$\mathcal{V}$-Polyhedral Cuts

Egon Balas and **Aleksandr M. Kazachkov**

Carnegie Mellon University

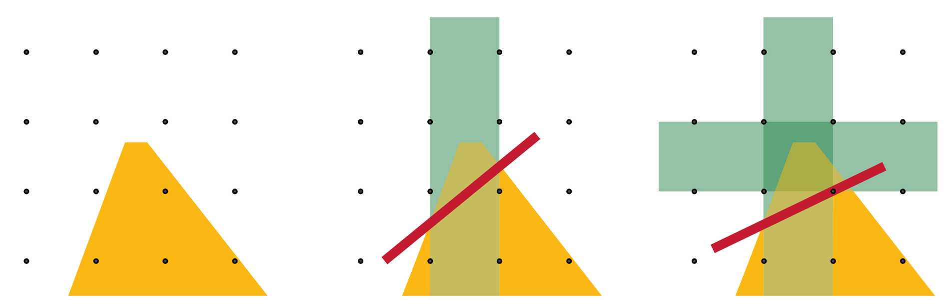
THEORY

Input: Rational mixed-integer linear program, where $I \subseteq [n]$ indexes the integer-constrained variables, and an optimal solution $\bar{x}$ to the LP relaxation.

$$\textbf{(IP)}\left[\textbf{(LP)}\left[\begin{array}{l}\min_{x\in\mathbb{R}^n} \; cx \\ Ax \geq b \end{array}\right] P \atop x_j \in \mathbb{Z} \text{ for all } j \in I\right] P_I$$

Let $\mathcal{D} := \vee_{t\in T}(D^t x \geq d^t)$ be a **disjunction** such that $P_I \subseteq \cup_{t\in T} P^t$, where $P^t := \{x \in P : D^t x \geq d^t\}$.

Disjunctions are used to obtain tighter relaxations of P_I by generating valid **cutting planes**: inequalities that cut $\bar{x}$ but no points in P_I.



Stronger disjunctions lead to stronger cuts. However, only the simplest disjunctions (splits) are typically used in practice.

Challenges:
(1) There are many possible disjunctions that can be used.
(2) Each stronger disjunction can lead to an enormous number of cuts.
(3) It is difficult to identify which cuts are most effective.

How strong are cuts from general disjunctions?
Can they be generated efficiently and non-recursively?

Idea: Find a collection $(\mathcal{P}, \mathcal{R})$ of points and rays that yield a $\mathcal{V}$-polyhedral relaxation of $\operatorname{conv}(\cup_{t\in T} P^t)$, then formulate and solve the following LP.

$$\min_{(\alpha,\beta)\in\mathbb{R}^n\times\mathbb{R}} \alpha v \qquad \textbf{(PRLP)}$$
$$\alpha p \geq \beta \quad \text{for all } p \in \mathcal{P}$$
$$\alpha r \geq 0 \quad \text{for all } r \in \mathcal{R}$$

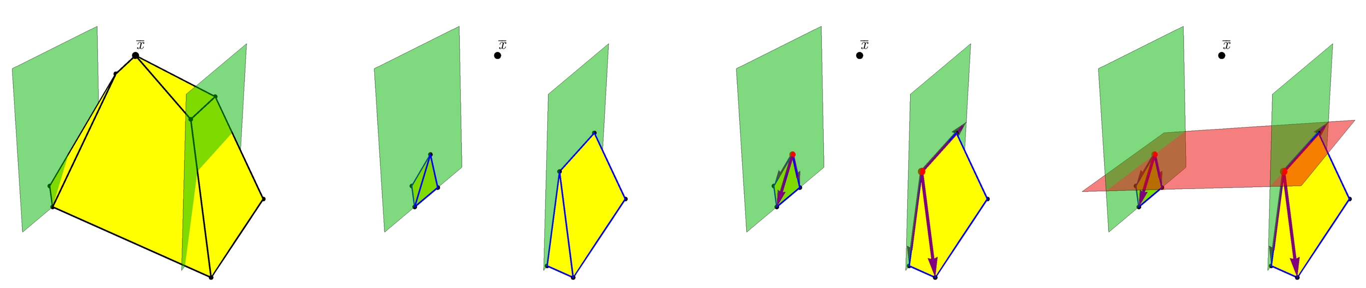
We use a variety of objective directions v for each PRLP.

We call the resulting cut, $\alpha x \geq \beta$, a **$\mathcal{V}$-polyhedral cut** (VPC).

THEOREM
VPCs from $(\mathcal{P}, \mathcal{R})$ are facet-defining inequalities for $\operatorname{conv}(\mathcal{P} \cup \mathcal{R})$. If every point in $\mathcal{P}$ belongs to some P^t, $t \in T$, and every ray in $\mathcal{R}$ has a point from some P^t, $t \in T$, in its relative interior, the VPCs from $(\mathcal{P}, \mathcal{R})$ define facets of $\operatorname{conv}(\cup_{t\in T} P^t)$.

Problem: $\mathcal{V}$-polyhedral descriptions can grow exponentially large in the size of the original problem.

Solution: Use a sparse relaxation formed from the optimal basis at each disjunctive term. For each $t \in T$, we use the simple polyhedral cone defined by a point $p^t \in \operatorname{argmin}\{cx : x \in P^t\}$ and the inequalities of the cobasis associated with p^t. In this case, $|\mathcal{P}| = |T|$ and $|\mathcal{R}| = n|T|$.



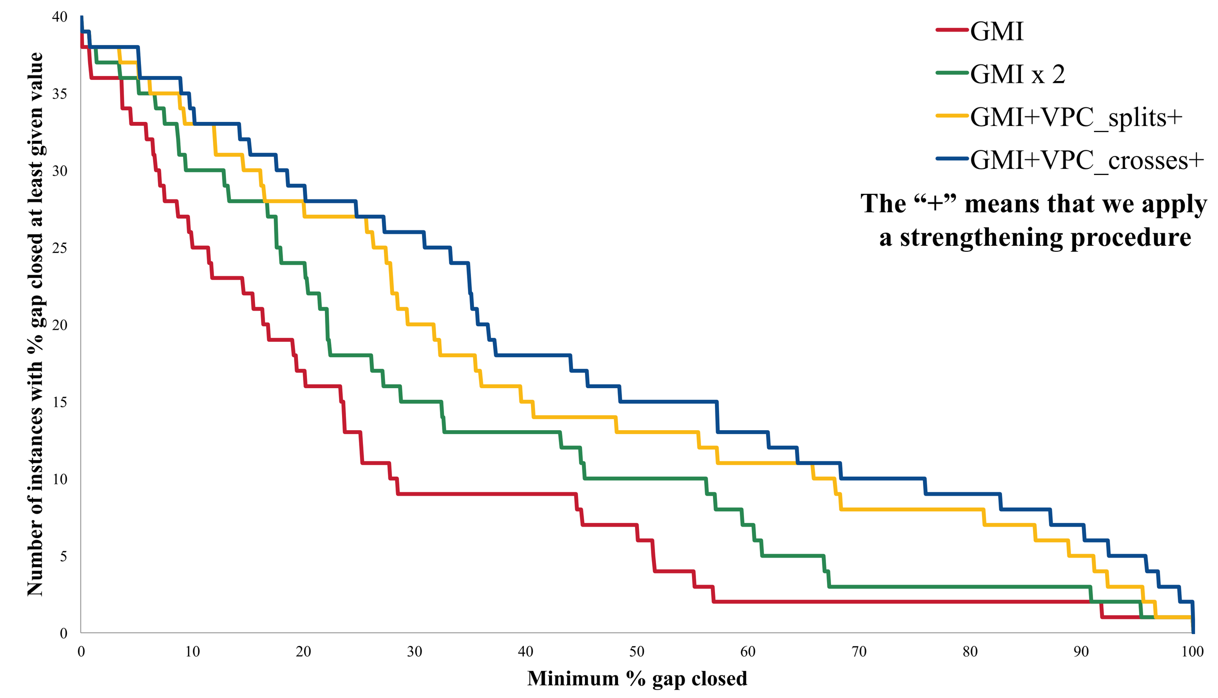
COMPUTATION

Experimental setup: Measured % of integrality gap closed by one round of VPCs added to the LP relaxation on 40 instances from MIPLIB (at most 500 rows and 500 columns). Time limit is set to one hour and a cut limit of 10,000 is used.

Which disjunctions to use?
Splits: $(x_j \leq \lfloor \bar{x}_j \rfloor) \vee (x_j \geq \lceil \bar{x}_j \rceil)$ for each $j \in I$ such that $\bar{x}_j \notin \mathbb{Z}$.
Crosses: For each pair of split disjunctions, use the intersection.

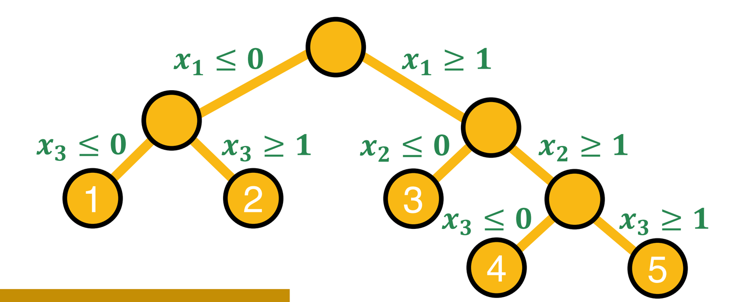
	GMI	GMI x 2	Splits	Crosses
% gap closed	23.6	31.8	40.7	45.6
# cuts / # GMI	1.00	1.91	24.97	253.49
Time (s) / cut			0.14	0.61

% Gap Closed Profile for Splits and Crosses



The results indicate that strong cuts *exist*. Finding them efficiently is a challenge, as it involves selecting the right disjunctions from $O(n^2)$ many options and then developing a way to only generate the strong cuts.

Potential solution: Instead of exploring the neighborhood of $\bar{x}$ via many cut-generating sets, use the effort to generate **one** strong disjunction.



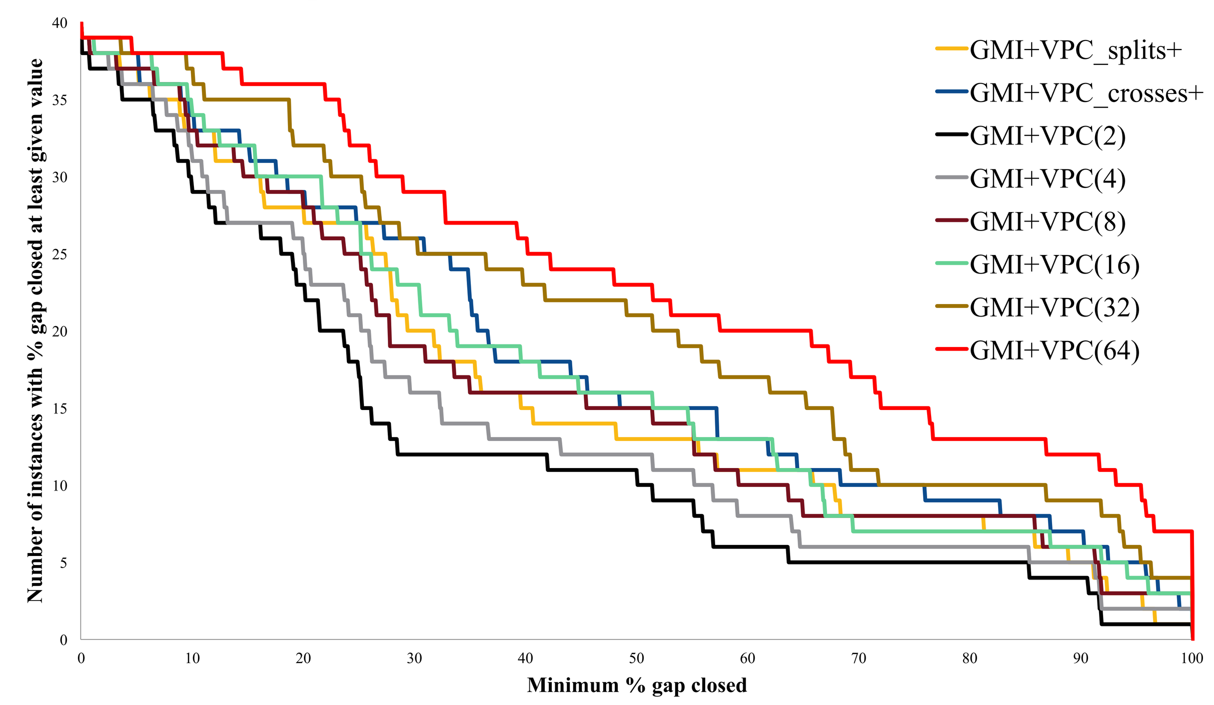
The leaves of any branch-and-bound tree yield a disjunction.

	2 leaves	4 leaves	8 leaves	16 leaves	32 leaves	64 leaves
% gap closed	31.4	35.4	40.9	43.4	52.2	59.0
# cuts / # GMI	5.59	15.15	34.14	77.09	130.40	144.06
Time (s) / cut	0.03	0.05	0.11	0.21	0.50	1.21

Includes time to generate B&B tree

Tree is obtained by full strong branching until reaching desired number of leaves

% Gap Closed Profile for VPCs from Partial B&B



EXTENSIONS

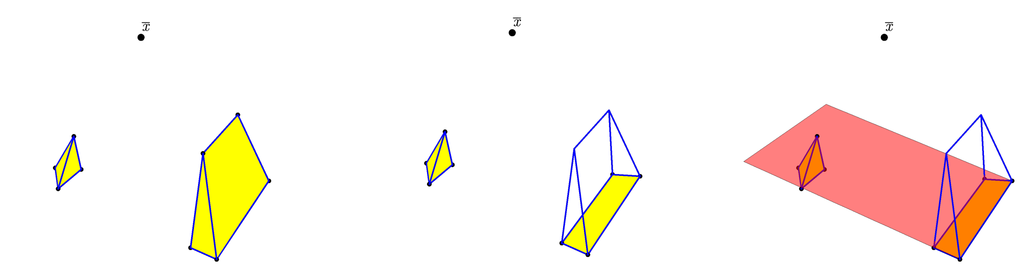
What happens when we only generate few cuts?

In these results, we set the cut limit to the number of splits (GMI cuts).

VPC_1	2 leaves	4 leaves	8 leaves	16 leaves	32 leaves	64 leaves
% gap closed	31.2	34.8	39.5	42.0	50.2	56.3
# cuts / # GMI	0.73	0.80	0.84	0.92	0.94	0.97
Time (s) / cut	0.03	0.04	0.10	0.17	0.37	0.92

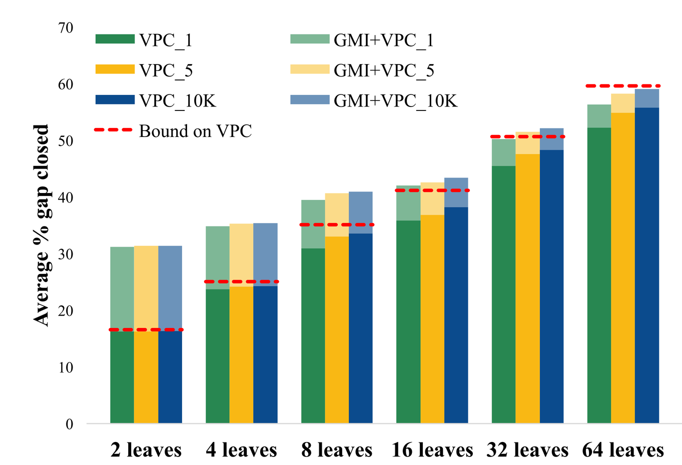
How can we strengthen the cuts?

The disjunction remains valid if we tighten the relaxation within each term. The simplest way to do this is by adding Gomory cuts to each term.

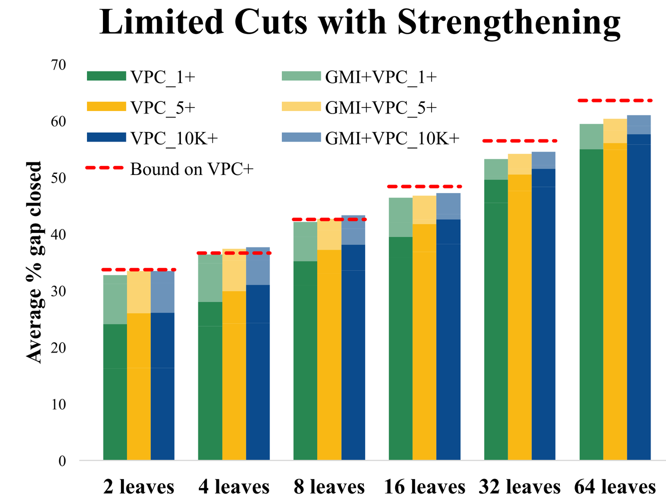


Finally, we test the strengthening (VPC+) along with the effect of limiting cuts as compared to the theoretical best % gap closed from each disjunction.

Limited Cuts



Limited Cuts with Strengthening



VPC_1: cut limit = # GMI cuts   VPC_5: cut limit = 5 × # GMI cuts   VPC_10K: cut limit = 10,000

CONCLUSION

Strong $\mathcal{V}$-polyhedral cuts can be generated from general disjunctions using an LP with the same dimension as the original problem and a number of constraints depending linearly on the number of disjunctive terms.

One strong disjunction can be used to avoid the paradox of choice. This disjunction is readily available from any partial branch-and-bound tree.

Future work:
1. Improve efficiency and test on larger instances.
2. Attempt to adapt traditional strengthening approaches to VPCs.
3. Incorporate and evaluate VPCs within a branch-and-bound framework.
4. Explore the effect of using tighter relaxations of the disjunctive terms.

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