the BIC-DIC equivalence in non-linear environments without increasing differences over distributions condition and environments with multidimensional types is an exciting prospect for future research.

[18] We also establish that the standard increasing differences property is necessary for dominant strategy incentive compatibility to be conveniently characterized in terms of a monotone-marginal condition and an envelope formula.

Appendix

Proof of Proposition 2. Our proof essentially extends the proof of Propositions 1, 2, and 3 in Mookherjee and Reichelstein (1992) to Bayesian settings. For the if statement, note that agent i does not deviate from the truth-telling Bayes-Nash equilibrium if and only if

$$\begin{aligned} U_i(x_i) &\geq E_{\mathbf{x}_{-i}}\left(v_i(q(x_i', \mathbf{x}_{-i}), x_i) + t_i(x_i', \mathbf{x}_{-i})\right) \\ &= U_i(x_i') + E_{\mathbf{x}_{-i}}\left(v_i(q(x_i', \mathbf{x}_{-i}), x_i) - v_i(q(x_i', \mathbf{x}_{-i}), x_i')\right) \end{aligned} \tag{A.1}$$

for all $x_i, x_i' \in X_i$. Using (2), this is equivalent to require that for all $x_i, x_i' \in X_i$,

$$\int_{x_i'}^{x_i} E_{\mathbf{x}_{-i}}\left(v_{ix}(q(s, \mathbf{x}_{-i}), s)\right) ds \geq E_{\mathbf{x}_{-i}}\left(v_i(q(x_i', \mathbf{x}_{-i}), x_i)\right) - E_{\mathbf{x}_{-i}}\left(v_i(q(x_i', \mathbf{x}_{-i}), x_i')\right),$$

which is true under the condition that $E_{\mathbf{x}_{-i}}(v_{ix}(q(s, \mathbf{x}_{-i}), x_i))$ is non-decreasing in s for all $x_i \in X_i$.

For the only if statement, suppose that mechanism (q, t) is BIC. We then have

$$U_i(x_i) = \max_{x_i' \in X_i}\big(E_{\mathbf{x}_{-i}}\left(v_i(q(x_i', \mathbf{x}_{-i}), x_i) + t_i(x_i', \mathbf{x}_{-i})\right)\big).$$

Since v_i is absolutely continuous in x_i and has a bounded derivative with respect to type x_i equation (2) follows from the envelope theorem (Milgrom and Segal, 2002). It remains to show that BIC also implies the monotone-expected-marginal condition. Suppose, in contradiction, $E_{\mathbf{x}_{-i}} v_{ix}(q(y, \mathbf{x}_{-i}), z) > E_{\mathbf{x}_{-i}} v_{ix}(q(x, \mathbf{x}_{-i}), z)$ for some agent i and $x, y, z \in X_i$, with $y < x$. The increasing differences over distributions property implies that

$$E_{\mathbf{x}_{-i}}\left(v_i(q(y, \mathbf{x}_{-i}), x) - v_i(q(y, \mathbf{x}_{-i}), y)\right) > E_{\mathbf{x}_{-i}}\left(v_i(q(x, \mathbf{x}_{-i}), x) - v_i(q(x, \mathbf{x}_{-i}), y)\right).$$

At the same time, the incentive compatibility implies

$$E_{\mathbf{x}_{-i}}\left(v_i(q(y, \mathbf{x}_{-i}), x) - v_i(q(y, \mathbf{x}_{-i}), y)\right) \leq U_i(x) - U_i(y) \leq E_{\mathbf{x}_{-i}}\left(v_i(q(x, \mathbf{x}_{-i}), x) - v_i(q(x, \mathbf{x}_{-i}), y)\right).$$

We thus reach a contradiction. □

Proof of Proposition 3. The sufficiency part is straightforward. Let us prove the necessity part. Consider some $x', y' \in X_i$ such that $x' > y'$ and let $\underline{a} \in \arg\min_{a \in A}\left(v_i(a, x') - v_i(a, y')\right)$ and $\overline{a} \in \arg\max_{a \in A}\left(v_i(a, x') - v_i(a, y')\right)$. Given our assumption that $v_i(a, x_i)$ is continuous in a, such $\underline{a}$ and $\overline{a}$ are guaranteed to exist. For each $a \in A$ we can then always find $\alpha(a, x', y') \in [0, 1]$ such that

$$v_i(a,x') - v_i(a,y') = \alpha(a,x',y')\big(v_i(\overline{a},x') - v_i(\overline{a},y')\big) + (1-\alpha(a,x',y'))\big(v_i(\underline{a},x') - v_i(\underline{a},y')\big).$$

Let us consider distribution G that puts the unit mass on allocation a and distribution F that puts probability $\alpha(a,x',y')$ on $\overline{a}$ and probability $1-\alpha(a,x',y')$ on $\underline{a}$. By construction, we have

$$\int v_i(a,x')dG - \int v_i(a,x')dF = \int v_i(a,y')dG - \int v_i(a,y')dF,$$

and the increasing differences over distributions property implies that $\int v_i(a,x)dG - \int v_i(a,x)dF$ is a constant function in x, which we denote as $g_i(a)$. Hence,

$$\begin{aligned} v_i(a,x) &= \alpha(a,x',y')v_i(\overline{a},x) + (1-\alpha(a,x',y'))v_i(\underline{a},x) + g_i(a) \\ &= f_i(a)M_i(x) + m_i(x) + g_i(a) \end{aligned}$$

where $f_i(a) = \alpha(a,x',y')$, $M_i(x) = v_i(\overline{a},x) - v_i(\underline{a},x)$, and $m_i(x) = v_i(\underline{a},x)$. The increasing differences over distributions and $v_i(\overline{a},x') - v_i(\underline{a},x') \geq v_i(\overline{a},y') - v_i(\underline{a},y')$ then implies that $M_i(x)$ is either an increasing or constant function. For the latter case, we redefine $\tilde{f}_i(a) = 0$, $\tilde{M}_i(x)$ to be any increasing function, and $\tilde{g}_i(a) = g_i(a) + f_i(a)M_i(x')$ to obtain expression (3). □

Proof of Theorem 1

LEMMA A1. *Suppose, for all $i \in \mathcal{I}$, $X_i = [0,1]$ and λ_i is the uniform distribution on X_i. Then, for any BIC allocation $\tilde{q}$ there exists a feasible allocation q satisfying (4) with $f_i(q(\cdot,\mathbf{x}_{-i}))$ being non-decreasing for all $i \in \mathcal{I}$ and $\mathbf{x}_{-i} \in X_{-i}$.*

PROOF. The proof essentially repeats the proof of Lemma 2 in Gershkov et al. (2013), and we only sketch it here. We consider a partition $[0,1]^I$ to 2^{nI} cubes of equal size. For each cube S in this partition, we approximate $\mathbf{f}(\tilde{q}(\mathbf{x}))$, $\mathbf{x} \in S$, by its average defined by

$$\mathbf{f}(\tilde{q}(S)) = 2^{nI} \int_S \mathbf{f}(\tilde{q}(\mathbf{x}))d\mathbf{x}.$$

Note allocation $\tilde{q}(S) \in A$ is well-defined, because mapping $\mathbf{f}$ is convex-valued. In addition, discrete allocation $\tilde{q}(S)$ inherits non-decreasing expected marginals from $\tilde{q}$. Lemma 1 then ensures that there exists an allocation $q(S)$ with non-decreasing marginals that can also be extended to piece-wise constant functions over $[0,1]^I$. Taking the limit with respect to the size of partition, we obtain the result of the lemma. For the details of the construction, we refer to Gershkov et al. (2013). □

Lemma A2. *Suppose, for all $i \in \mathcal{I}$, $X_i \subseteq \mathbb{R}$ and λ_i is some distribution on X_i. Then, for any BIC allocation $\tilde{q}$ there exists a feasible allocation q satisfying (4) with $f_i(q(\cdot, \mathbf{x}_{-i}))$ being non-decreasing for all $i \in \mathcal{I}$ and $\mathbf{x}_{-i} \in X_{-i}$.*

Proof. The proof repeats the proof of Lemma 3 in Gershkov et al. (2013). Its main idea is to relate the uniform distribution covered by Lemma A1 to the case of a general distribution. In particular, if random variable Z_i is uniformly distributed, then $\lambda_i^{-1}(Z_i)$ is distributed according to λ_i.[19] Hence, for a given BIC allocation $\tilde{q}$ we use transformation λ_i^{-1} to construct an allocation $\tilde{q}'$ defined on uniformly distributed types that also has a non-decreasing expected marginals. For allocation $\tilde{q}'$, we then apply the results of Lemmas 1 and A1 to obtain an allocation q' with non-decreasing marginals defined on uniformly distributed types. We then use transformation λ_i to recover an allocation q with non-decreasing marginals defined on types distributed according to λ_i. For the details of the construction, we refer to Gershkov et al. (2013). □

Example A1. We now show that the assumption that mapping $\mathbf{f}$ being convex-valued is generally indispensable for the BIC-DIC equivalence result of Theorem 1.

Consider a two-agent example with the set of possible allocations $A = [0, 1]$. Each agent i's type x_i is drawn independently from the uniform distribution over $[0, 1]$. For an allocation $q \in A$ and transfers $t_1, t_2 \in \mathbb{R}$, agent 1's utility equals to $qx_1 + t_1$, and agent 2's utility is $q^2 x_2 + t_2$. This environment satisfies all conditions of Theorem 1 except for the assumption that mapping (f_1, f_2) is convex-valued, where $f_1(q) = q$ and $f_2(q) = q^2$. Let us consider the following allocation rule:

$$q(x_1, x_2) = \begin{cases} 1 & \text{if } \max\{x_1, x_2\} \leq \frac{1}{2} \text{ or } \min\{x_1, x_2\} > 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

This allocation rule is Bayesian implementable because its expected marginals $\int_0^1 f_i(x_i, x_j) dx_j$ are non-decreasing everywhere. It is, however, not dominant-strategy implementable because marginals $f_i(x_1, x_2)$ are strictly decreasing for some $(x_1, x_2) \in X$. We now show that there does not exist an equivalent DIC mechanism for any BIC mechanism with allocation rule q.

Suppose, in contradiction, that for some BIC mechanism (q, t) there exists an equivalent DIC mechanism $(\hat{q}, \hat{t})$. Let $U_i(x_i)$ and $\hat{U}_i(x_i)$ be agent i's interim expected utilities in mechanisms (q, t) and $(\hat{q}, \hat{t})$, respectively. Since the two mechanisms are equivalent, we have $U_i(x_i) = \hat{U}_i(x_i)$ for all

[19] Where $\lambda_i^{-1}(z_i) = \inf\{x_i \in X_i | \lambda_i(x_i) \geq z_i\}$.

$x_i \in X_i$ and $i = 1, 2$. The envelope formula then implies that $\forall x_i, x_i' \in X_i$,

$$\begin{aligned} U_i(x_i) &= U_i(x_i') + \int_{x_i'}^{x_i} \int_0^1 f_i(q(s, x_j))dx_j ds \\ &= \hat{U}_i(x_i') + \int_{x_i'}^{x_i} \int_0^1 f_i(\hat{q}(s, x_j))dx_j ds = \hat{U}_i(x_i). \end{aligned}$$

Therefore, we must have for almost all $x_i \in [0, 1]$, and for all $i, j \in \{1, 2\}$, $i \neq j$,

$$\int_0^1 f_i(\hat{q}(x_i, x_j))dx_j = \int_0^1 f_i(q(x_i, x_j))dx_j = \frac{1}{2}. \tag{A.2}$$

Integrating (A.2) over x_j, we have for all $i \in \{1, 2\}$,

$$\int_0^1 \int_0^1 f_i(\hat{q}(x_1, x_2))dx_1 dx_2 = \frac{1}{2}, \tag{A.3}$$

which further implies that

$$0 = \int_0^1 \int_0^1 [f_1(\hat{q}(x_1, x_2)) - f_2(\hat{q}(x_1, x_2))]dx_1 dx_2 = \int_0^1 \int_0^1 \left[\hat{q}(x_1, x_2) - (\hat{q}(x_1, x_2))^2\right] dx_1 dx_2. \tag{A.4}$$

Since $q(x_1, x_2) \in A = [0, 1]$, equation (A.4) implies that $\hat{q}(x_1, x_2) \in \{0, 1\}$ for almost every type profile $(x_1, x_2) \in X$. In addition, allocation $\hat{q}$ being dominant strategy implementable implies that $f_2(\hat{q}(x_1, x_2)) = (\hat{q}(x_1, x_2))^2$ must be non-decreasing in x_2. The equal-expected-marginal condition (A.2) for agent 1 then implies that for almost all $x_1 \in [0, 1]$, $\hat{q}(x_1, x_2) = 0$ for $x_2 \in [0, 1/2]$ and $\hat{q}(x_1, x_2) = 1$ for $x_2 \in (1/2, 1]$.[20]

This allocation rule, however, does not satisfy the equal-expected-marginal condition (A.2) for agent 2. In particular, $\int_0^1 (\hat{q}(x_1, x_2))^2 dx_1 = 0$ for all $x_2 \in [0, 1/2]$, and $\int_0^1 (\hat{q}(x_1, x_2))^2 dx_2 = 1$ for all $x_2 \in (1/2, 1]$. We thus reach a contradiction. □

Proof of Theorem 2. Consider an arbitrary BIC mechanism $(\tilde{q}, \tilde{t})$ and the corresponding DIC mechanism (q, t) constructed in Theorem 1. Since equation (7) holds for any $\mathbf{g}$, the first part of Theorem 2 immediately follows. The idea behind the proof of the second part of the theorem is to show that if functions $\check{f}_i$ and g_i satisfy conditions (i) or (ii), the DIC mechanism constructed in Theorem 1 also satisfies

$$E_{\mathbf{x}} \left(\sum\nolimits_i g_i(q(\mathbf{x})) \right) \geq E_{\mathbf{x}} \left(\sum\nolimits_i g_i(\tilde{q}(\mathbf{x})) \right). \tag{A.5}$$

[20]Because of the monotonicity of the allocation rule and $\hat{q} \in \{0, 1\}$, the only indeterminacy in $\hat{q}(x_1, x_2)$ could happen at $x_2 = 1/2$.

Suppose condition (i) is satisfied. Let us first consider the case where types are discrete and uniformly distributed (as in Lemma 1). If the marginals of allocation $\tilde{q}$ are not non-decreasing, then $\check{f}_j(\tilde{q}_j(x'_j, \mathbf{x}_j)) < \check{f}_j(\tilde{q}_j(x_j, \mathbf{x}_{-j}))$ for some j, $x'_j > x_j$, and $\mathbf{x}_{-j}$. Using the construction of the algorithm in Lemma 1 we then obtain an allocation $\hat{q} \in A$ satisfying the equal-marginal conditions in (4) and delivering strictly smaller value to objective $E_{\mathbf{x}}||\mathbf{f}(\cdot)||^2$. Since function $\check{f}_j$ is non-decreasing and concave (or non-increasing and convex), we also have

$$\hat{q}_j(x_j, \mathbf{x}_{-j}) = \hat{q}_j(x'_j, \mathbf{x}_{-j}) \leq \frac{1}{2}\tilde{q}_j(x_j, \mathbf{x}_{-j}) + \frac{1}{2}\tilde{q}_j(x'_j, \mathbf{x}_{-j}),$$

$$\hat{q}_j(x_j, \mathbf{x}'_{-j}) \leq (1-\delta)\tilde{q}_j(x_j, \mathbf{x}'_{-j}) + \delta\tilde{q}_j(x'_j, \mathbf{x}'_{-j}),$$

$$\hat{q}_j(x'_j, \hat{\mathbf{x}}_{-j}) \leq (1-\delta)\tilde{q}_j(x'_j, \mathbf{x}'_{-j}) + \delta\tilde{q}_j(x_j, \mathbf{x}'_{-j}).$$

Since function g_i is non-increasing and concave in each component, this further implies

$$g_i(\hat{q}(x_j, \mathbf{x}_{-j})) + g_i(\hat{q}(x'_j, \mathbf{x}_{-j})) \geq g_i(\tilde{q}(x_j, \mathbf{x}_{-j})) + g_i(\tilde{q}(x'_j, \mathbf{x}_{-j})),$$

$$g_i(\hat{q}(x_j, \mathbf{x}'_{-j})) + g_i(\hat{q}(x'_j, \mathbf{x}'_{-j})) \geq g_i(\tilde{q}(x_j, \mathbf{x}'_{-j})) + g_i(\tilde{q}(x'_j, \mathbf{x}'_{-j})),$$

for each $i \in \mathcal{I}$ and, hence, $E_{\mathbf{x}}(\sum_i g_i(\hat{q}(\mathbf{x}))) \geq E_{\mathbf{x}}(\sum_i g_i(\tilde{q}(\mathbf{x})))$. We iterate this procedure to obtain a sequence of allocations $q^n \in A$ and a decreasing numerical sequence $s^n = E_{\mathbf{x}}||\mathbf{f}(q^n(\mathbf{x}))||^2$, $n = 1, 2, ...$ If we find that $\check{f}_j(q^n_j(\cdot, \mathbf{x}_{-j}))$ is non-decreasing for all j and $\mathbf{x}_{-j}$, we set $q^{n+1} \equiv q^n$ and $s^{n+1} \equiv s^n$. Since s^n is a weakly decreasing sequence bounded below by 0, it has a limit, which we denote as s. Since set A is compact, there also exists a convergent subsequence q^n with a limit q such that $q(\mathbf{x}) \in A$ for all $\mathbf{x} \in X$. Clearly, $s = E_{\mathbf{x}}(||\mathbf{f}(q(\mathbf{x})||^2)$ and $\check{f}_j(q_j(\cdot, \mathbf{x}_{-j}))$ is non-decreasing for each j and $\mathbf{x}_{-j}$. Since functions g_i are continuous, we also have $E_{\mathbf{x}}\big(\sum_i g_i(q(\mathbf{x}))\big) \geq E_{\mathbf{x}}\big(\sum_i g_i(\tilde{q}(\mathbf{x}))\big)$.

The result can then be further extended to continuous space with an arbitrary distribution similar to Lemmas A1 and A2. We then use equation (6) to define payment rule t delivering the same interim expected utilities. Finally, we derive that the social surplus in the constructed allocation

$$\begin{aligned} E_{\mathbf{x}}\left(\sum\nolimits_i v_i(q(\mathbf{x}), x_i)\right) &= E_{\mathbf{x}}\left(\sum\nolimits_i f_i(q(\mathbf{x}))M_i(x_i) + m_i(x_i) + g_i(q(\mathbf{x}))\right) \\ &\geq E_{\mathbf{x}}\left(\sum\nolimits_i f_i(\tilde{q}(\mathbf{x}))M_i(x_i) + m_i(x_i) + g_i(\tilde{q}(\mathbf{x}))\right) = E_{\mathbf{x}}\left(\sum\nolimits_i v_i(\tilde{q}(\mathbf{x}), x_i)\right), \end{aligned}$$

where the inequality follows from the equal-marginal conditions in (4) and inequality (A.5). This establishes the claim of the theorem. The proof is analogous when condition (ii) is satisfied. □

Proof of Corollaries 1 and 5. The statements follow from Theorem 1. □

Proof of Corollary 2, 3, and 4. The statements follow from Theorem 2. □

Proof of Corollary 6. Consider any BIC mechanism $(\tilde{q}, \tilde{t})$ and the equivalent DIC mechanism (q, t), constructed in Theorem 1. Since we have $g_i(q) = 0$ for each $i \in \mathcal{I}$ in the public good provision setting, the same ex ante expected utilities in both mechanisms implies that both mechanism yield the same expected transfers, i.e., $E_{\mathbf{x}}\left(\sum_{i\in\mathcal{I}} t_i(\mathbf{x})\right) = E_{\mathbf{x}}\left(\sum_{i\in\mathcal{I}} \tilde{t}_i(\mathbf{x})\right)$.

To prove the claim of the corollary, we need to show that the expected costs for the DIC mechanism is lower than the expected costs for the BIC mechanism, i.e., $E_{\mathbf{x}}(K(q(\mathbf{x}))) \leq E_{\mathbf{x}}(K(\tilde{q}(\mathbf{x})))$. This statement follows from applying the argument of the proof of Theorem 2 to function $-K$ instead of functions g_i, $i \in \mathcal{I}$. In particular, consider the sequence of allocation q^n constructed in the algorithm of Theorem 1. Since function K is non-decreasing and convex, the expected cost of allocations q^n is non-increasing in n, i.e., $E_{\mathbf{x}}(K(q^{n+1}(\mathbf{x}))) \leq E_{\mathbf{x}}(K(q^n(\mathbf{x}))) \leq E_{\mathbf{x}}(K(\tilde{q}(\mathbf{x})))$. The continuity of K then implies that the inequality holds in the limit. Finally, the result further extends to continuous type space with an arbitrary distribution similar to Lemmas A1 and A2. □

Proposition A1. *If function v_i violates the increasing differences property for some agent $i \in \mathcal{I}$, then there exists a dominant-strategy incentive compatible mechanism (q, t) that does not have non-decreasing marginals $v_{ix}(q(\cdot, \mathbf{x}_{-i}), x_i)$ for all $\mathbf{x}_{-i} \in X_{-i}$ and $x_i \in X_i$.*

Proof. Suppose $v_i(a, x)$ does not satisfy the increasing differences property. Then, there must exist $a, a' \in A$, and $x, y, z \in X$ with $x < y < z$ such that either

$$v_i(a, x) - v_i(a', x) \leq v_i(a, y) - v_i(a', y) \text{ and } v_i(a, y) - v_i(a', y) \geq v_i(a, z) - v_i(a', z), \qquad \text{(A.6)}$$

with at least one inequality being strict, or

$$v_i(a, x) - v_i(a', x) \geq v_i(a, y) - v_i(a', y) \text{ and } v_i(a, y) - v_i(a', y) \leq v_i(a, z) - v_i(a', z), \qquad \text{(A.7)}$$

with at least one strict inequality. We consider only case (A.6). Case (A.7) can be treated similarly.

Let us assume that the utility of agent i satisfies (A.6). We consider a mechanism with an allocation rule q and a payment rule t that are functions of agent i's reports only, i.e., $q : X_i \to A$ and $t : X_i \to \mathbb{R}^I$. In particular, we assign $q(x) = q(z) = a'$, $q(y) = a$, and $\forall s \neq x, y, z$,

$$q(s) = \begin{cases} a & \text{if } v_i(a, s) - v_i(a', s) \geq \bar{t}_i, \\ a' & \text{otherwise,} \end{cases}$$

where $\bar{t}_i = v_i(a, y) - v_i(a', y)$. Agent i receives no transfers if allocation a is chosen and $\bar{t}_i$ otherwise, i.e., $t_i(s) = 0$ if $q(s) = a$ and $t_i(s) = \bar{t}_i$ if $q(s) = a'$. All other agents receive no transfers, i.e., $t_j(s) \equiv 0$ for all $j \neq i$ and $s \in X_i$. It is straightforward to check that (q, t) is dominant-strategy incentive compatible.

We now show that agent i's marginals induced by allocation rule q cannot be all non-decreasing. Suppose, in contradiction, that $v_{ix}(q(\cdot), s)$ is non-decreasing for all $s \in X_i$. Then, we have

$$v_{ix}(q(x), s) \leq v_{ix}(q(y), s) \leq v_{ix}(q(z), s),\ \forall s \in X_i$$

or, equivalently, $v_{ix}(a', s) \leq v_{ix}(a, s) \leq v_{ix}(a', s),\ \forall s \in X_i$. But then $v_{ix}(a', s) = v_{ix}(a, s), \forall s \in X_i$, and by integration over s we have

$$v_i(a', y) - v_i(a', x) = v_i(a, y) - v_i(a, x) \text{ and } v_i(a', z) - v_i(a', y) = v_i(a, z) - v_i(a, y),$$

which contradicts (A.6). □

PROPOSITION A2. *Suppose that there exist two agents whose type distributions are absolutely continuous. If function v_i violates the increasing differences over distributions property for some agent $i \in \mathcal{I}$, then there exists a Bayesian incentive compatible mechanism (q, t) that does not have non-decreasing expected marginals $E_{\mathbf{x}_{-i}}[v_{ix}(q(\cdot, \mathbf{x}_{-i}), x_i)]$ for all $x_i \in X_i$.*

PROOF. $\forall G, F \in \Delta(A)$ and $\forall s \in X_i$, let

$$\Delta(G, F, s) = \int v_i(a, s) dG - \int v_i(a, s) dF.$$

Suppose $v_i(a, x)$ does not satisfy the increasing differences over distributions property. Then, there must exist $G, F \in \Delta(A)$, and $x, y, z \in X$ with $x < y < z$ such that either

$$\Delta(G, F, x) \leq \Delta(G, F, y) \text{ and } \Delta(G, F, y) \geq \Delta(G, F, z) \tag{A.8}$$

with at least one inequality being strict, or

$$\Delta(G, F, x) \geq \Delta(G, F, y) \text{ and } \Delta(G, F, y) \leq \Delta(G, F, z) \tag{A.9}$$

with at least one strict inequality. We consider only case (A.8). Case (A.9) can be treated similarly.

Assume that the utility of agent i satisfies (A.8). Let $a_G, a'_G, a_F, a'_F \in A$, and G_α (F_β) be the binary probability distribution that puts a weight α (β) on the allocation a_G (a_F) and the remaining weight $1 - \alpha$ ($1 - \beta$) on the allocation a'_G (a'_F), where $\alpha, \beta \in [0, 1]$.

Lemma A3. *There exists a pair of binary distributions G_α, F_β such that*

$$\Delta(G_\alpha, F_\beta, x) \leq \Delta(G_\alpha, F_\beta, y) \text{ and } \Delta(G_\alpha, F_\beta, y) \geq \Delta(G_\alpha, F_\beta, z) \tag{A.10}$$

with at least one inequality being strict.

Proof. Since both G_α and F_β can be deterministic, the claim of the lemma is clearly true if the increasing differences property is violated. Thus, it is without loss to assume that this property is satisfied by v_i. We want to first show that $\exists a, a', a'' \in A$ that satisfy the two following conditions simultaneously:

(i) $v_i(a, x) - v_i(a'', x) \neq v_i(a, y) - v_i(a'', y) \neq v_i(a, z) - v_i(a'', z)$.[21]

(ii) $\nexists \lambda \in \mathbb{R}$ such that

$$(v_i(a, y) - v_i(a', y)) - (v_i(a, x) - v_i(a', x)) = \lambda[(v_i(a, y) - v_i(a'', y)) - (v_i(a, x) - v_i(a'', x))],$$
$$(v_i(a, z) - v_i(a', z)) - (v_i(a, y) - v_i(a', y)) = \lambda[(v_i(a, z) - v_i(a'', z)) - (v_i(a, y) - v_i(a'', y))].$$

To see this, suppose such a tripe of allocations does not exist. Then, $\forall a, a', a'' \in A$, either

$$v_i(a, x) - v_i(a'', x) = v_i(a, y) - v_i(a'', y) = v_i(a, z) - v_i(a'', z),$$

or $\exists \lambda(a, a', a'') \in \mathbb{R}$ such that

$$(v_i(a, y) - v_i(a', y)) - (v_i(a, x) - v_i(a', x)) = \lambda(a, a', a'')[(v_i(a, y) - v_i(a'', y)) - (v_i(a, x) - v_i(a'', x))],$$
$$(v_i(a, z) - v_i(a', z)) - (v_i(a, y) - v_i(a', y)) = \lambda(a, a', a'')[(v_i(a, z) - v_i(a'', z)) - (v_i(a, y) - v_i(a'', y))].$$

Fix any $a', a'' \in A$ such that $v_i(a', x) - v_i(a'', x) \neq v_i(a', y) - v_i(a'', y) \neq v_i(a', z) - v_i(a'', z)$.[22] Let

$$A_{a''} = \{a \in A : v_i(a, x) - v_i(a'', x) = v_i(a, y) - v_i(a'', y) = v_i(a, z) - v_i(a'', z)\},$$

and $\bar{A}_{a''} = A \setminus A_{a''}$. Note that $\forall s, s' \in X_i$, we have

$$\Delta(G, F, s) - \Delta(G, F, s') = \int_{A_{a''} \cup \bar{A}_{a''}} \int [(v_i(a, s) - v_i(\tilde{a}, s)) - (v_i(a, s') - v_i(\tilde{a}, s'))] dF(\tilde{a}) dG(a).$$

[21] Note that because of the increasing differences property, $\forall a, a'' \in A$ we can only have either $v_i(a, x) - v_i(a'', x) \neq v_i(a, y) - v_i(a'', y) \neq v_i(a, z) - v_i(a'', z)$ or $v_i(a, x) - v_i(a'', x) = v_i(a, y) - v_i(a'', y) = v_i(a, z) - v_i(a'', z)$.

[22] If such allocations do not exist, we will have $\Delta(G, F, x) = \Delta(G, F, y) = \Delta(G, F, z)$, which violates (A.8).

Hence,

$$
\begin{aligned}
&\Delta(G,F,y)-\Delta(G,F,x)\\
=&\int_{\bar{A}_{a''}}\int \lambda(a,\tilde{a},a'')[(v_i(a,y)-v_i(a'',y))-(v_i(a,x)-v_i(a'',x))]dF(\tilde{a})dG(a)\\
&+\int_{A_{a''}}\int [(v_i(a'',y)-v_i(\tilde{a},y))-(v_i(a'',x)-v_i(\tilde{a},x))]dF(\tilde{a})dG(a)\\
=&\int_{\bar{A}_{a''}}\int -\lambda(a,\tilde{a},a'')\lambda(a'',a,a')[(v_i(a'',y)-v_i(a',y))-(v_i(a'',x)-v_i(a',x))]dF(\tilde{a})dG(a)\\
&+\int_{A_{a''}}\int \lambda(a'',\tilde{a},a')[(v_i(a'',y)-v_i(a',y))-(v_i(a'',x)-v_i(a',x))]dF(\tilde{a})dG(a)\\
=&[(v_i(a'',x)-v_i(a',x))-(v_i(a'',y)-v_i(a',y))]K,
\end{aligned}
$$

where

$$
K=\int_{\bar{A}_{a''}}\int \lambda(a,\tilde{a},a'')\lambda(a'',a,a')dF(\tilde{a})dG(a)-\int_{A_{a''}}\int \lambda(a'',\tilde{a},a')dF(\tilde{a})dG(a),
$$

and, similarly,

$$
\begin{aligned}
&\Delta(G,F,z)-\Delta(G,F,y)\\
=&\int_{\bar{A}_{a''}}\int \lambda(a,\tilde{a},a'')[(v_i(a,z)-v_i(a'',z))-(v_i(a,y)-v_i(a'',y))]dF(\tilde{a})dG(a)\\
&+\int_{A_{a''}}\int [(v_i(a'',z)-v_i(\tilde{a},z))-(v_i(a'',y)-v_i(\tilde{a},y))]dF(\tilde{a})dG(a)\\
=&\int_{\bar{A}_{a''}}\int -\lambda(a,\tilde{a},a'')\lambda(a'',a,a')[(v_i(a'',z)-v_i(a',z))-(v_i(a'',y)-v_i(a',y))]dF(\tilde{a})dG(a)\\
&+\int_{A_{a''}}\int \lambda(a'',\tilde{a},a')[(v_i(a'',z)-v_i(a',z))-(v_i(a'',y)-v_i(a',y))]dF(\tilde{a})dG(a)\\
=&[(v_i(a'',y)-v_i(a',y))-(v_i(a'',z)-v_i(a',z))]K.
\end{aligned}
$$

Since $v_i(a'',s)-v_i(a',s)$ is monotone in s $\forall s\in X_i$, we have

$$
sign\left[\Delta(G,F,y)-\Delta(G,F,x)\right]=sign\left[\Delta(G,F,z)-\Delta(G,F,y)\right],
$$

which violates (A.8). Hence, there must exist $a,a',a''\in A$ that satisfy both (i) and (ii). Note that for any such a triple of allocations (a, a', a''), we must also have

$$v_i(a,x) - v_i(a',x) \neq v_i(a,y) - v_i(a',y) \neq v_i(a,z) - v_i(a',z) \text{ and}$$
$$v_i(a'',x) - v_i(a',x) \neq v_i(a'',y) - v_i(a',y) \neq v_i(a'',z) - v_i(a',z),$$

since otherwise the two equations of (ii) will hold for either $\lambda = 0$ or $\lambda = 1$. Consequently, any triple of allocations that is a permutation of (a, a', a'') will also satisfy conditions (i) and (ii), which suggests that the order of selecting a, a' and a'' does not matter. Hence, without loss of generality, we can assume further that

$$v_i(a,y) - v_i(a,x) < \min\{v_i(a',y) - v_i(a',x), v_i(a'',y) - v_i(a'',x)\}. \tag{A.11}$$

Next, for all $s \in X_i$, let us denote

$$\begin{aligned}\Delta(\alpha,\beta,s) &= [\alpha v_i(a'',s) + (1-\alpha)v_i(a,s)] - [\beta v_i(a',s) + (1-\beta)v_i(a,s)] \\ &= \alpha[v_i(a'',s) - v_i(a,s)] + \beta[v_i(a,s) - v_i(a',s)]\end{aligned}$$

and

$$\begin{aligned}\hat{\Delta}(\alpha,\beta,s) &= [\alpha v_i(a',s) + (1-\alpha)v_i(a,s)] - [\beta v_i(a'',s) + (1-\beta)v_i(a,s)] \\ &= \alpha[v_i(a',s) - v_i(a,s)] + \beta[v_i(a,s) - v_i(a'',s)].\end{aligned}$$

Given (A.11) and the increasing differences property, $\Delta(\alpha,\beta,y) - \Delta(\alpha,\beta,x) \geq 0$ if and only if

$$\alpha \geq \beta \left[\frac{(v_i(a',y) - v_i(a,y)) - (v_i(a',x) - v_i(a,x))}{(v_i(a'',y) - v_i(a,y)) - (v_i(a'',x) - v_i(a,x))}\right], \tag{A.12}$$

while $\Delta(\alpha,\beta,y) - \Delta(\alpha,\beta,z) \geq 0$ if and only if

$$\alpha \leq \beta \left[\frac{(v_i(a',y) - v_i(a,y)) - (v_i(a',z) - v_i(a,z))}{(v_i(a'',y) - v_i(a,y)) - (v_i(a'',z) - v_i(a,z))}\right]. \tag{A.13}$$

Similarly, we have $\hat{\Delta}(\alpha,\beta,y) - \hat{\Delta}(\alpha,\beta,x) \geq 0$ if and only if

$$\alpha \geq \beta \left[\frac{(v_i(a'',y) - v_i(a,y)) - (v_i(a'',x) - v_i(a,x))}{(v_i(a',y) - v_i(a,y)) - (v_i(a',x) - v_i(a,x))}\right], \tag{A.14}$$

while $\hat{\Delta}(\alpha, \beta, y) - \Delta(\alpha, \beta, z) \geq 0$ if and only if

$$\alpha \leq \beta \left[\frac{(v_i(a'', y) - v_i(a, y)) - (v_i(a'', z) - v_i(a, z))}{(v_i(a', y) - v_i(a, y)) - (v_i(a', z) - v_i(a, z))} \right]. \tag{A.15}$$

Note that again because of the increasing differences property, the R.H.S. of the inequalities (A.12), (A.13), (A.14) and (A.15) are all positive. Hence, if

$$\frac{(v_i(a', y) - v_i(a, y)) - (v_i(a', x) - v_i(a, x))}{(v_i(a'', y) - v_i(a, y)) - (v_i(a'', x) - v_i(a, x))} < \frac{(v_i(a', y) - v_i(a, y)) - (v_i(a', z) - v_i(a, z))}{(v_i(a'', y) - v_i(a, y)) - (v_i(a'', z) - v_i(a, z))},$$

one can always find $\alpha, \beta \in [0, 1]$ such that both (A.12) and (A.13) are satisfied, and with at least one of them holds strictly. Otherwise, if the above strict inequality holds the other way round, then one can always find $\alpha, \beta \in [0, 1]$ such that both (A.14) and (A.15) are satisfied, and with at least one of them holds strictly. In conclusion, we can always construct a pair of binary probability distributions G_α, F_β that satisfies $\Delta(G_\alpha, F_\beta, x) \leq \Delta(G_\alpha, F_\beta, y)$ and $\Delta(G_\alpha, F_\beta, y) \geq \Delta(G_\alpha, F_\beta, z)$, with at least one inequality being strict. □

Lemma A3 shows that if v_i violates the property of increasing differences over distributions for some arbitrary pair of probability distributions (G, F), it must also violate this property for some pair of binary probability distributions (G_α, F_β). Given this important observation, we now turn to construct a Bayesian incentive compatible mechanism that violates the monotone-expected-marginal condition.

Let (G_α, F_β) be a pair of binary distributions that satisfies (A.10). By assumption, there must exist an agent $j \neq i$ whose type distribution is absolutely continuous (and hence atomless). By continuity, we can always find transfers $t_j^G, t_j^F \in \mathbb{R}$, and partitions $X_j^G \cup X_j^{G'} = X_j$ and $X_j^F \cup X_j^{F'} = X_j$ such that

(i) $\Pr\left(x_j \in X_j^G\right) = 1 - \Pr\left(x_j \in X_j^{G'}\right) = \alpha$, $\Pr\left(x_j \in X_j^F\right) = 1 - \Pr\left(x_j \in X_j^{F'}\right) = \beta$,

(ii) $v_j(a_G, x_j) \geq v_j(a'_G, x_j) + t_j^G \; \forall x_j \in X_j^G$ and $v_j(a_G, x_j) \leq v_j(a'_G, x_j) + t_j^G$ otherwise, and

(iii) $v_j(a_F, x_j) \geq v_j(a'_F, x_j) + t_j^F \; \forall x_j \in X_j^F$ and $v_j(a_F, x_j) \leq v_j(a'_F, x_j) + t_j^F$ otherwise.

Consider a mechanism with an allocation rule q and a payment rule t that are functions of the reports of agents i and j. In particular, we let

$$q(x_i, \mathbf{x}_{-i}) = \begin{cases} a_G & \text{if } x_i = y, \text{ and } x_j \in X_j^G, \\ a'_G & \text{if } x_i = y, \text{ and } x_j \in X_j^{G'}, \\ a_F & \text{if } x_i \in \{x, z\}, \text{ and } x_j \in X_j^F, \\ a'_F & \text{if } x_i \in \{x, z\}, \text{ and } x_j \in X_j^{F'}, \end{cases}$$

and $\forall s \neq x, y, z$,

$$q(s, \mathbf{x}_{-i}) = \begin{cases} a_G & \text{if } \Delta(G_\alpha, F_\beta, s) \geq \bar{t}_i \text{ and } x_j \in X_j^G, \\ a'_G & \text{if } \Delta(G_\alpha, F_\beta, s) \geq \bar{t}_i \text{ and } x_j \in X_j^{G'}, \\ a_F & \text{if } \Delta(G_\alpha, F_\beta, s) < \bar{t}_i \text{ and } x_j \in X_j^F, \\ a'_F & \text{if } \Delta(G_\alpha, F_\beta, s) < \bar{t}_i \text{ and } x_j \in X_j^{F'}, \end{cases}$$

where $\bar{t}_i = \Delta(G_\alpha, F_\beta, y)$. Agent i receives $\bar{t}_i$ if either allocation a_F or a'_F is chosen, and $t_i = 0$ otherwise. Agent j receives t_j^G (t_j^F) if allocation a'_G (a'_F) is chosen, and $t_j = 0$ otherwise. For all agents $k \neq i, j,$, $t_k(\mathbf{x}) = 0$ $\forall \mathbf{x} \in \mathbf{X}$. It is straightforward to check that (q, t) is a Bayesian incentive compatible mechanism.

We now show that agent i's expected marginals induced by allocation rule q cannot be all non-decreasing. Suppose, in contradiction, that $E_{\mathbf{x}_{-i}} v_{ix}(q(\cdot, \mathbf{x}_{-i}), s)$ is non-decreasing for all $s \in X_i$. Then, we have

$$E_{\mathbf{x}_{-i}}[v_{ix}(q(x, \mathbf{x}_{-i}), s)] \leq E_{\mathbf{x}_{-i}}[v_{ix}(q(y, \mathbf{x}_{-i}), s)] \leq E_{\mathbf{x}_{-i}}[v_{ix}(q(z, \mathbf{x}_{-i}), s)], \ \forall s \in X_i$$

or, equivalently,

$$\int v_{ix}(a, s) dF_\beta \leq \int v_{ix}(a, s) dG_\alpha \leq \int v_{ix}(a, s) dF_\beta, \ \forall s \in X_i,$$

which implies $\int v_{ix}(a, s) dG_\alpha = \int v_{ix}(a, s) dF_\beta, \ \forall s \in X_i$. Then, by the integration over s we have

$$\Delta(G_\alpha, F_\beta, x) = \Delta(G_\alpha, F_\beta, y) \text{ and } \Delta(G_\alpha, F_\beta, y) = \Delta(G_\alpha, F_\beta, z),$$

which contradicts to (A.10). □

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