An Economic Interpretation of Optimal Control Theory

By ROBERT DORFMAN*

Capital theory is the economics of time. Its task is to explain if, and why, a lasting instrument of production can be expected to contribute more to the value of output during its lifetime than it costs to produce or acquire. From the explanation, it deduces both normative and descriptive conclusions about the time-path of the accumulation of capital by economic units and entire economies.

Traditionally, capital theory, like all other branches of economics, was studied in the context of stationary equilibrium. For example, the stationary state of the classical economists, and the equilibrium of Böhm-Bawerk's theory of the period of production, both describe the state of affairs in which further capital accumulation is not worthwhile. A mode of analysis that is confined to a distant, ultimate position is poorly suited to the understanding of accumulation and growth,[1] but no other technique seemed available for most of the history of capital theory.

For the past fifty years it has been perceived, more or less vaguely, that capital theory is formally a problem in the calculus of variations.[2] But the calculus of variations is regarded as a rather arcane subject by most economists and, besides, in its conventional formulations appears too rigid to be applied to many economic problems. The application of this conceptual tool to capital theory remained peripheral and sporadic until very recently, and capital theory remained bound by the very confining limitations of the ultimate equilibrium.

All this has changed abruptly in the past decade as a result of a revival, or rather reorientation, of the calculus of variations prompted largely by the requirements of space technology.[3] In its modern version, the calculus of variations is called optimal control theory. It has become, deservedly, the central tool of capital theory and has given the latter a new lease on life. As a result, capital theory has become so profoundly transformed that it has been rechristened growth theory, and has come to grips with numerous important practical and theoretical issues that previously could not even be formulated.

The main thesis of this paper is that optimal control theory is formally identical with capital theory, and that its main insights can be attained by strictly economic reasoning. This thesis will be supported by deriving the principal theorem of optimal control theory, called the maximum principle, by means of economic analysis.

I. *The Basic Equations*

In order to have a concrete vocabulary, consider the decision problem of a firm that wishes to maximize its total profits over some period of time. At any date t, this firm will have inherited a certain stock of capital and other conditions from its

* The author is professor of economics at Harvard University.

[1] A point made most forceably by Joan Robinson in [9] and elsewhere.

[2] Notable examples are Hotelling [6] and Ramsey [8].

[3] The twin sources of the new calculus of variations are R. Bellman [4] and L. S. Pontryagin, et al. [7]. Bellman emphasized from the first the implications of his work for economics.

past behavior. Denote these by $k(t)$. With this stock of capital and other facilities k and at that particular date t, the firm is in a position to take some decisions which might concern rate of output, price of output, product design, or whatnot. Denote the decisions taken at any date by $x(t)$. From the inherited stock of capital at the specified date together with the specified current decisions the firm derives a certain rate of benefits or net profits per unit of time. Denote this by $u(k(t), x(t), t)$.[4] This function u determines the rate at which profits are being earned at time t as a result of having k and taking decisions x.

Now look at the situation as it appears at the initial date $t=0$. The total profits that will be earned from then to some terminal date T is given by:

$$W(k_0, \vec{x}) = \int_0^T u(k, x, t)dt$$

which is simply the sum of the rate at which profit is being earned at every instant discounted to the initial date (if desired) and added up for all instants.[5] In this notation $\vec{x}$ does not denote an ordinary number but the entire time path of the decision variable x from the initial date to T. This notation asserts that if the firm starts out with an initial amount of capital k_0 and then follows the decision policy denoted by $\vec{x}$, it will obtain a total result, W, which is the integral of the results obtained at each instant; these results in turn depending upon the date of the pertinent instant, the capital stock then and the decision applicable to that moment. The firm is at liberty, within limits, to choose the time path of the decision variable $\vec{x}$ but it cannot choose independently the amount of capital at each instant; that is a consequence of the capital at the initial date and the time path chosen for decision variable. This constraint is expressed by saying that the rate of change of the capital stock at any instant is a function of its present standing, the date, and the decisions taken. Symbolically:[6]

$$(1) \qquad \dot{k} = \frac{dk}{dt} = f(k, x, t).$$

Thus the decisions taken at any time have two effects. They influence the rate at which profits are earned at that time and they also influence the rate at which the capital stock is changing and thereby the capital stock that will be available at subsequent instants of time.

These two formulas express the essence of the problem of making decisions in a dynamic context. The problem is to select the time path symbolized by $\vec{x}$ so as to make the total value of the result, W, as great as possible taking into account the effect of the choice of x on both the instantaneous rate of profit and the capital stock to be carried into the future. This is truly a difficult problem, and not only for beginners. The essential difficulty is that an entire time path of some variable has to be chosen. The elementary calculus teaches how to choose the best possible number to assign to a single variable or the best numbers for a few variables by differentiating some function and setting partial derivatives equal to zero. But finding a best possible time path is an entirely different matter and leads into some very advanced mathematics. The strategy of the solution is to reduce the problem which, as it stands, requires us to find an entire time path, to a problem which demands us to determine only a single number (or a few numbers), which is something we know how to do from the ordinary cal-

[4] In the sequel we shall often omit the time-arguments in the interest of simplicity, and thus write simply $u(k, x, t)$.

[5] The argument t allows the introduction of any discounting formula that may be appropriate.

[6] The dot will be used frequently to denote a rate of change with respect to time.

culus. This transformation of the problem can be performed in a number of ways. One way, which dates back to the eighteenth century, leads to the classical calculus of variations. Another way, which will be followed here, leads to the maximum principle of optimal control theory. This method depends very heavily on introducing the proper notation. First, introduce a formula for the value that can be obtained by the firm starting at an arbitrary date t with some amount of capital k and then following an arbitrary decision policy $\vec{x}$ until the terminal date. It is

$$W(k_t, \vec{x}, t) = \int_t^T u[k, x, \tau]d\tau$$

which, of course, is just a generalization of the W formula introduced previously.

Now break W up into two parts. Think of a short time interval of length Δ beginning at time t. Δ is to be thought of as being so short that the firm would not change x in the course of it even if it could. Then we can write

$$W(k, \vec{x}, t) = u(k, x_t, t)\Delta + \int_{t+\Delta}^T u[k(t), x, \tau]d\tau. \tag{2}$$

This formula says that if the amount of capital available at time t is k and if the policy denoted by $\vec{x}$ is followed from then on, then the value contributed to the total sum from date t on consists of two parts. The first part is the contribution of a short interval that begins at date t. It is the rate at which profits are earned during the interval times the length of the interval. It depends on the current capital stock, the date, and the current value of the decision variable, here denoted by x_t. The second part is an integral of precisely the same form as before but beginning at date t+Δ. It should be noticed that the starting capital stock for this last integral is not k(t) but k(t+Δ). This fact, that the capital stock will change during the interval in a manner influenced by x_t, will play a very significant role. We can take advantage of the fact that the same form of integral has returned by writing

$$W(k_t, \vec{x}, t) = u(k, x_t, t)\Delta + W(k_{t+\Delta}, \vec{x}, t + \Delta)$$

where the changes in the subscripts are carefully noted.

Now some more notation. If the firm knew the best choice of $\vec{x}$ from date t on, it could just follow it and thereby obtain a certain value. We denote this value, which results from the optimal choice of $\vec{x}$ by V^*, as follows

$$V^*(k_t, t) = \max W(k_t, \vec{x}, t).$$

Notice that V^* does not involve $\vec{x}$ as an argument. This is because $\vec{x}$ has been maximized out. The maximum value that can be obtained beginning at date t with capital k does not depend on $\vec{x}$ but is the value that can be obtained in those conditions from the best possible choice of $\vec{x}$. Now suppose that the policy designated by x_t is followed in the short time interval from t to t+Δ and that thereafter the best possible policy is followed. By formula (2) the consequence of this peculiar policy can be written as

$$V(k_t, x_t, t) = u(k_t, x_t, t)\Delta + V^*(k_{t+\Delta}, t + \Delta).$$

In words, the results of following such a policy are the benefits that accrue during the initial period using the decision x_t plus the maximum possible profits that can be realized starting from date t+Δ with capital k(t+Δ) which results from the decision taken in the initial period.

Now we have arrived at the ordinary calculus problem of finding the best possible value for x_t. If the firm adopts this value, then V of the last formula will be equal to V^*. The calculus teaches us that one frequently effective way to discover a value of a variable that maximizes a given function is to differentiate the function with respect to the variable and equate the partial derivative to zero. This is the method that we shall use. But first we should be warned that this method is not sure-fire. It is quite possible for the partial derivatives to vanish when the function is not maximized (for example, they may vanish when it is minimized), and cases are not rare in which the partial derivatives differ from zero at the maximum. We shall return to these intricacies later. For the present we assume that the partial derivative vanishes at the maximum, differentiate $V(k_t, x_t, t)$ with respect to x_t, and obtain

$$\Delta \frac{\partial}{\partial x_t} u(k, x_t, t) + \frac{\partial}{\partial x_t} V^*(k(t+\Delta), t+\Delta) = 0. \tag{3}$$

The trouble with that formula, aside from the fact that the function V^* is still unknown, is that we are told to differentiate V^* with respect to x_t, whereas it does not involve x_t explicitly. To get around this, notice

$$\frac{\partial V^*}{\partial x_t} = \frac{\partial V^*}{\partial k(t+\Delta)} \frac{\partial k(t+\Delta)}{\partial x_t} .$$

Both of these expressions merit some analysis and we shall start with the second. Since we are dealing with short time periods we can use the approximation

$$k(t+\Delta) = k(t) + \dot{k}\Delta.$$

That is, the amount of capital at t+Δ is equal to the amount of capital at t plus the rate of change of capital during the interval times the length of the interval. Remembering formula (1), $\dot{k}$ depends on x_t:

$$\dot{k} = f(k, x_t, t).$$

Thus we can write

$$\frac{\partial k(t+\Delta)}{\partial x_t} = \Delta \frac{\partial f}{\partial x_t} .$$

Turn, now, to the first factor, $\partial V^*/\partial k$. This derivative is the rate at which the maximum possible profit flow from time t+Δ on changes with respect to the amount of capital available at t+Δ. It is, therefore, the marginal value of capital at time t+Δ, or the amount by which a unit increment in capital occurring at that time would increase the maximum possible value of W. We denote the marginal value of capital at time t by λ(t), defined by

$$\lambda(t) = \frac{\partial}{\partial k} V^*(k, t).$$

Inserting these results in formula (3), we obtain

$$\Delta \frac{\partial u}{\partial x_t} + \lambda(t+\Delta)\Delta \frac{\partial f}{\partial x_t} = 0 \tag{4}$$

and furthermore, the constant Δ can be cancelled out. We have one more simplification to make before arriving at our first important conclusion. The marginal value of capital changes gradually over time and so, to a sufficiently good approximation,

$$\lambda(t+\Delta) = \lambda(t) + \dot{\lambda}(t)\Delta.$$

That is, the marginal value of capital at t+Δ is the marginal value at t plus the rate at which it is changing during the interval multiplied by the length of the interval. Insert this expression in equation (4), after cancelling the common factor Δ in the equation as written, to obtain

$$\frac{\partial u}{\partial x_t} + \lambda(t)\frac{\partial f}{\partial x_t} + \dot{\lambda}(t)\Delta \frac{\partial f}{\partial x_t} = 0.$$

Now allow Δ to approach zero. The third term becomes negligibly small in comparison with the other two. Neglecting it, there results:

$$\frac{\partial u}{\partial x_t} + \lambda \frac{\partial f}{\partial x_t} = 0. \tag{5}$$

This is our first major result and constitutes about half of the maximum principle. It makes perfectly good sense to an economist. It says that along the optimal path of the decision variable at any time the marginal short-run effect of a change in decision must just counter-balance the effect of that decision on the total value of the capital stock an instant later. We see that because the second term in the equation is the marginal effect of the current decision on the rate of growth of capital with capital valued at its marginal worth, λ. The firm should choose x at every moment so that the marginal immediate gain just equals the marginal long-run cost, which is measured by the value of capital multiplied by the effect of the decision on the accumulation of capital.

Now suppose that x_t is determined so as to satisfy equation (5). On the assumption that this procedure discovers the optimal value of x_t, $V(k_t, x_t, t)$ will then be equal to its maximum possible value or $V^*(k, t)$. Thus

$$V^*(k, t) = u(k, x_t, t)\Delta + V^*(k(t+\Delta), t+\Delta).$$

Now differentiate this expression with respect to k. The derivative of the left-hand side is by definition $\lambda(t)$. The differentiation of the right-hand side is very similar to the work that we have already done and goes as follows:

$$\begin{aligned}\lambda(t) &= \Delta\frac{\partial u}{\partial k} + \frac{\partial}{\partial k}V^*(k(t+\Delta), t+\Delta)\\ &= \Delta\frac{\partial u}{\partial k} + \frac{\partial k(t+\Delta)}{\partial k}\lambda(t+\Delta)\\ &= \Delta\frac{\partial u}{\partial k} + \left(1 + \Delta\frac{\partial f}{\partial k}\right)(\lambda + \dot{\lambda}\Delta)\\ &= \Delta\frac{\partial u}{\partial k} + \lambda + \Delta\lambda\frac{\partial f}{\partial k} + \Delta\dot{\lambda} + \dot{\lambda}\frac{\partial f}{\partial k}\Delta^2.\end{aligned}$$

We can ignore the term in Δ^2 and make the obvious cancellations to obtain

$$(6)\qquad -\dot{\lambda} = \frac{\partial u}{\partial k} + \lambda\frac{\partial f}{\partial k}.$$

This is the second major formula of the maximum principle and possesses an illuminating economic interpretation.

To a mathematician, $\dot{\lambda}$ is the rate at which the value of a unit of capital is changing. To an economist, it is the rate at which the capital is appreciating. $-\dot{\lambda}(t)$ is therefore the rate at which a unit of capital depreciates at time t. Accordingly the formula asserts that when the optimal time path of capital accumulation is followed, the decrease in value of a unit of capital in a short interval of time is the sum of its contribution to the profits realized during the interval and its contribution to enhancing the value of the capital stock at the end of the interval. In other words, a unit of capital loses value or depreciates as time passes at the rate at which its potential contribution to profits becomes its past contribution.

This finding is reminiscent of the figure of speech employed by the nineteenth century capital theorists. They said that a capital good embodied a certain amount of value which it imparted gradually to the commodities that were made with its assistance. That is just what is going on here. Each unit of the capital good is gradually decreasing in value at precisely the same rate at which it is giving rise to valuable outputs, either currently saleable or stored for the future in accumulated capital. We can also interpret $-\dot{\lambda}$ as the loss that would be incurred if the acquisition of a unit of capital were postponed for a short time.

II. *The Maximum Principle*

In effect we have been led to construct the auxiliary or Hamiltonian function

$$H = u(k, x, t) + \lambda(t)f(k, x, t),$$

to compute its partial derivative with respect to x, and to set that partial derivative equal to zero. This construction has substantial economic significance. If we imagine H to be multiplied by Δ, we can see that it is the sum of the total profits earned in the interval Δ plus the accrual of capital during the interval valued at its marginal value. $H\Delta$ is thus the total contribution of the activities that go on during the interval Δ, including both its direct contribution to the integral W, and the value of the capital accumulated during the interval. Naturally, then, the decision variable x during the current interval should be chosen so as to make H as great as possible. It is for this reason that the procedure we are describing is called the maximum principle. A simple and frequently effective way to do this is to choose a value of the control variable for which the partial derivative vanishes, as we have done.

We have also, in effect, computed the partial derivative of H with respect to k and equated that partial derivative to $-\dot{\lambda}$. The common sense of this operation can be seen best from a modified Hamiltonian,

$$H^* = u(k, x, t) + \frac{d}{dt}\lambda k$$
$$= u(k, x, t) + \lambda k + \dot{\lambda} k.$$

$H^*\Delta$ is the sum of the profits realized during an interval of length Δ and the increase in the value of the capital stock during the interval, or in a sense, the value of the total contribution of activities during the interval to current and future profits.[7] If we maximize H^* formally with respect to x and k we obtain:

$$\frac{\partial u}{\partial x} + \lambda \frac{\partial f}{\partial x} = 0,$$

$$\frac{\partial u}{\partial k} + \lambda \frac{\partial f}{\partial k} + \dot{\lambda} = 0,$$

[7] H^* differs from H by including capital gains.

which are equations (5) and (6).

Of course, the firm cannot maximize H^* with respect to k since k is not a variable subject to choice. But we now see that equations (5) and (6) advise the firm to choose the time-paths of x and λ so that the resultant values of k are the ones it would choose, if it could do so, to make the sum of profits and increment in capital value as great as possible in every short time interval.

As a technical note, in differentiating H, the marginal value λ is not regarded as a function of x and k, but as a separate time path which is to be determined optimally.

Now we have before us the basic ideas of the maximum principle. There is naturally much more to the method than these two formulas. A good deal of mathematical elaboration is required before the two formulas can be implemented, and we shall indicate later some of the complications that can arise. But there is one additional feature that has to be mentioned before we have finished dealing with fundamentals. This concerns the boundary conditions; for example, the amount of capital available at the beginning of the planning period and the amount required to be on hand on the terminal date.

To see how these boundary data affect the solution to the problem, consider how the three basic formulas operate. They are:

$$\text{(I)} \qquad k = f(k, x, t)$$

$$\text{(II)} \qquad \frac{\partial u}{\partial x} + \lambda \frac{\partial f}{\partial x} = 0$$

$$\text{(III)} \qquad \frac{\partial u}{\partial k} + \lambda \frac{\partial f}{\partial k} = -\dot{\lambda}.$$

The first of these is part of the data of the problem. It specifies how capital grows at any instant as a result of its current standing and the choices made. The other two formulas are the main results of the maximization principle. Formula II says that the choice variable at every instant should be selected so that the marginal immediate gains are in balance with the value of the marginal contribution to the accumulation of capital. Formula III says that capital depreciates at the same rate that it contributes to useful output.

The three formulas are conveniently written and remembered in terms of the Hamiltonian. In this form they are:

$$\text{(I}'\text{)} \qquad \frac{\partial H}{\partial \lambda} = \dot{k}$$

$$\text{(II}'\text{)} \qquad \frac{\partial H}{\partial x} = 0$$

$$\text{(III}'\text{)} \qquad \frac{\partial H}{\partial k} = -\dot{\lambda}.$$

Notice the reciprocal roles played by k and λ in these equations. The partial derivative of H with respect to either is simply related to the time-derivative of the other.

These three formulas jointly determine completely the time paths of the choice variable, the capital stock, and the value of capital. We shall start at time zero with a certain capital stock and a certain initial value for capital. Now look at formula II written out a bit more explicitly:

$$\text{(II)} \quad \frac{\partial}{\partial x} u(k, x, \mathrm{t}) + \lambda(\mathrm{t}) \frac{\partial}{\partial x} f(k, x, \mathrm{t}) = 0.$$

With k and λ known, this formula determines the value of x, the choice variable.[8] Putting this value in formula I we obtain $\dot{k}$, the rate at which the capital stock is changing. Putting it in formula III we similarly obtain $\dot{\lambda}$ the rate at which the value of a unit of capital is changing. Thus we know the capital stock and the value of a unit of capital a short time later. Using these new values, we can repeat our substitutions in the three formulas and so find, in order, a new value of the choice variable, a new rate for the change in the capital stock and a new rate for the change in the value of capital. Repeating this cycle over and over again, we can trace through the evolution of all the variables from time zero to time T.

In short, these three formulas working together determine the optimal paths of all the variables starting out from any given initial position. In another sense, then, the problem of the choice of an optimal path has been reduced to a much simpler problem, the problem of choosing an optimal initial value for the value of a unit of capital. This is not by any means an easy problem, but it is obviously a great deal easier than finding an entire optimal path without the aid of these formulas.

III. *The Boundary Conditions*

We can now mention the role of boundary conditions. They are of two sorts. Initial conditions describe the state of the firm or economy at the initial date, t=0. In particular, they set forth the initial stock of capital. Terminal conditions prescribe the values of some, or all, of the variables at the terminal date, t=T. For example, the problem may require that the firm have at least some specified stock of capital, say $\overline{K}$ on hand at the terminal date, which can be imposed by including $k(\mathrm{T}) \geq \overline{K}$ in the conditions of the problem. Or, again, if the problem is strictly one of maximizing profits during a finite interval, 0 to T, it is clear that capital on hand at date T cannot contribute to that objective; it exists too late to be of service before date T. Such a problem gives rise to the terminal condition $\lambda(\mathrm{T})=0$.

Now we have seen that the three equations (I), (II), (III) jointly determine the entire evolution of x, k, and λ once the starting values have been prescribed. In particular they determine the terminal

[8] Some mathematical complications arise here. We assume that with k, λ, and t given, formula (II) is satisfied by a unique value of x.

values. We have only[9] to determine a set of starting values that leads to acceptable terminal values to find an entire time path that satifies the necessary conditions for being optimal. In our example, since the initial capital stock is given, the critical initial value to be determined is $\lambda(0)$, the marginal value of capital at the initial date. The three basic formulas, abstract though they may appear, in fact constitute a constructive solution to the problem of choosing an optimal time path. They are a solution, in principle, of the problem of optimal capital accumulation.

We have now found that the old-fashioned technique of equating margins, used with a little ingenuity, leads to the maximum principle, which is the fundamental theorem of optimal control theory.

IV. *An Example*

About the simplest known example of the application of these principles to an economic problem is the derivation of the socially optimal path of capital accumulation for a one-sector economy with an exponentially growing population and production under constant returns to scale.[10]

Let us set forth some notation and data. $N(\mathrm{t})$ is population at date t. Since population grows exponentially, at rate n, say,

$$N(\mathrm{t}) = N(0)e^{n\mathrm{t}}.$$

It will save clutter if we assume $N(0)=1$ (measured in hundreds of millions of people). Denote per capita consumption by c and the utility enjoyed by a person consuming at rate c by $u(c)$. The total utility enjoyed by all the persons alive at time t with per capita consumption at rate c is

$$e^{n\mathrm{t}}u(c).$$

Let ρ be the social rate of time preference. Then the importance at time 0 of the consumption achieved at time t is

$$e^{-\rho\mathrm{t}}e^{n\mathrm{t}}u(c) = e^{(n-\rho)\mathrm{t}}u(c). \tag{7}$$

A defensible social objective for a society with time horizon T (conceivably infinite) is to maximize

$$W = \int_0^{\mathrm{T}} e^{(n-\rho)\mathrm{t}}u(c)\,d\mathrm{t}, \tag{8}$$

or the sum of the utilities enjoyed between 0 and T.[11]

Consumption is limited by output and output by capital stock. Let $K(\mathrm{t})$ denote the capital stock at date t and let $k(\mathrm{t})=K(\mathrm{t})/N(\mathrm{t})$ denote capital per capita. By virtue of constant returns to scale, we can write the production function of the economy as

$$Y(\mathrm{t}) = N(\mathrm{t})f(k(\mathrm{t}))$$

or, omitting the confusing time-arguments,

$$Y = Nf(k) = e^{n\mathrm{t}}f(k).$$

Gross investment equals output minus consumption, or Y-Nc. Net investment equals gross investment minus physical depreciation. Suppose that physical capital deteriorates at the rate δ per unit per annum so that the total rate of decay of the physical stock, when it is K, is δK. Then net capital accumulation is

$$\begin{aligned}\dot{K} &= Y - Nc - \delta K = N(f(k) - c) - \delta K\\ &= N(f(k) - c) - \delta Nk\\ &= N(f(k) - c - \delta k).\end{aligned}$$

Finally, eliminate $\dot{K}$ by noticing:

$$\begin{aligned}\dot{k} &= \frac{d}{d\mathrm{t}}\,\frac{K}{N} = \frac{K}{N}\left(\frac{\dot{K}}{K} - \frac{\dot{N}}{N}\right)\\ &= k\left(\frac{\dot{K}}{Nk} - n\right) \qquad (9)\\ &= f(k) - c - \delta k - nk\\ &= f(k) - c - (n+\delta)k.\end{aligned}$$

[9] Only! Reputations have been made by solving this problem in important instances.

[10] A more extended discussion of a very similar model can be found in Arrow [1].

[11] It is best to assume $\rho > n$ or else the integral will be infinite for $\mathrm{T}=\infty$.

Equations (8) and (9) constitute our simple example. Equation (9) is an example of equation (I). To derive equation (II), differentiate equations (7) and (9) with respect to the choice variable, c:

$$\frac{\partial}{\partial c} e^{(n-\rho)t} u(c) = e^{(n-\rho)t} u'(c),$$

$$\frac{\partial}{\partial c} [f(k) - c - (n + \delta)k] = -1.$$

Hence equation (II) is:

$$(10) \qquad e^{(n-\rho)t} u'(c) - \lambda = 0,$$

or the value of a unit of capital at time t is the marginal utility of consumption at that time, adjusted for population growth and the social rate of time preference.

Equation (III) is obtained similarly by differentiating equations (7) and (9) with respect to k. There results:

$$-\dot{\lambda} = 0 + \lambda[f'(k) - (n + \delta)],$$

or

$$(11) \qquad f'(k) = n + \delta - \frac{\dot{\lambda}}{\lambda}.$$

Equation (10) can be used to eliminate the unfamiliar λ. Differentiating it with respect to time:

$$\frac{\dot{\lambda}}{\lambda} = n - \rho + \frac{u''(c)}{u'(c)} \frac{dc}{dt}.$$

Thus equation (11) becomes

$$f'(k) = \rho + \delta - \frac{u''(c)}{u'(c)} \frac{dc}{dt}.$$

This is our final equation for the optimal path of capital accumulation. It asserts that along such a path the rate of consumption at each moment must be chosen so that the marginal productivity of capital is the sum of three components:

(1) ρ, the social rate of time preference,
(2) δ, the rate of physical deterioration of capital, and
(3) the rather formidable looking third term which, however, is simply the percentage rate at which the psychic cost of saving diminishes through time. This can be seen by noting that the psychic cost of saving at any time is $u'(c)$, its time rate of change is $u''(c)dc/dt$, and its percentage time rate of change is the negative of the third term in the sum.

In other words, along the optimum path of accumulation the marginal contribution of a unit of capital to output during any short interval of time must be just sufficient to cover the three components of the social cost of possessing that unit of capital, namely, the social rate of time-preference, the rate of physical deterioration of capital, and the additional psychic cost of saving a unit at the beginning of the interval rather than at the end. All of these are expressed as percents per unit of time, which is also the dimension of the marginal productivity of capital.

The evolution of this economy along its optimal path of development can be visualized most readily by drawing a phase diagram as shown in Figure 1. We have found that the rates of change of k and c can be written:

$$(9) \qquad k = f(k) - (n + \delta)k - c,$$

$$\dot{c} = \frac{u'(c)}{u''(c)} [\rho + \delta - f'(k)].$$

Thus $\dot{k}=0$ whenever c and k satisfy the equation

$$c = f(k) - (n + \delta)k.$$

In Figure 1, k is plotted horizontally and c vertically. The curve labelled $\dot{k}=0$ shows the combinations of c and k that satisfy this equation. It has the shape drawn because of the conventional assumptions that the marginal productivity of capital is positive but diminishing (i.e., $f'(k)>0$,

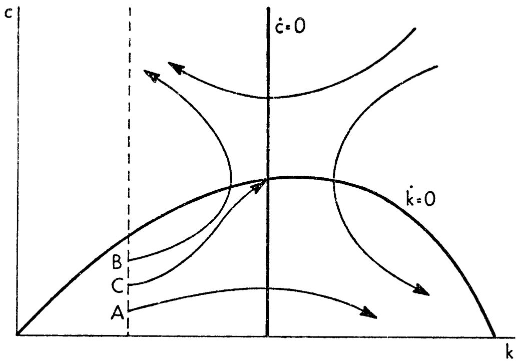

FIGURE 1

$f''(k) < 0$), and the very plausible assumption that for very low levels of capital per worker, $f'(k) > n+\delta$. We also assume that no output is possible without some capital, i.e., $f(0) = 0$. If consumption per capita is less than the rate on the locus just described, capital per capita increases ($\dot{k} > 0$). Above the locus $\dot{k} < 0$.

Similarly, consumption per capita is unchanging ($\dot{c} = 0$) if

$$f'(k) = \rho + \delta.$$

The vertical line in Figure 1, labelled $\dot{c} = 0$, is drawn at this level of k. If we accept the usual assumptions of positive but diminishing marginal utility $u'(c) > 0$, $u''(c) < 0$. Then $\dot{c} > 0$, i.e., per capita consumption grows, to the left of this line. The reason is that with low levels of capital per capita the amount of depreciation is small and the amount of capital needed to equip the increment in population with the current level of capital per capita is also small.

These considerations enable us to depict qualitatively the laws of motion of the system. Imagine an initial low level of capital per capita, represented by the dashed vertical in the diagram. The entire evolution of the system is determined by the choice of the initial level of per capita consumption. If a low initial level is chosen, such as at point A in the figure, both consumption and capital per capita will increase for some time, following the curved arrow that emanates from point A. But when the level of capital per capita reaches the critical level, consumption per capita will start to fall though capital per capita will continue to increase. This is a policy of initial generosity in consumption followed by increasing abstemiousness intended, presumably to attain some desired ultimate level of capital per capita.

Similarly, the path emanating from point B represents a policy of continually increasing consumption per capita, with capital initially being accumulated and eventually being consumed. The other paths drawn have similar interpretations.

The path originating at point C is of particular interest. It leads to the intersection of the two critical loci, the steady state of the system in which neither per capita consumption nor per capita income changes. Once at this point all the absolute values grow exponentially at the common rate n.

It is now seen that if the initial capital per capita is given, the entire course of the economy is determined by the choice of the initial level of per capita consumption. This choice determines, among other things, the amount of capital per capita at any specified date.[12] If the conditions of the problem prescribe a particular amount of capital at some date, the initial c must be the one with a path that leads to the specified point. If there is no such prescription for capital accumulation, the initial c will be the one that causes the capital stock to be exhausted at the terminal date under consideration. And if there is no terminal date (i.e., $T = \infty$) the problem becomes much trickier mathematically and, indeed, the theory of optimization with an infinite time horizon is not yet completely established. But, in this simple case, we can see that the only

[12] The position of the economy at particular dates cannot be read off the phase diagram.

possible solution is the path that originates at point C and terminates at the point where $\dot{c}=\dot{k}=0$. For, the figure shows that all other paths that satisfy the optimizing conditions lead eventually to situations in which either c or k is negative. Since such paths cannot be realized, the only feasible optimizing path is the one that approaches $\dot{c}=\dot{k}=0$.

This result is quite characteristic of infinite horizon problems: the optimal growth paths, under many conditions, approach the situation in which consumption and the capital stock grow exponentially at a rate determined by the rate of population growth and the rate of technical progress (here assumed zero), just as in this case.

For finite horizon problems, it can be shown that the more remote the terminal date considered, the closer the path will come to the steady state position ($\dot{c}=\dot{k}=0$) before veering away to either high consumption or high capital accumulation as the case may be. This is a version of the turnpike theorem.

V. *Derivation via Finite Maximizing*

Those who distrust clever, intuitive arguments, as I do, may find some comfort in seeing the same results deduced from the more familiar method of maximizing subject to a finite number of constraints. Let us suppose that the entire period of T months is divided into n subperiods of m months each. $u(x_t, k_t, t)$ then denotes the rate at which profits are being earned or other benefits derived during the t-th subperiod, with x_t being the value of the decision variable during that subperiod, and k_t the value of the state variable at its beginning. Since the subperiod is m months long, the total profit earned is $u(x_t, k_t, t)\ m$.

The rate of change of the state variable during the t-th subperiod is $f(x_t, k_t, t)$. Then the values of the state variable at the beginnings of successive subperiods are connected by the equation

$$k_{t+1} = k_t + f(x_t, k_t, t)m. \tag{12}$$

Finally, the finite version of our problem is to choose $2n$ values, x_t, k_t so as to maximize the total profit over the entire period,

$$\sum_{t=1}^{n} u(x_t, k_t, t)m$$

subject to the n constraints (12), and to any boundary conditions that may apply. To be specific, suppose that initial and terminal values for the state variable are preassigned. These give rise to the side conditions

$$k_1 = K_0$$
$$k_{n+1} = K_T.$$

This problem is solved by setting up the Lagrangean function

$$\begin{aligned} L = &\sum_{t=1}^{n} u(x_t, k_t, t)m \\ &+ \sum_{1}^{n} \lambda_t[k_t + f(x_t, k_t, t)m - k_{t+1}] \\ &+ \lambda_0[K_0 - k_1] + \mu[k_{n+1} - K_T] \end{aligned}$$

and setting each of its partial derivatives equal to zero. The Greek symbols in this formula are the Lagrange multipliers, one for each constraint. We shall interpret them after we have completed our calculations.

The same Hamiltonian expression that we encountered before is beginning to emerge, so it is convenient to write

$$H(x_t, k_t, t) = u(x_t, k_t, t) + \lambda_t f(x_t, k_t, t)$$

and

$$\begin{aligned} L = m \sum_{1}^{n} H(x_t, k_t, t) + \sum_{1}^{n} \lambda_t(k_t - k_{t+1}) \\ + \lambda_0(K_0 - k_1) + \mu(k_{n+1} - K_T). \end{aligned}$$

Now differentiate and equate derivatives to zero:

$$(13)\quad \frac{\partial L}{\partial x_t} = m\frac{\partial}{\partial x_t}H(x_t, k_t, t) = [u_1(x_t, k_t, t) + \lambda_t f_1(x_t, k_t, t)]m = 0 \quad \text{for } t = 1, \ldots, n,$$

which is analogous to equation (5). And

$$\frac{\partial L}{\partial k_t} = m\frac{\partial}{\partial k_t}H(x_t, k_t, t) + \lambda_t - \lambda_{t-1} = 0$$

or

$$(14)\quad -\frac{\lambda_t - \lambda_{t-1}}{m} = u_2(x_t, k_t, t) + \lambda_t f_2(x_t, k_t, t), \text{ for } t = 1, \ldots, n,$$

which is the discrete analog of equation (6).

Finally

$$\frac{\partial L}{\partial k_{n+1}} = -\lambda_n + \mu = 0.$$

Thus $\mu = \lambda_n$ and can be forgotten.

These equations are applicable to problems in which time is regarded as a discrete variable. The Lagrange multipliers have their usual interpretation. In particular, λ_t is the amount by which the maximum attainable value of $\sum u(x_t, k_t, t)m$ would be increased if an additional unit of capital were to become available by magic at the end of the t-th period. In other words, λ_t is the marginal value of capital on hand at date mt.

The maximizing conditions found previously should be the limit of these equations as m approaches zero and n approaches infinity, and they are. To show this, we have to revise our notation slightly. The subscripted variables now denote the values that the variables have in the t-th period. When m changes, the dates included in the t-th period change also. So we need symbols for the values of the variables at fixed dates. To this end, let τ denote any date and $x(\tau)$, for example, the value of x at that date. The connection between x_t and $x(\tau)$ is easy. Any date τ is in the subperiod numbered t where t is given by

$$t = 1 + [\tau/m].$$

In this formula, [ ] is an old-fashioned notation meaning "integral part of." For example: [3.14159] = 3. Then $x(\tau)$ is defined by

$$x(\tau) = x_{1+[\tau/m]},$$

and similarly for the other variables. Equations (13) and (14) can now be written in terms of τ:

$$(15)\quad u_1[x(\tau), k(\tau), \tau] + \lambda(\tau)f_1[x(\tau), k(\tau), \tau] = 0,$$

$$(16)\quad -\frac{\lambda(\tau) - \lambda(\tau-m)}{m} = u_2[x(\tau), k(\tau), \tau] + \lambda(\tau)f_2[x(\tau), k(\tau), \tau].$$

Notice in equation (16) that λ_{t-1} has been replaced by $\lambda(\tau-m)$, reflecting that the beginnings of the intervals are m months apart.

Equation (15) is identical with equation (II). As m approaches zero, the left-hand side of equation (16) approaches $-\dot{\lambda}(\tau)$, taking for granted that it approaches a limit and applying the definition of the derivative. The whole equation, therefore, approaches equation (III). Equation (I) is similarly and obviously the limiting form of equation (12).

Thus the basic equations of the maximum principle are seen to be the limiting forms of the ordinary first-order necessary conditions for a maximum applied to the same problem, and the auxiliary variables of the maximum principle are the limiting values of the Lagrange multipliers.

VI. *Qualifications and Extensions*

This entire development has been exceedingly informal, to put it kindly. The calculus of variations is a difficult and delicate subject, so that a choice always has to be made between stating a proposition correctly, with all the qualifications that it deserves, and stating it forceably and clearly so that the essential idea can be grasped at a glance. The more intelligible alternative has been chosen throughout this paper since all the theorems have been stated and proved rigorously elsewhere in the literature.[13] This choice, as it happens, has especial drawbacks in the present context because much of the virtue of the maximum principle lies precisely in the qualifications that have been suppressed: it is valid under more general conditions than the classical methods that yield almost the same theorems.

As an example of the alternative mode of exposition, our main conclusions can be stated more formally and correctly as follows:[14]

THEOREM 1. Let it be desired to find a time-path of a control variable $x(t)$ so as to maximize the integral

$$\int_0^T u[k(t), x(t), t]dt$$

where

$$\frac{dk}{dt} = f[k(t), x(t), t],$$

where $k(0)$ is preassigned, and where it is required that $k(T) \geq \bar{K}$. It is assumed that the functions $u(k, x, t)$ and $f(k, x, t)$ are twice continuously differentiable with respect to k, differentiable with respect to x, and continuous with respect to t. Then if $x^*(t)$ is a solution to this problem, there exists an auxiliary variable $\lambda(t)$ such that:

(a) For each t, $x^*(t)$ maximizes $H[k(t), x(t), \lambda(t), t]$ where $H(k, x, \lambda, t) = u(k, x, t) + \lambda f(k, x, t)$;

(b) $\lambda(t)$ satisfies

$$\frac{d\lambda}{dt} = -\frac{\partial H}{\partial x}$$

evaluated at $k = k(t)$, $x = x^*(t)$, $\lambda = \lambda(t)$; and

(c) $k(T) \geq \bar{K}$, $\lambda(T) \geq 0$, $\lambda(t)[k(T) - \bar{K}] = 0$.

This theorem applies to the type of problem that we have been considering, with the useful elaboration that a lower limit has been imposed on the terminal value of the state variable, k. Part (c) of the conclusion, called the transversality condition, arises from this added requirement. It asserts that the terminal value of the auxiliary variable cannot be negative and that it will be zero if, at the end of the optimal path, $k(T)$ exceeds the required value.

The principle difference between this formal statement and our previous conclusions lies in conclusion (a) of the Theorem. The assertion that the Hamiltonian function, H, is maximized at each instant of time is not the same as the assertion that its partial derivatives vanish, made in our equations (II) and (II′). Equating partial derivatives to zero is neither necessary nor sufficient for maximization, though it is especially illuminating to economists, when it is appropriate, because conditions on partial derivatives translate readily into marginal equalities. There are three complications that can make the vanishing of partial derivatives an inadequate indication of the location of a maximum.

First, there are the so-called higher order conditions. First partial derivatives can vanish at a minimum or at a saddle-point as well as at a maximum. To guard against this possibility, second partial derivatives, and even higher ones, have to be taken into account.

Second, the vanishing of partial derivatives, even when higher order con-

[13] For example, in Arrow and Kurz [3] and Halkin [5].

[14] The given theorem is adapted from Arrow [2], Propositions 1 and 2. More elaborate theorems can be found in that source.

ditions are satisfied, establishes only a local maximum. It does not preclude that there may be some other value of the variables, a finite distance away, for which the function to be maximized has a still higher value. For reassurance on this point, one must inspect global rather than merely differential or local properties of the functions involved.

Finally, where the range of variation of the functions involved is limited in some manner, the maximum may be attained at a point where the partial derivatives do not vanish. This is a frequent occurence in economic applications, made familiar by linear programming. For example, it may be optimal for a firm with great growth possibilities to reduce its dividends to zero, though negative dividends are not permissible. In terms of our formulas this would be indicated by finding

$$\partial H/\partial x_t < 0 \quad \text{for all } x_t \geq 0,$$

where x_t denotes dividend payments per year at time t. H would be maximized by choosing $x_t=0$, its smallest permissible value, although the partial derivative does not vanish there.[15] This maximum could not be found by the ordinary methods of the calculus. Other methods are available, of course, for example those of mathematical programming. It is in just these circumstances that the maximum principle yields more elegant and manageable theorems than the older calculus of variations, which is more closely akin to the differential calculus.

For all these reasons, the fundamental condition for an optimal growth path is the maximization of $H(k, x, \lambda, t)$ at all moments of time, and the vanishing of $\partial H/\partial x$ is only an imperfectly reliable device for locating this maximum. It is, however, a very illuminating device and contains the conceptual essence of the matter, which is why we have concentrated on it.

[15] Technically this is called a "corner solution."

Throughout the discussion we have tried to be ambiguous about the exact nature of the time paths, $x(t)$ and $k(t)$. We have treated x and k as if they were one-dimensional variables, such as the quantity of capital or the rate of consumption. In many economic problems, however, there are several state variables and several choice variables. In such problems, it is profitable to think of $x(t)$, $k(t)$, their derivatives, and so on, as vectors. Then $\lambda(t)$ should also be regarded as a vector, with one component for each component of $k(t)$. When this viewpoint is taken, all our conclusions and the theorem still apply with scarcely a change in notation. That is why we were so ambiguous: it is easiest to think about ordinary numbers, but our conclusions and even most of our arguments are applicable when the variables are vectors.

The last remark raises some important new possibilities. Many economic problems concern the time paths of interconnected variables. For example, a problem may deal with the growth paths of consumption (c), investment (i), government expenditure (g), and income (y) in an economy. These four variables can be regarded as four components of a decision vector, x, connected by an income accounting identity $c(t)+i(t)+g(t)=y(t)$. Then the optimizing growth-path problem requires finding optimal growth paths for these four variables (and perhaps others) that satisfy the income accounting identity.

The new feature that we have encountered is the introduction of constraints or side conditions on the values of the decision variables. The same line of reasoning that we have been using applies, with the sole modification that when the function $V(k, x_t, t)$ is maximized, the vector x_t has to be chosen so as to satisfy all the side constraints. The algebra becomes somewhat more complicated but leads to conclusions like those discussed above and with the same economic import. In 1968, Kenneth Arrow derived a lucid version of the formal statement of the theorem applicable to problems in which the decision variables are constrained. See [2, Proposition 3, p. 90]. Of course, this argument, too, presumes that circumstances are such that the proper partial derivatives vanish at the maximum.

References

[1] K. J. Arrow, "Discounting and Public Investment Criteria," in A. V. Kneese and S. C. Smith, eds., *Water Research*, Washington 1966, pp. 13–32.

[2] ———, "Applications of Control Theory to Economic Growth," American Mathematical Society, *Mathematics of the Decision Sciences, Part 2*. Providence 1968, pp. 85–119.

[3] ———, and M. Kurz, *Public Investment, the Rate of Return, and Optimal Fiscal Policy*. Stanford University Institute for Mathematical Studies in the Social Sciences, 1968.

[4] R. Bellman, *Dynamic Programming*. Princeton 1957.

[5] H. Halkin, "On the Necessary Condition for Optimal Control of Nonlinear Systems," *Journal D'Analyse Mathématique*, 1964, *12*, 1–82.

[6] H. Hotelling, "A General Mathematical Theory of Depreciation," *J. Amer. Statist. Ass.*, Sept. 1925, *20*, 340–53.

[7] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes*, (tr. by K. N. Trirogoff). New York 1962.

[8] F. P. Ramsey, "A Mathematical Theory of Saving," *Econ. J.* Dec. 1942, *38*, 543–59.

[8] J. Robinson, *The Accumulation of Capital*. Homewood, 1956.