90-906
Spring 2000

FINAL SOLUTIONS

1. a) We compare the regressions on pg 2 and pg 5:

$$F\text{-stat} = \frac{(7.67\times10^{11} - 7.45\times10^{11})/3}{7.45\times10^{11}/852}$$

$$= 8.39$$

Pick 1% signif and look on the $F_{3,\infty}$ table to get 3.80.

We reject linearity & are 99% confident that the true model is non-linear.

1 b) Salary paths diverge if the coefficient on age*yrschool is non-zero. So, we test the null hypothesis that $H_0: \beta_{AGEYRS} = 0$.

$$t\text{-stat} = \frac{75 - 0}{48} = 1.57$$

At the 5% signif level, the table value is 1.96. We do not reject. There is not enough information in these data to be 95% confident that the salary paths diverge.

1 c) Well, $\frac{\partial \text{Salary}}{\partial \text{age}} = \beta_{age} + 2\beta_{age2} \cdot age + \beta_{ageyrs} \cdot yrschool$

(Note: In light of the results of the previous ~~section~~, you could omit the β_{ageyrs} term & use the results on page 4).

So, $\widehat{\frac{\partial \text{Salary}}{\partial \text{age}}} = 1067 + 2(-23.4)(37.5) + 75.2(14)$

$= 365$

We need the variance of $\widehat{\frac{\partial \text{Salary}}{\partial \text{age}}}$, which we get from:

$$V\left(\widehat{\frac{\partial \text{Sal}}{\partial \text{age}}}\right) = V\left((0\ 0\ 0\ 1\ 75\ 14)\left(\hat{\beta}_{OLS}\right)\right)$$

$$= (0\ 0\ 0\ 1\ 75\ 14)\, V(\hat{\beta}_{OLS}) \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 75 \\ 14 \end{pmatrix}$$

$$= (1\ 75\ 14) \begin{bmatrix} 79{,}1679 & -4{,}173 & -32{,}885 \\ -4{,}173 & 48.8 & 17.9 \\ -32{,}885 & 17.9 & 2308 \end{bmatrix} \begin{pmatrix} 1 \\ 75 \\ 14 \end{pmatrix}$$

$= 9407$

95% CI: $\frac{\partial \text{Salary}}{\partial \text{age}} = 365 \pm 1.96\sqrt{9407} = 365 \pm 190$

1 d) Salary is maximized where $\frac{\partial Salary}{\partial age} = 0$

$$\frac{\partial Salary}{\partial age} = \beta_{age} + 2\beta_{age2}\, age + \beta_{ageyrs}\, yrschool =$$

$$age = \frac{1}{-2\beta_{age2}}\left(\beta_{age} + \beta_{ageyrs}\, yrschool\right)$$

Note: Again, β_{ageyrs} could be omitted.

$$\widehat{age} = \frac{1}{-2(-23.4)}\left(1067 + 75.2(14)\right)$$

$$\widehat{age} = 45.3$$

We must use the delta-method to calculate $V(\widehat{age})$.

$$\frac{\partial age}{\partial \beta_{age}} = \frac{1}{-2\beta_{age2}} \qquad \frac{\partial age}{\partial \beta_{yrsage}} = \frac{1}{-2\beta_{age2}} \cdot yrschool$$

$$\frac{\partial age}{\partial \beta_{age2}} = \left(\left(\beta_{age} + \beta_{ageyrs}\, yrschool\right)\right)\frac{1}{2(\beta_{age2})^2}$$

$$\widehat{\frac{\partial age}{\partial \beta_{age}}} = \frac{1}{-2(-23.4)} = \frac{1}{46.8} = 0.021$$

$$\widehat{\frac{\partial age}{\partial \beta_{yrsage}}} = 0.021(14) = 0.30$$

$$\frac{\partial age}{\partial \beta_{age2}} = (1067 + 75.2(14))\,\frac{1}{2(-23.4)^2} = 1.94$$

[illegible], Spruce, [illegible] Volumes

$$V(\widehat{age}) = (0.021 \quad 1.94 \quad 0.30) \begin{bmatrix} 791679 & -4173 & -32885 \\ -4173 & 48.8 & 17.9 \\ -32885 & 17.9 & 2308 \end{bmatrix} \begin{pmatrix} 0.021 \\ 1.94 \\ 0.30 \end{pmatrix}$$

$$= 6.98$$

95% CI: $\widehat{age} = 45.3 \pm 1.96\sqrt{6.98} = 45.3 \pm 5.2$

2. a) $\hat{\beta}_{OLS} = (X'X)^{-1}X'Y$

$$= \frac{1}{\Sigma X^2} \Sigma XY$$

$$V(\hat{\beta}_{OLS}) = \frac{e'e}{N}(X'X)^{-1}$$

$$= \frac{e'e}{N}\frac{1}{\Sigma X^2}$$

X	Y	X^2	XY
-3	-3	9	9
-2	-1	4	2
-1	-2	1	2
0	-1	0	0
1	1	1	1
2	0	4	0
3	5	9	15
		28	29

$\hat{\beta}_{OLS} = 29/28 \cong \underline{1}$

X	Y	e	e^2
-3	-3	0	0
-2	-1	-1	1
-1	-2	1	1
0	-1	1	1
1	1	0	0
2	0	2	4
3	5	-2	4

$\frac{e'e}{N} = \frac{11}{7} = 1.57$

$\hat{\beta}_{OW} = 1 \qquad V(\hat{\beta}_{OW}) = 1.57/28 = 0.056$

$H_0 : \beta = 0 \qquad t\text{-stat} = \dfrac{1}{\sqrt{0.056}} = 4.2$

We will reject this null at any reasonable signif level.

2. b) Must regress e^2 on $X-\bar{X}$ and $X^2-\overline{X^2}$

e^2	$X-\bar{X}$	$X^2-\overline{X^2}$
0	-3	5
1	-2	0
1	-1	-3
1	0	-4
0	1	-3
4	2	0
4	3	5

$(\bar{X}=0,\ \overline{X^2}=4)$

$Z = [1 \quad X-\bar{X} \quad X^2-\overline{X^2}]$

$\hat{\alpha}_{OLS} = (Z'Z)^{-1} Z' e^2$

$$Z'Z = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 28 & 0 \\ 0 & 0 & 88 \end{bmatrix}$$

$(X-\bar{X})^2$	$(X^2-\overline{X^2})^2$	$(X-\bar{X})(X^2-\overline{X^2})$
9	25	-15
4	0	0
1	9	3
0	16	0
1	9	-3
4	0	0
9	25	15
28	88	0

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e^2	$(X-\bar{X})$	$(X^2-\overline{X^2})$	$e^2(X-\bar{X})$	$e^2(X^2-\overline{X^2})$
0	-3	5	0	0
1	-2	0	-2	0
1	-1	-3	-1	-3
1	0	-4	0	-4
0	1	-3	0	0
4	2	0	8	0
4	3	5	12	20
			17	13

$$Z'e^2 = \begin{bmatrix} 11 \\ 17 \\ 13 \end{bmatrix}$$

$$(Z'Z)^{-1}Z'e^2 = \begin{bmatrix} \frac{1}{7} & 0 & 0 \\ 0 & \frac{1}{28} & 0 \\ 0 & 0 & \frac{1}{88} \end{bmatrix} \begin{bmatrix} 11 \\ 17 \\ 13 \end{bmatrix} = \begin{bmatrix} 1.57 \\ 0.61 \\ 0.15 \end{bmatrix}$$

e^2	$X-\bar{X}$	$X^2-\overline{X^2}$	$1.57+0.61(X-\bar{X})+0.15(X^2-\overline{X^2})$	resid
0	-3	5	0.49	-0.49
1	-2	0	0.35	0.65
1	-1	-3	0.51	0.49
1	0	-4	0.97	0.03
0	1	-3	1.73	-1.73
4	2	0	2.79	1.21
4	3	5	4.15	-0.15

$$SSE = (-0.49)^2 + (0.65)^2 + (0.49)^2 + 0.03^2 + (-1.73)^2 + 1.21^2 + (-0.15)^2$$

$$= 5.38$$

$$\overline{e^2} = 1.57$$

$$TSS = (0-1.57)^2 + (1-1.57)^2 + (1-1.57)^2 + (1-1.57)^2 + (0-1.57)^2 + (4-1.57)^2 + (4-1.57)^2$$

$$= 17.71$$

$$R^2 = \frac{17.71 - 5.38}{17.71} = 0.70$$

$$NR^2 = 7 \cdot 0.70 = 4.9$$

χ^2_2 - table at 5% is 5.99

So, we would accept the null of homoscedasticity.

2 a) Let's use robust std errors:

$$\widehat{V(\hat{\beta}_{OLS})} = (X'X)^{-1} \; X' \operatorname{diag}(e^2) X \; (X'X)^{-1}$$

$$= \frac{1}{\Sigma X^2} \; \Sigma X^2 e^2 \; \frac{1}{\Sigma X^2}$$

$$= \frac{1}{28} \; \Sigma X^2 e^2 \; \frac{1}{28}$$

e^2	X^2	e^2X^2
0	9	0
1	4	4
1	1	1
1	0	0
0	1	0
4	4	16
4	9	36
		57

$$\widehat{V(\hat{\beta}_{OLS})} = \frac{1}{28} \; 57 \; \frac{1}{28} = 0.073$$

$$t\text{-stat} = \frac{1}{\sqrt{0.073}} = \frac{1}{0.27} = 3.7$$

Still reject null of no effect X on Y.

3 a) Recall the formula for omitted variables:

$$\begin{pmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{pmatrix} = \begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix}^{-1} \begin{pmatrix} X_1'Y \\ X_2'Y \end{pmatrix} = \cdots$$

$$E\begin{pmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{pmatrix} = \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} + \begin{bmatrix} X_1'X_1 & X_1'X_2 \\ X_2'X_1 & X_2'X_2 \end{bmatrix}^{-1} \begin{pmatrix} X_1'X_3 \\ X_2'X_3 \end{pmatrix} \beta_3$$

The fact that $Cov(X_1, X_3) > 0$ only establishes that $X_1'X_3 > 0$. This is little help in signing the bias. The $(X'X)^{-1}$ matrix may scramble $X_1'X_3$ and $X_2'X_3$ making the bias on β_1 +/- or 0.

b) Similarly, knowing $Cov(X_1, X_3) > 0$ and $Cov(X_2, X_3) > 0$ does not help much. Now, we know that $X_1'X_3 > 0$, $X_2'X_3 > 0$. Of course, we also know that the 11 and 22 elements of $(X'X)^{-1}$ above are > 0. But, if the 12 and 21 elements are < 0, still the bias in β_1, β_2 could be +/- or 0.

3\. c) Now, we can sign the bias:

$$E\begin{pmatrix}\hat\beta_{1OLS}\\ \hat\beta_{2OLS}\end{pmatrix} = \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} + \begin{bmatrix}X_1'X_1 & \\ & X_2'X_2\end{bmatrix}^{-1}\begin{pmatrix}X_1'X_3\\ X_2'X_3\end{pmatrix}\beta_3$$

$$= \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} + \begin{pmatrix}X_1'X_3/X_1'X_1\\ X_2'X_3/X_2'X_2\end{pmatrix}\beta_3$$

Both $\hat\beta_{1OLS}$ and $\hat\beta_{2OLS}$ are biased up since $X_1'X_3$, $X_1'X_1$, $X_2'X_3$, $X_2'X_2 > 0$.

d) Again, we can sign the bias:

$$E\begin{pmatrix}\hat\beta_{1OLS}\\ \hat\beta_{2OLS}\end{pmatrix} = \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} + \frac{1}{X_1'X_1X_2'X_2 - (X_1'X_2)^2}\begin{bmatrix}X_2'X_2 & -X_1'X_2\\ -X_1'X_2 & X_1'X_1\end{bmatrix}\begin{pmatrix}X_1'X_3\\ X_2'X_3\end{pmatrix}\beta_3$$

$$= \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} + \frac{\beta_3}{\cdots}\begin{pmatrix}X_2'X_2\,X_1'X_3 - X_1'X_2\,X_2'X_3\\ X_1'X_1\,X_2'X_3 - X_1'X_2\,X_1'X_3\end{pmatrix}$$

Since β_3 and $\frac{1}{\cdots}$ are >0 and $X_1'X_1$, $X_1'X_3$, $X_2'X_2$, $X_2'X_3 > 0$, and $X_1'X_2 < 0$, both $\hat\beta_{1OLS}$ and $\hat\beta_{2OLS}$ are biased up.

3e) $E\begin{pmatrix}\hat\beta_1\\ \hat\beta_2\end{pmatrix} = \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} + \begin{bmatrix}4 & 3\\ 3 & 10\end{bmatrix}^{-1}\begin{pmatrix}2\\ -1\end{pmatrix}$

$= \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} + \frac{1}{40-9}\begin{bmatrix}10 & -3\\ -3 & 4\end{bmatrix}\begin{pmatrix}2\\ -1\end{pmatrix}$

$= \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} + \frac{1}{31}\begin{pmatrix}23\\ -10\end{pmatrix}$

$\hat\beta_1$ is biased up by $23/31$

$\hat\beta_2$ is biased down by $10/31$

4 a) Yes, it is.

$$\overline{\Delta \ln Y} = \frac{1}{T-1} \sum_{t=2}^{T} \ln Y_t - \ln Y_{t-1}$$

$$= \frac{1}{T-1}\left(\ln Y_2 - \ln Y_1 + \ln Y_3 - \ln Y_2 + \ln Y_4 - \ln Y_3 + \cdots + \ln Y_T - \ln Y_{T-1}\right)$$

$= 0 \qquad = 0$ etc

$$= \frac{1}{T-1}\left(\ln Y_T - \ln Y_1\right)$$

$$= \frac{1}{T-1}\left(\beta_1 + \beta_2 T + \epsilon_T - \beta_1 - \beta_2 1 - \epsilon_1\right)$$

$$= \frac{1}{T-1}\left(\beta_2 (T-1) + \epsilon_T - \epsilon_1\right)$$

$$= \beta_2 + \frac{1}{T-1}\left(\epsilon_T - \epsilon_1\right)$$

So, $\overline{\Delta \ln y}$ is unbiased $E(\overline{\Delta \ln y}) = \beta_2$.
Also, its variance $\Downarrow 0$ as $T \nearrow \infty$:
$V(\overline{\Delta \ln y}) = \frac{2}{(T-1)^2}\sigma^2$.

4 b) No, it is not. When the model is differenced:

$$\Delta \ln Y_t = \beta_1 + \epsilon_t - \epsilon_{t-1}$$

we introduce serial correlation in the errors. So, the usual OLS estimator will be CAN but usual OLS std errors will be wrong.

4 c) It is given above $\frac{2}{(T-1)^2}\sigma^2$.

Also, we could use the estimator for the variance of OLS in the GLS model:

$$V(\beta_{OLS}) = (X'X)^{-1} X'\Sigma X (X'X)^{-1}$$

$$X'X = (T-1)$$

$$X'\Sigma X = (1\ 1, \ldots,) \begin{bmatrix} 2\sigma^2 & -\sigma^2 & 0 & - & 0 \\ -\sigma^2 & 2\sigma^2 & -\sigma^2 & & 0 \\ & -\sigma^2 & 2\sigma^2 & \ddots & \\ & 0 & & \ddots & -\sigma^2 \\ & & & -\sigma^2 & 2\sigma^2 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\cancel{(\sigma^2 \ldots \sigma^2)\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = (T-1)\sigma^2}$$

$$\cancel{(X'X)^{-1}(X'\Sigma X)(X'X)^{-1} = \sigma^2/(T-1)}$$

$$X'\Sigma X = (\sigma^2\ 0\ 0 \cdots 0\ \sigma^2)\begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix} = 2\sigma^2$$

$$(X'X)^{-1} X'\Sigma X\ (X'X)^{-1} = \frac{1}{T-1}\, 2\sigma^2\, \frac{1}{T-1}$$

$$= 2\sigma^2/(T-1)^2$$

4 d) Yes, by the G-M Thm.