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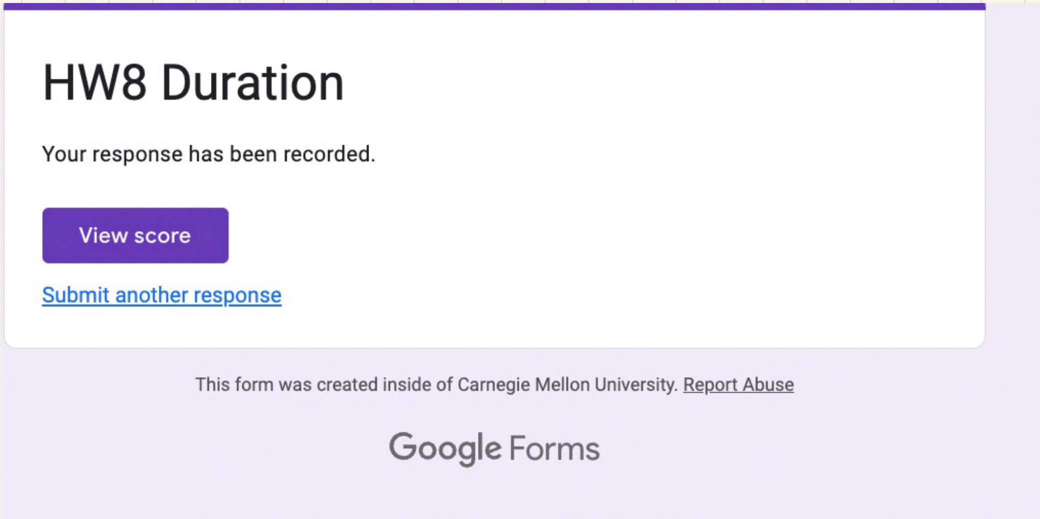
1. (*2 points*) Please fill out the survey here on how long you spent on HW8. You will receive two points for your response; to receive them, please upload a screenshot of the confirmation page.



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2. (*14 points*) Consider the complex logarithm $\ln(z)$ for a complex number $z \in \mathbb{C}$. Recall that in lecture, we defined $\mathrm{Ln}(z) = \ln(|z|) + j\mathrm{Arg}(z)$, where $\mathrm{Arg}(z) \in [-\pi, \pi)$.

a.(*3 points*) Find $\mathrm{Ln}(e^j)$, $\mathrm{Ln}(-e)$, and $\mathrm{Ln}(-e^j)$

b.(*3 points*) Does $\mathrm{Ln}(e^j) + \mathrm{Ln}(-e) = \mathrm{Ln}\left((-e)(e^j)\right)$? Please show your work.

c.(*3 points*) Does $\mathrm{Ln}(e^j) + \mathrm{Ln}(-e^j) = \mathrm{Ln}\left(e^j \cdot (-e^j)\right)$? Please show your work.

d.(*5 points*) Now suppose we define the complex logarithm as $\ln(z) = \ln|z| + j(-2\pi + \mathrm{Arg}(z))$. Does $e^{\ln(z)} = z$ still hold for all $z \in \mathbb{C}$? Briefly explain your answer.

EXPLAIN

2a.) $\underline{\mathrm{Ln}(e^j)}$ $\quad r = 1$

$\theta = 1$

$= \ln(1) + j(1) = \boxed{j}$

$\underline{\mathrm{Ln}(-e)}$ $\quad r = -1$

$\theta = \pi$

$= \ln(|-1|) + j(\pi) = \ln(1) + j\pi = \boxed{j\pi}$

$\underline{\mathrm{Ln}(-e^j)}$ $\quad r = -1 \quad \theta = 1$

$= \ln(|-1|) + j(1) = \boxed{j}$

b.(*3 points*) Does $\text{Ln}(e^j) + \text{Ln}(-e) = \text{Ln}\left((-e)(e^j)\right)$? Please show your work.

$\text{Ln}(e^j) = j$

$\text{Ln}(-e) = \pi$

$\text{Ln}((-e)(e^j)) \Rightarrow$ $r = -e$

$\theta = 1$

$\therefore$ $\ln(-e) + j = \underline{1 + j}$

$j + \pi \neq 1 + j$

$\boxed{\text{No}}$

c.(*3 points*) Does $\mathrm{Ln}(e^j) + \mathrm{Ln}(-e^j) = \mathrm{Ln}\left(e^j \cdot (-e^j)\right)$? Please show your work.

$$\mathrm{Ln}(e^j \cdot (-e^j)) = \mathrm{Ln}(-e^{2j})$$

$$\theta = 2$$

$$r = -1$$

$$\Rightarrow \mathrm{Ln}(-e^{2j}) = \ln(|-1|) + 2j$$

$$\Rightarrow \ln(1) + 2j = \underline{2j}$$

$$\mathrm{Ln}(e^j) = j$$

$$\mathrm{Ln}(-e^j) = j$$

$$\underline{\mathrm{Ln}(e^j) + \mathrm{Ln}(-e^j) = 2j}$$

$\boxed{\text{YES}}$ THEY ARE EQUAL

d.(*5 points*) Now suppose we define the complex logarithm as $\ln(z) = \ln|z| + j(-2\pi + \text{Arg}(z))$. Does $e^{\ln(z)} = z$ still hold for all $z \in \mathbb{C}$? Briefly explain your answer.

$$e^{\ln(z)} = e^{(\ln(z) + j(-2\pi + \text{Arg}(z)))}$$

$$= e^{\ln(z)}\left(e^{j(-2\pi + \text{Arg}(z))}\right)$$

$$= z\left(e^{j(-2\pi + \text{Arg}(z))}\right)$$

if the angle of z is not equal to 2π then $e^{\ln(z)} \neq z$ because $e^{j(-2\pi + \text{Arg}(z))}$ would have a nonzero value which would have $e^{\ln(z)} \neq z$

3. (*20 points*) In the following sections, please show all work.

a.(*10 points*) Find expressions for $\sinh(z)$ and $\cosh(z)$ in terms of e^z and e^{-z}, and prove them using the Taylor series expansions.
[Hint]: Use the power series definition of the complex exponential e^z. You may also use the fact that $\sinh'(x) = \cosh(x)$, $\cosh'(x) = \sinh(x)$.

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

$$\sinh(x) = x + \frac{1}{3!} \cdot x^3 + \frac{1}{5!} \cdot x^5 + \cdots = \sum \frac{x^{1+2n}}{(1+2n)!}$$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} \cdots$$

$$-e^{-x} = -1 + x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} \cdots$$

$$\frac{e^x}{2} = \frac{1}{2} + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{12} \cdots$$

$$-\frac{e^{-x}}{2} = -\frac{1}{2} + \frac{x}{2} - \frac{x^2}{4} + \frac{x^3}{12} \cdots$$

$$\frac{e^x}{2} - \frac{e^{-x}}{2} = \frac{e^x - e^{-x}}{2} = \left(\frac{1}{2} - \frac{1}{2}\right) + \left(\frac{x}{2} + \frac{x}{2}\right) + \left(\frac{x^2}{4} - \frac{x^2}{4}\right) + \left(\frac{x^3}{12} + \frac{x^3}{12}\right)$$
$$+ \left(\frac{x^4}{24} - \frac{x^4}{24}\right) + \left(\frac{x^5}{240} + \frac{x^5}{240}\right)$$

3a.) (contined)

$$\frac{e^x - e^{-x}}{2} = x + \frac{x^3}{6} + \frac{x^5}{120}$$

$$= \sinh(x)$$

$$\cosh(x) = \frac{e^x + e^{-x}}{2} \qquad \cosh(x) = 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots$$

$$\frac{e^x}{2} = \frac{1}{2} + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{12} \cdots$$

$$\frac{e^{-x}}{2} = +\frac{1}{2} - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{12} \cdots$$

$$\frac{e^x}{2} + \frac{e^{-x}}{2} = \frac{e^x + e^{-x}}{2} = \left(\frac{1}{2} + \frac{1}{2}\right) + \left(\frac{x}{2} - \frac{x}{2}\right) + \left(\frac{x^2}{4} + \frac{x^2}{4}\right) + \left(\frac{x^3}{12} - \frac{x^3}{12}\right) + \left(\frac{x^4}{24} + \frac{x^4}{24}\right)$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{1}{2}x^2 + \frac{1}{12}x^4$$

$$= \cosh(x)$$

b.(*10 points*) Show the following two identities, using the fact that $\sin(a) = \frac{(e^{ja}-e^{-ja})}{2j}$, $\cos(a) = \frac{(e^{ja}+e^{-ja})}{2}$ for any $a \in \mathbb{C}$, as well as the complex identities you found in part (a).

i) $\sin(p + jq) = \sin(p)\cosh(q) + j\cos(p)\sinh(q)$

ii) $\cos(p + jq) = \cos(p)\cosh(q) - j\sin(p)\sinh(q)$

i.) $\sin(p+jq) = (\sin(p)\cosh(q)) + j(\cos q \sinh(q))$

LHS: $\sin(p+jq) = \dfrac{e^{j(p+jq)} - e^{-j(p+jq)}}{2j}$

RHS: $\sin(p)\cosh(q) + (\cos(p)\sinh(q))j$

$$\left(\frac{e^{jp}-e^{-jp}}{2j}\left(\frac{e^{q}-e^{-q}}{2}\right)\right) + \frac{e^{jp}+e^{-jp}}{2}\left(\frac{e^{q}-e^{-q}}{2}\right)j$$

$$= \frac{\cancel{e^{pj+q}} - e^{q-pj} - e^{pj-q} \cancel{+e^{-q-pj}}}{4j} + \frac{e^{jp+q} + e^{q-jp} - e^{jp-q} - e^{-q-jp}}{4}j$$

$$= \frac{e^{j(p+jq)2}}{4j} + \frac{2e^{p+jqj}}{4} \cdot \frac{j}{j}$$

$$= \frac{e^{j(p+jz)}}{2j} - \frac{e^{-j(p+jz)}}{2j}$$

$$= \frac{e^{j(p+jz)} - e^{-j(p+jz)}}{2j}$$

LHS = RHS ✓

ii.) ii) $\cos(p + jq) = \cos(p)\cosh(q) - j\sin(p)\sinh(q)$

LHS

$$\cos(p+jq) = \frac{e^{j(p+jq)} + e^{-j(p+jq)}}{2}$$

RHS

$$\cos(p)\cosh(q) - j\sin(p)\sinh(q) =$$

$$\frac{e^{jp} + e^{-j(p)}}{2}\left(\frac{e^{q} + e^{-q}}{2}\right) - j\left(\frac{e^{pj} - e^{-pj}}{2j}\right)\left(\frac{e^{q} + e^{-q}}{2}\right)$$

$$\frac{e^{q+pj} + e^{q-pj} + e^{pj-q} + e^{-q-pj}}{4} - j\left(\frac{\cancel{e^{pj+q}} \cancel{- e^{q-pj}} \cancel{- e^{-qpj}} \cancel{+ e^{-q-pj}}}{4}\right.$$

$$= \frac{2e^{p+qj} + 2e^{-j(p+jq)}}{4} = \frac{e^{j(p+jq)} + e^{-j(p+qj)}}{2}$$

RHS = LHS ✓

4. (*18 points*) In the following sections, please show all work.

a.(*12 points*) Let $z_1 = 3 + 2j$. Solve for the real and imaginary parts of $\sin(z_1)$, $\cos(z_1)$, and e^{z_1}. You may find it helpful to use the identities that you derived in **3b**.
Note: *Some work or evaluation is expected - do not simply plug in the values of z and report the answer.*

b.(*3 points*) Find the polar form of $z_2 = 1 - j$.

c.(*3 points*) Let $z_3 = 2e^{j\frac{\pi}{4}}$. Find $\arg(\frac{z_2}{z_3})$ (z_2 is given in **4b**).

a) $\sin(3+2j) = \sin(3)\cosh(2) + j(\cos(3)\sinh(2))$

$= (0.1411)\left(\frac{e^{2}+e^{-2}}{2}\right) + j(-0.989)\left(\frac{e^{2}-e^{-2}}{2}\right)$

$= 0.531 + (-3.591)j$

$$\boxed{\begin{array}{ll} \text{For } \sin(z_1) & \text{Real} = 0.531 \\ & \text{Imaginary} = -3.591j \end{array}}$$

$$\cos(z_1) = \cos(3)\cosh(2) - j\sin(3)\sinh(2)$$

$$= (-0.99)\left(\frac{e^{2}+e^{-2}}{2}\right) - j(0.141)\left(\frac{e^{2}-e^{-2}}{2}\right)$$

$$= -3.72 - j(0.512)$$

$$\boxed{\begin{array}{ll} \text{For } \cos(z_1) & \text{Real} = -3.72 \\ & \text{Imaginary} = -0.512 \end{array}}$$

b.) $z_2 = 1 - 1j$

$\theta = \tan^{-1}\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$

$r = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}$

$\boxed{\text{Polar Form: } \sqrt{2}\, e^{-\pi/4 j}}$

c.) $z_3 = 2e^{j\pi/4}$

$\arg\left(\frac{z_2}{z_3}\right) = \theta_2 - \theta_3 = -\frac{\pi}{4} - \frac{\pi}{4} = \boxed{-\frac{\pi}{2}}$

5. (*18 points*) For parts a. and b., use $f(z) = \frac{\sin(z)}{z}$ centered at $z = \frac{\pi}{2}$.

Note: The first order truncation of a Taylor series centered at a point a is given by

$$P_1(z) = f(a) + f'(a)(z-a).$$

Similarly, the n^{th} order truncation of a Taylor series, $P_n(z)$, centered at a point a is given by:

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3 \cdot 2}(x-a)^3 + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n(n-1)(n-2)\cdots(3)(2)}.$$

a.(*10 points*) Write the first, second and third order truncations of the Taylor series for the function $f(z)$, $z \in \mathbb{C}$.

b.(*8 points*) Plot the function and its three truncated approximations. You should make four plots corresponding to the function itself, its first-order truncation, its second-order truncation, and its third-order truncation. Each plot should be three dimensional, with coordinates corresponding to $(\Re(z), \Im(z), |f(z)|)$. You can use `surf` in Matlab to make your plots. Plot from -2 to 2 with a 0.15 step size for each axis.

a.) First Order Truncation:

$$P_1(z) = f(a) + f'(a)(z-a)$$

$$= \frac{\sin(a)}{a} + \frac{a\cos(a) - \sin(a)}{a^2}(z-a)$$

$$= \boxed{\frac{2}{\pi} - \frac{4}{\pi^2}\left(z - \frac{\pi}{2}\right)}$$

Second order truncation:

$$P_2(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2}(x-a)^2$$

$$= \frac{2}{\pi} - \frac{4}{\pi^2}\left(z - \frac{\pi}{2}\right) + \frac{a^2\sin(a) + 2(a\cos a - \sin a)}{2 \cdot a^3}(x-a)^2$$

$$= \boxed{\frac{2}{\pi} - \frac{4}{\pi^2}\left(z - \frac{\pi}{2}\right) + \frac{(8-\pi^2)(x-\frac{\pi}{2})^2}{\pi^3}}$$

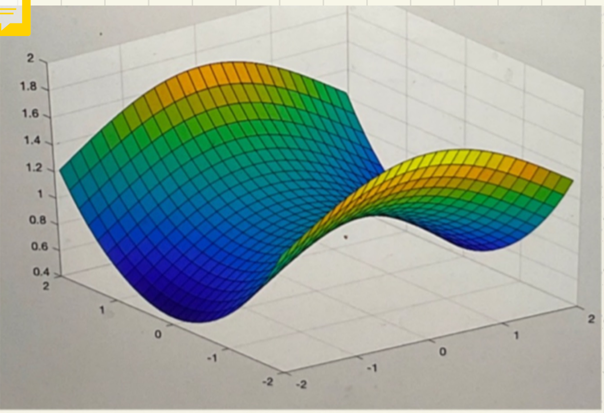
5c.) (cont.)

Third Order Truncation: $f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3 \cdot 2}(x-a)^3$

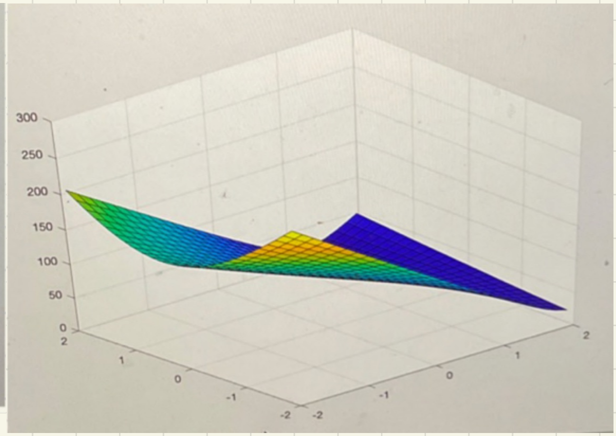
$$\boxed{\frac{2}{\pi} - \frac{4\left(x-\frac{\pi}{2}\right)}{\pi^2} + \frac{(8-\pi^2)\left[x-\frac{\pi}{2}\right]^2}{\pi^3} + \frac{2(\pi^2-8)\left(x-\frac{\pi}{2}\right)^3}{\pi^4}}$$

b.(*8 points*) Plot the function and its three truncated approximations. You should make four plots corresponding to the function itself, its first-order truncation, its second-order truncation, and its third-order truncation. Each plot should be three dimensional, with coordinates corresponding to $(\Re(z), \Im(z), |f(z)|)$. You can use `surf` in Matlab to make your plots. Plot from -2 to 2 with a 0.15 step size for each axis.

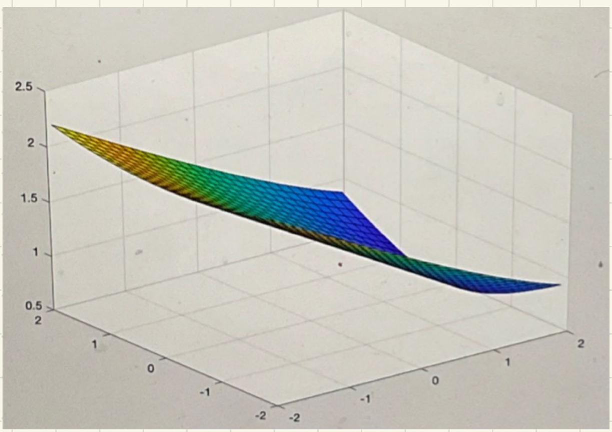
Itself



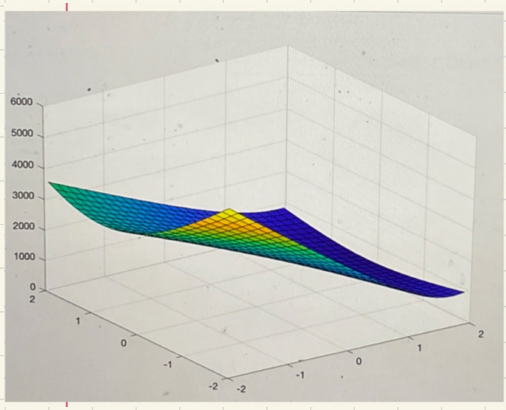
Second Order



First Order



Third Order



6. (*18 points*) Consider $z \in \mathbb{C}$ and then:

a.(*6 points*) Find the derivative of $f(z) = \frac{2z-1}{z^2-2z+10}$ for $z \in \mathbb{C}$. For which values of z is this function differentiable?

b.(*12 points*) By explicitly evaluating the Cauchy-Riemann equations, determine whether the following complex functions are analytic or not.

i) $f(z) = 2z^3 + 3z + 5$

ii) $f(z) = \frac{\bar{z}}{z}$

Do we Just use Quadratic formula

a.) $f'(z) = \dfrac{2(x^2-2x+10)-(2x-2)(2x-1)}{(x^2-2x+10)^2}$

$= \dfrac{2x^2-4x+20-4x^2+2x+4x-2}{(x^2-2x+10)^2}$

$= \boxed{\dfrac{-2x^2+2x+18}{(x^2-2x+10)^2}}$

f is differentiable when $x^2-2x+10 \neq 0$

$x^2-2x+10=0$ $\quad \dfrac{2 \pm \sqrt{4-4(10)}}{2} = \dfrac{2 \pm \sqrt{-36}}{2}$

$= 1 \pm 3j$

It is differentiable when $\boxed{z \neq 1 \pm 3j}$

b.) i.) $f(z) = 2z^3 + 3z + 5$

$z = x + jy$

$= 2(x+jy)^3 + 3(x+jy) + 5$

$= 2((x+jy)(x+jy)(x+jy)) + 3(x+jy) + 5$

$= 2((x^2 + 2xy(j) - y^2)(x+jy)) + 3(x+jy) + 5$

$= 2(x^3 + 2x^2y(j) - xy^2 + x^2y(j) - 2xy^2 - y^3(j)) + 3(x+jy) + 5$

$= 2((x^3 - 3xy^2) + j(3x^2y - y^3)) + 3(x+jy) + 5$

$= 2x^3 - 6xy^2 + 6x^2y(j) - 2y^3 j + 3x + 3yj + 5$

$= \underbrace{(2x^3 - 6xy^2 + 3x + 5)}_{u} + j\underbrace{(6x^2y - 2y^3 + 3y)}_{v}$

$\frac{\partial u}{\partial x} = 6x^2 - 6y^2 + 3$ $\quad$ $\frac{\partial v}{\partial x} = 12xy$

SAME

-1

$\frac{\partial u}{\partial y} = -12xy$ $\quad$ $\frac{\partial v}{\partial y} = 6x^2 - 6y + 3$

$\Rightarrow$ YES, IT IS ANALYTIC SINCE IT PASSES CAUCHY-RIEMANN

iii) $f(z) = \dfrac{\bar{z}}{z} = \dfrac{x - yj}{x + yj} \cdot \dfrac{x - yj}{x - yj}$ $\quad z = x + yj$

$$= \frac{x^2 - 2xyj - y^2(j)}{x^2 + y^2} = \underbrace{\frac{x^2 - y^2(8)}{x^2 + y^2}}_{u} - \underbrace{\frac{2xy}{x^2 + y^2}}_{v}(j)$$

$$\frac{\partial u}{\partial x} = \frac{12xy^2}{(x^2 + y^2)^2} \qquad \frac{\partial v}{\partial x} = \frac{-2y(-x^2 + y^2)}{(x^2 + y^2)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-12x^2y}{(x^2 + y^2)^2} \qquad \frac{\partial v}{\partial y} = \frac{-2x(x^2 - y^2)}{(x^2 + y^2)^2}$$

Since $\dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$

& $\dfrac{\partial u}{\partial y} \neq -\dfrac{\partial v}{\partial x}$

then this function is NOT analytic

7. (*10 points*) **Power Systems Application Problem**

Voltage (V) and current (I) are two well known sinusoidal elements that are often represented as *phasors*. Phasors are sinusoidal signals with time-invariant amplitude, phase, and angular velocity ($2\pi f$). Phasors have their own notation, *phasor notation*, which represents the magnitude and phase of the signal. For a given signal with magnitude A, frequency f, and phase θ, phasor notation would look like the following:

$$x(t) = Acos(2\pi ft + \theta°) \rightarrow X = A\underline{/\theta°}$$

In electric power systems, voltage and current are also typically represented using their *root mean square* (RMS) values, where the relationship between a sinusoidal and RMS is the following:

$$y = X\cos(2\pi ft + \theta°) \rightarrow Y_{rms} = \frac{X}{\sqrt{2}}\underline{/\theta°}$$

In power systems, we use V_{rms} (the phasor representation of v(t) with RMS values) and I_{rms} (the phasor representation of i(t) with RMS values) to find *complex power* (S) where $S = V_{rms}I^*_{rms}$. As a result, S is then also a complex number where $\Re(S)$ is called real or active power (P) (which is the power you're most likely familiar with) and $\Im(S)$ is called reactive power (Q).

a.(*4 points*) Find the phasor representation of V_{rms} and I_{rms} when $v(t) = 170\cos(2\pi ft - 30°)$ and $i(t) = 7\cos(2\pi ft - 60°)$ for $f = 60$ Hz. Please round to the nearest whole number.

b.(*6 points*) Find $|S|$, P, and Q.

a.) $V_{rms} = \frac{170}{\sqrt{2}} \angle -30°$

$\boxed{V_{rms} = 120 \angle -30°}$

$I_{rms} = \frac{7}{\sqrt{2}} \angle -60$

$\boxed{I_{rms} = 5.0 \angle -60}$

b.(*6 points*) Find $|S|$, P, and Q.

$V_{rms} \to$ rectangular form $= 120\cos(30) + 120\sin(30)j$

$V_{rms} = 18.54 + 118.8j$

$I_{rms} \to$ rectangular form $= 5\cos(-60) + 5\sin(-60)j$

$I_{rms} = -4.71 + 1.51j$

$|S| = V_{rms} I_{rms}^* = (18.54 + 118.8j)(-4.71 - 1.51j)$

$S = 91.8 - 587.9j$

$$\boxed{\begin{aligned} |S| &= 595 \\ P &= 91.8 \\ Q &= -587.9 \end{aligned}}$$

$|S| = \sqrt{(91.8)^2 + (587.9)^2}$

$|S| = 595$