Investment and rate of return for the regulated firm

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The subject of this paper is a public utility's optimal dynamic response to the rate of return allowed by a regulatory commission. The problem is formulated in terms of the firm's capital-budgeting policy, and a dynamic analysis is made using modern control theory. This permits the utility's financial policies to be related directly to (a) the way the capital market values the utility and (b) the restriction on earnings imposed by rate-of-return regulation. The solution to the resulting nonlinear control problem provides a quantitative basis for analyzing the impact that small changes in the rate of return have on market valuation and the firm's investment policy. The results furnish one measure of what is commonly called "capital attraction capability," or the firm's ability to attract investment capital in a competitive capital market, and display the relation between that capability and the allowed rate of return.

1. Introduction

■ Because regulatory policy judgements are casuistic in substance they have generally been considered beyond quantitative representation or assessment. However, recent years have seen attempts to apply "scientific method" to the regulatory process through the use of mathematical models. This does not mean human judgement can or should be eliminated. But it may be possible to supplement and buttress judgement so as to improve the fairness and consistency of regulatory decisions.

The purpose of this paper is to present a quantitative framework showing what effect a rate-of-return specification can have on the structure and composition of the regulated firm's long-range investment decisions and the value the capital market assigns to the firm. From this some consequences of regulatory rate specifications can be seen.

The central task is to construct and solve a formulation for a regulated utility in terms of its capital budgeting. This makes it possible to link the utility's investment decisions to (1) the way the capital market values the utility and (2) the restriction that rate-of-return regulation imposes on earnings. This rate-of-return specification appears explicitly in the optimal strategies of the solution to the capital-budgeting model. The allowed rate not only affects the

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values of the strategies, it also affects the very structure of the solution in that totally different solutions result from different ranges in the rate of return. The impact of small changes in the rate of return on capital budgeting policy and on the market's valuation of the firm has long been observed in practice. This paper furnishes a theoretical framework from which these findings can be analyzed quantitatively.

The results yield one quantitative measure of a "fair" rate of return. It has long been recognized that this rate must be sufficient to attract capital for the growth requirements of utility demand. This appears explicitly in some of the solution cases.

2. Rate-of-return regulation

■ Determining the "fair return" to be allowed a privately owned, regulated company is a complex but important task. In recent times regulatory agencies have regulated earnings by (1) determining the value of the utility's outstanding investment and (2) deciding upon a "fair" rate to apply to that value. The product of value and rate of return is the net revenue constituting a fair return. Price schedules are then specified to yield this net revenue. As could be expected, this vital series of calculations has been subject to an array of political, judicial, economic, and traditional influences whose subtleties almost defy comprehension.

The question what constitutes "fair" in fair rate of return has been the subject of controversy between utilities and government agencies since regulation by statute appeared on the American scene a little over a hundred years ago. Since then a long series of court decisions, culminating with the Hope case, have set forth the "comparable earnings" and "attraction of capital" standards with which the allowed rate of return must comply. An excellent discussion of these standards, and of others implied by them, appears in [17].

The usual way of calculating a fair rate of return is to (1) determine the rate of interest paid on long-term debt, and then (2) combine this with a rate on equity capital. The two must be weighted according to the relative proportions of the two components in the utility's total capitalization. The heart of the rate-of-return question obviously lies in the determination of the rate on equity capital. The complex problem of finding the utility's total capitalization is not treated here. It is assumed that the assets of the utility are properly valued as equal to their opportunity costs in production in some other activity in the economy.

3. General description of problem

■ Attempts to analyze mathematically regulation's impact on the operations and investment policies of utilities can be generally divided into two classes. First, there are formulations using the neo-classical economic models of the firm in terms of production functions and profit maximization. Second, there are formulations concerned with static analysis of the utility's financial structure; models of this type characteristically involve no optimization operations.

Probably the best-known paper in the first class is one by H. Averch and L. Johnson [2], who use the neo-classical model of the firm to demonstrate that rate-of-return regulation tends to produce capital-intensive investment decisions when the firm's objective is

profit maximization. Westfield has presented a similar model [24], analyzing the effects of changes in the price of capital and demonstrating conditions under which overinvestment is optimal. Recent investigation by Tasch [23] using this same framework showed the crucial nature of many of the assumptions in [2] and [24].

Among financial models of regulated utilities, a paper by Sparrow [22] presents an analysis specifying what the rate of return would have to be to support specified growth rates of financial indicators recognized as important by investors. Sparrow's model requires no optimization operations and is a generalization and synthesis of two earlier papers [4] and [20].

An econometric approach has been presented by Gordon [7]. His financial model does involve optimization. The formulation is a static analysis on an econometric model of share price.

A recent published attempt to construct a financial model of a firm that integrates capital budgeting with the firm's valuation is a paper by Lerner and Carlton [16]. Their formulation is a static analysis using an investment-opportunities schedule under common-stock price maximization. Their results have been criticized, partly because they depend on the specific form of the investment-opportunities schedule the authors used.

There appear to be no published efforts at a dynamic financial analysis of the firm comprehending both market valuation and capital-expansion considerations. Such an effort, for the special case of a regulated public utility, is the purpose of this paper.

The basic economic problem is to determine the amounts of retained earnings and new equity capital (proceeds from securities issues) allocated to investment under rate-of-return regulation. Investment will increase future earnings. But retention of earnings reduces current dividends, and new equity issues tend to dilute the current owner's equity. Thus, the management's control problem is to choose the investment program that most benefits the owners.

This capital-budgeting problem is formally posed in section 4 as a problem in optimal control. The model is a dynamic, continuous-time model of the utility, encompassing operations and investment in an analysis of financial activity. The formulation is entirely financial in nature, which distinguishes it from an economic model of the firm in that it subsumes the production function and assumes that the utility operates along its optimal expansion path. Optimal expansion in this analysis is concerned exclusively with capital accumulation and its development over time.

The choice of the optimal investment program is constrained by two differential equations describing the change in stock price and equity per share (net worth divided by the number of shares outstanding). These constraints include behavioral assumptions pertaining to market valuation and the utility's operations. Retained earnings and/or new outside capital increase the firm's investment capital. The maximum investment in one period is limited by an upper bound, which can be regarded as the maximum rate at which the utility can efficiently increase its capital and still generate its allowed rate of return.

4. The model

■ Three fundamental differential equations are developed. These show the changes in financial structure under conditions of internal and external financing. The utility chooses the investment program that is most beneficial to the owners, and this is stated in mathematical terms. The development uses the following definitions:

$B(t)$ = total equity,
$N(t)$ = number of shares of common stock,
$E(t)$ = equity per share ($= B(t)/N(t)$),
$X(t)$ = total net earnings,
$d(t)$ = dividend per share,
$r(t)$ = rate of return to equity capital,
$P(t)$ = market price per share of common stock.

All of these variables are functions of time, t.

First, consider the effects of investment on equity. Retained earnings are an effective increase in equity and can be expressed as

$$X(t) - N(t)d(t)$$

or, equivalently, as

$$N(t)[E(t)r(t) - d(t)] .$$

Equity can also be increased by issuing new stock. Let δ denote the discount on market price to enhance the marketability of the new issue. The quantity δ should be interpreted as the average or effective discount, since discounting can take many forms. Internally it represents the cost per share of marketing the issue. Thus

$$0 \leq \delta < 1 ,$$

with

$$(1 - \delta)P(t) = \text{the effective price of the new issue,}$$

and

$$(1 - \delta)P(t)\dot{N}(t) = \text{dollar value of new outside equity raised.}^{1}$$

Thus, the change in equity can be expressed as

$$\dot{B}(t) = N(t)[E(t)r(t) - d(t)] + (1 - \delta)P(t)\dot{N}(t) .$$

It follows directly from the definitions that

$$\dot{B}(t) = N(t)\dot{E}(t) + \dot{N}(t)E(t)$$

is an alternative form of this change. Combining these two expressions yields

$$\dot{E}(t) = E(t)r(t) - d(t) + [(1 - \delta)P(t) - E(t)]\frac{\dot{N}(t)}{N(t)}$$

which denotes the change in equity per share. This equation describes the change in owner's equity when the firm both retains some earnings and also raises new equity capital. The only behavioral assumption is embedded in the assumption that δ is a known constant.

Investment by retention and equity financing also affects the price of a share. Unlike the equity equation above, the price equation to be developed is almost exclusively behavioral. It is an attempt to represent mathematically the mechanism of market valuation of publicly owned firms.

[1] A dot over any variable denotes the first time derivative of that variable; i.e., $\dot{N} = dN/dt$.

Proceeding along traditional lines,[2] the market mechanism can be stated as follows. The marketability of a stock is governed by (1) the expectation of post-purchase price appreciation and (2) the present level of dividends. This mechanism operates in any trading period to adjust the price of a share to equal the combination of expected price appreciation and current dividends, discounted by a rate applicable to investments of comparable risk. The following traditional assumptions will facilitate a more formal representation of this mechanism.

1. Investors always prefer more wealth to less and are indifferent between dividends and capital gains.

2. No single buyer or seller of shares is large enough to influence the market price significantly. Also, no special costs are incurred in the actual trading mechanism, and there are no tax differentials that could influence preference between dividends and capital gains. (The assumption that any new stock issued by the utility at a discounted market price will not affect the prevailing market price is implied.)

Under these assumptions, and given sufficient time for trading, the following must hold [20]:

$$\hat{P}(t+1) - P(t) + d(t) = \rho(t)P(t)\,,$$

where

$\hat{P}(t+1)$ = the expected price at the end of the trading period (assumed one unit in length)

and

$\rho(t)$ = investor discount rate.

This is an equilibrium condition and is called the "principle of valuation." It states that in any trading period the market will, given enough time, adjust the price so that dividends plus expected capital gains equal the rate of return the investor requires on an expenditure of $P(t)$.

The "sufficient time for trading" condition means that if

$$\hat{P}(t+1) - P(t) + d(t) - \rho(t)P(t) > 0,$$

then the market will react to increase the current price, and if

$$\hat{P}(t+1) - P(t) + d(t) - \rho(t)P(t) < 0,$$

then the market will react to decrease the current price.

To capture the above character of these equilibrating price adjustments, which counteract tendencies for inequality in the principle of valuation, a continuous-time representation will be used. The following differential equation exhibits this character:

$$\dot{P}(t) = c[d(t) - \rho(t)P(t)],\ c > 0\,,$$

where c can be thought of as a trading activity factor denoting how quickly the market responds to tendencies for inequality in the principle of valuation. This is the mathematical description of price change that will be used in the capital-budgeting model. The traditional assumption of "market certainty" has been made.

The rate $\rho(t)$ is often called the investor rate of return or investor capitalization rate for investments of comparable risk. It should not be confused with $r(t)$, the rate of return on equity capital.

[2] An example that uses this traditional point of departure is [20].

Normally, there are three sources of capital funds for investment:

1. retained earnings,
2. new outside equity,
3. debt or senior capital.

It is assumed that the utility has the option to use all three sources.

Let $A(t)$ represent the net assets of the utility at time t.

Then

$$A(t) = B(t) + Q(t),$$

where

$$Q(t) = \text{debt at time } t$$

and

$$B(t) = \text{equity at time } t.$$

Define

$$h(t) = \frac{Q(t)}{B(t)} = \text{debt-equity ratio.}$$

Then

$$A(t) = B(t)[1 + h(t)]$$

and

$$\dot{A}(t) = \dot{B}(t)[1 + h(t)] + B(t)\dot{h}(t).$$

Recall the development of the relationship for change in equity—

$$\dot{B}(t) = N(t)[E(t)r(t) - d(t)] + (1 - \delta)P(t)\dot{N}(t).$$

Substituting this into the relationship for change in assets—

$$\dot{A}(t) = [1 + h(t)]\dot{B}(t) + B(t)\dot{h}(t)$$

yields

$$\dot{A}(t) = [1 + h(t)]N(t)[E(t)r(t) - d(t)] + (1 - \delta)P(t)\dot{N}(t)[1 + h(t)] + B(t)\dot{h}(t).$$

Factoring out $[1 + h(t)]B(t) = A(t)$ and using the identity $B(t) = N(t)E(t)$ yields the relationship

$$\dot{A}(t) = A(t)\left[r(t) - \frac{d(t)}{E(t)} + (1 - \delta)\frac{P(t)}{E(t)}\frac{\dot{N}(t)}{N(t)} + \frac{\dot{h}(t)}{1 + h(t)}\right].$$

This is how the total assets of the utility will increase when any or all of the three sources of capital are used for investment.[3] The rate of change of the utility's asset base is represented by

$$\frac{\dot{A}(t)}{A(t)} = \text{rate of change of net assets,}$$

and this is the quantity which will be bounded under the assumptions to be imposed on conditions for maximum growth rates.

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[3] The instantaneous rate of change of assets, $\dot{A}(t)/A(t)$, is composed of the three capital sources available to the utility:

1 $r(t) - \frac{d(t)}{Et}$, the contribution from retained earnings;

2 $(1 - \delta)\frac{P(t)}{E(t)}\frac{\dot{N}(t)}{N(t)}$, the contribution from new equity issues;

3 $\frac{\dot{h}(t)}{1 + h(t)}$, the contribution from debt or senior capital.

The capital budgeting activities of a utility must be centrally concerned with preserving the financial integrity of that utility while attempting to meet the investment requirements to satisfy demand for service. The objective of its investment program must be one that is most beneficial to the suppliers of investment funds—old equity owners for internal sources, and new equity owners for external sources. The composition of any investment program should be structured to produce the greatest increase in the worth of their present holdings. This compensation principle is necessary to ensure confidence in the financial integrity of the utility so as to maintain its ability to attract and retain capital. This can be quantitatively stated

$$\text{Max} P(T) \exp[-\rho(T)T] + \int_{t_0}^{T} d(t) \exp\left[-\int \rho(u)du\right] dt,$$

where (t_0, T) represents the planning period of the capital budgeting program with t_0 representing the present time. (The assumption of investor (and owner) indifference between dividends and capital gains is also implied by this objective function.) This will be considered the objective of management in planning the investment program. The planning horizon, T, is quite arbitrary and can be considered infinite if desired. For the case of a bounded terminal price, infinite planning horizon, and constant market capitalization rate, the objective would appear as

$$\text{Max} \int_{t_0}^{\infty} d(t) \exp[-\rho t] dt.$$

Without the maximization operation this function is the classical definition of the price of a share of common stock.[4] Thus as the planning horizon tends to infinity, the objective of management is to choose investments to maximize the price of its stock, an objective function used frequently in static analysis of firm valuation.[5]

Standard and traditional assumptions have been made concerning the mechanism of firm valuation and the objective of the utility. Assumptions will now be made pertaining to the operations of the utility which yield quantitative instruments of control in capital expansion decisions. Special characteristics of a utility which limit some of the variables previously defined will be noted.
The operating assumptions are:

1. Dividends will constitute a proportion of earnings.[6] That is,

$$d(t) = [1 - u_r(t)]E(t)r(t),$$

where

$u_r(t)$ = the retention rate (a management control variable)

and

$$0 \leq u_r(t) \leq 1.$$

The retention rate is defined as the investment accrued from earnings, expressed as a fraction of current earnings.

2. The dollar value of new outside equity subscribed will be deter-

[4] The classical reference is [25].
[5] See [8], [9], and [16].
[6] The importance of dividend policy in financial models is not a settled issue. This model tends to emphasize its role. For another view, see [20].

mined as a proportion of current earnings.[7] This assumption allows retained earnings and new outside equity to be determined under the same conceptual structure. The assumption yields

$$u_s(t)N(t)E(t)r(t) = (1 - \delta)P(t)\dot{N}(t)\,,$$

where

$u_s(t)$ = stock financing rate (a management control variable),

and

$$u_s(t) \geq 0\,.$$

The stock financing rate is defined as the investment accrued from new equity issues expressed as a function of current earnings.

Limitations and restrictions on variables are:

3. The utility has a binding and meaningful rate-of-return constraint in force. Let

r = maximum allowable rate of return to equity capital.

For this constraint to be meaningful its value must exceed the interest rate on debt, and it is always advantageous for the utility to force its actual rate of return to the constraint. That is

$$r(t) = r \text{ for all } t$$

and

$$r > i \text{ where } i = \text{interest on long-term debt.}$$

4. There is presently in the literature no conclusive preference for describing the mechanism underlying changes in the debt-equity ratios of firms. However, once a debt-equity ratio is arrived at there is a tendency to maintain this ratio in future capital-expansion decisions. A constant debt-equity ratio will be assumed in this analysis. Empirical justification for this assumption can be found in [18] and [19].

5. The rate of growth of assets is constrained from above. Due to the special nature of the services and markets of a utility, this bound is assumed known and constant. That is,

$$\frac{\dot{A}(t)}{A(t)} \leq k.$$

6. The conditions of specified maximum growth rate, constant rate of return on equity, constant debt-equity ratio, and a utility type firm collectively infer that $\rho(t)$ should be very stable. If it is assumed that the market rate of interest on long-term debt is constant over the period of interest, then under most generally accepted hypotheses on what influences changes in $\rho(t)$,[8] it can be assumed that $\rho(t)$ will be constant. That is

$$\rho(t) = \rho.$$

7. Assume that the investor discount rate exceeds the growth rate. That is

$$\rho > k.$$

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[7] This assumption artificially simplifies the difficulty underlying the choice of size of an equity issue. Alternative views of a similar nature appear in [8] and [20].

[8] See [16] for a recent example.

If this relationship does not hold, the "growth paradox" can be encountered, which is not characteristic of firms in the utility sector.

Assumptions 1 and 2 deal with the internal operations of the utility. They are behavioral assumptions which lead to quantitative specification of managerial instruments of control for capital-budgeting decisions. Assumptions 3 through 7 collectively establish the conditions of stable economic stationarity in that the economic conditions encountered in the future will be much the same as they are at present. Thus the framework of the analysis to follow, while dynamic in context, assumes a stationary economic environment.

The capital budgeting problem to be solved can be descriptively stated: "Find the investment rate and composition which will maximize the present value of the owner's holdings when the company is a regulated public utility." Under the assumptions above, this can be mathematically stated as a problem in optimal control,[9] which is

$$\text{Max} P(T)\exp[-\rho T] + \int_{t_0}^{T} [1 - u_r(t)]rE(t)\exp[-\rho t]dt,$$

subject to

$$\dot{P}(t) = c[rE(t)(1 - u_r(t)) - \rho P(t)],$$

$$\dot{E}(t) = rE(t)\left[u_r(t) + u_s(t)\left(1 - \frac{E(t)}{(1-\delta)P(t)}\right)\right],$$

$$u_r(t) + u_s(t) \leq \frac{k}{r},$$

$$0 \leq u_r(t), \quad u_s(t),$$

and the initial conditions

$$P(t_0) = P_0 \quad \text{and} \quad E(t_0) = E_0.$$

The terminal condition is given by the fixed planning horizon T.

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[9] The control problem results directly from substituting assumptions (1) through (7) into the model developed. This can be demonstrated best by considering the differential equation

$$\frac{\dot{A}(t)}{A(t)} = r(t) - \frac{d(t)}{E(t)} + (1-\delta)\frac{P(t)}{E(t)}\frac{\dot{N}(t)}{N(t)} + \frac{\dot{h}(t)}{1+h(t)}.$$

Assumption 1) requires $d(t) = [1 - u_r(t)]E(t)r(t)$, thus

$$\frac{\dot{A}(t)}{A(t)} = r(t)u_r(t) + (1-\delta)\frac{P(t)}{E(t)}\frac{\dot{N}(t)}{N(t)} + \frac{\dot{h}(t)}{1+h(t)}.$$

Assumption 2) requires $u_s(t)N(t)E(t)r(t) = (1-\delta)P(t)\dot{N}(t)$, or $\dfrac{(1-\delta)P(t)\dot{N}(t)}{E(t)N(t)} = r(t)u_s(t)$. Making this substitution yields

$$\frac{\dot{A}(t)}{A(t)} = r(t)u_r(t) + r(t)u_s(t) + \frac{\dot{h}(t)}{1+h(t)}.$$

Assumption 3) requires $r(t) = r$ for all t.
Assumption 4) requires $\dot{h}(t) = 0$ for all t, thus

$$\frac{\dot{A}(t)}{A(t)} = ru_r(t) + ru_s(t).$$

Assumption 5) requires this quantity to be bounded by k, i.e.,

$$\frac{\dot{A}(t)}{A(t)} \leq k.$$

It follows directly from this that the controls $u_s(t)$ and $u_r(t)$ are constrained by the relation

$$u_r(t) + u_s(t) \leq \frac{k}{r}.$$

5. Preliminaries

■ Applications of control theoretic techniques to models of management control systems have not been plentiful. One reason may be the difficulty associated with the synthesis after the candidate solutions have been found. We deal with this difficulty here by an application of the construction technique of [11], cast in the control theoretic structure of the Maximum Principle [21]. This reverse time construction technique is particularly effective in solution synthesis in that the complete solution is "built up" quantitatively by construction from solution cases. The solution cases are identified by treating the analysis by the Maximum Principle as a mathematical programming problem at each instant of time. This integration of the analysis and the synthesis provides the systematic method for solution.

In constructing the solution, the terminal manifold is parameterized in terms of the state variables. By starting here and moving backwards in time, the entire state space is filled with optimal trajectories solving the problem for arbitrary initial states. This reverse time synthesis is guided by examining the solution cases encountered in the application of the Maximum Principle. For this model the solution cases are determined by a linear program solved by inspection. The approach here is similar to that of [6]. See also [11]. The Maximum Principle is also modified by an approach [1] to allow ease in handling discounted objective functionals. To demonstrate this, and to briefly review the general approach, the necessary conditions for optimal control are reviewed below.

Without the exponential discount term in the objective function, the capital budgeting problem would be of the form

$$\begin{aligned} \underset{U(t)}{\text{Max}}\ V(X,t_0) &= G[X(T),T] + \int_{t_0}^{T} L[X(t),U(t),t]dt \\ \dot{X}(t) &= f[X(t),U(t),t] \\ U(t) &\in \Omega \\ X(t_0) &= X_0 \\ T &\text{ fixed}. \end{aligned}$$

The optimal control function, $U^*(t)$, can be found by applying the Maximum Principle [10], [21], formally stated as follows .Let $U^*(t)$ be the optimal control function which maximizes $V(X,t_0)$. Then for $U^*(t)$ to be optimal it must necessarily satisfy

$$\dot{X}^*(t) = \frac{\partial H}{\partial \lambda},$$

$$\dot{\lambda}^*(t) = -\frac{\partial H}{\partial X},$$

$$\lambda^*(T) = \frac{\partial}{\partial X}[G[X^*(T),T]],$$

$$\dot{H} = \frac{\partial H}{\partial t},$$

$$X^*(t_0) = X_0,$$

and

$$H[X^*(t), U^*(t), \lambda^*(t), t] \geq H[X^*(t), U(t), \lambda^*(t), t]$$

for all

$$U(t) \in \Omega$$

(vector notation is used), where H is known as the Hamiltonian function defined as

$$H = G[X(t), U(t), t] + \lambda(t) \cdot f[X(t), U(t), t]$$

and $\lambda(t)$ is commonly referred to as the adjoint vector. In using the Maximum Principle, one attempts to derive a relationship between the optimal control and the state and adjoint variables. In general this relationship may be expected to yield

$$U^*(t) = g[X^*(t), \lambda^*(t)] .$$

If this relationship can be uniquely determined, then this represents the nonsingular case. If, however, the direct application of the Maximum Principle does not lead to a unique determination of $U^*(t)$, then a singular situation results. The most prevalent case is when the Hamiltonian is trivially satisfied for any $U(t)$ over some finite time interval. This is traditionally known as the problem of singular controls. This condition has been investigated in a number of articles, of which [12] is representative. Probably the most extensive treatment of singularities is contained in [11].

When a singular control situation is encountered, additional analysis is required to generate the singular trajectory. It is necessary that

$$\frac{\partial}{\partial U} \frac{d^2}{dt^2} \left[\frac{\partial H}{\partial U} \right] \geq 0 .$$

This relation is known as the Generalized Legendre-Clebsch Condition: [12] and [14].

The capital budgeting problem as stated, however, has an exponential discount term in the objective functional. The introduction of this exponential into the objective functional can be troublesome in terms of analysis by the Maximum Principle. However, a technique discussed in [1] mitigates this situation and will be treated below.

Recall the definition of $V(X, t_0)$. Let $V^*(X, t_0)$ be the maximum value of $V(X, t_0)$. Then from the Hamiltonian-Jacobi partial differential equation [3] for the optimal V, the following holds,

$$\frac{\partial V^*}{\partial X} = \lambda^* ,$$

if a unique $V^*(X, t_0)$ solution of sufficient smoothness exists. Now consider the case where the objective function is discounted. That is, consider

$$V(X, t_0) = G[X(T), T] \exp[-\rho T] + \int_{t_0}^{T} L[X(t), U(t), t] \exp[-\rho t] dt.$$

This formulation requires that the returns L and G be discounted to time 0. It is more natural to discount them to t_0 since this is the beginning of the range of interest. To do this, define the current

value function

$$W(X, t_0) = \frac{V(X, t_0)}{\exp[-\rho t_0]}.$$

Assuming a unique optimal solution of sufficient smoothness exists, then it appears more natural to define the adjoint variables in terms of the current-value function:

$$\lambda = \frac{\partial W}{\partial X} = \frac{\partial V}{\partial X}\left[\frac{1}{\exp[-\rho t]}\right].$$

Now application of the Maximum Principle to the discounted objective functional yields

$$\hat{H} = \exp[-\rho t]H(X, U, \lambda, t)$$

where H is the Hamiltonian for the undiscounted case. Application of the necessary conditions requires a change only in the adjoint equation. The condition now must be stated as

$$\frac{d}{dt}[\exp(-\rho t)\lambda(t)] = -\frac{\partial}{\partial X}\hat{H},$$

from which the following results:

$$\dot{\lambda}^*(t) = -\frac{\partial H}{\partial X} + \rho\lambda^*.$$

Now equation $\hat{H}$ is maximized by choice of $U(t)$, the control vector, and that is equivalent to maximizing H, the undiscounted Hamiltonian. The introduction of a discount term in the objective functional can be handled by a change in the equation defining the adjoint variable and then proceeding as required in the undiscounted case.

6. Solution of the control problem

■ In order to center attention on the economic evaluation of the solution of the capital-budgeting problem, this section bypasses the analytical complexities of solution synthesis in nonlinear optimal control problems. An outline of the synthesis technique which leads to the solution is presented in the appendix, and the synthesis is available in full detail in [5].

The Maximum Principle provides a means of finding the optimal control function under the implicit assumption that optimal control exists. This question of existence arises since the Maximum Principle requires the existence of the adjoint vector. If optimal control exists, then the existence of the adjoint vector is assured. This adjoint vector has a central role in the actual determination of the optimal control function. In general, existence of optimal control functions for nonlinear systems is difficult to assure. However, the existence of the optimal control function for this problem is established.[10]

[10] See [15], page 262, for proof of this statement.

Application of the Maximum Principle with the modification to remove the discount term requires that the optimal u must satisfy[11]

$$\text{Max}H = Er + c\lambda_P[Er - \rho P] + Er[\lambda_E - c\lambda_P - 1]u_r + Er\lambda_E\left[1 - \frac{E}{(1-\delta)P}\right]u_s$$

where

$$\dot{\lambda}_P = (c+1)\rho\lambda_P - \left(\frac{E}{P}\right)\frac{\lambda_E Er}{(1-\delta)P}u_s\,, \tag{1}$$

$$\dot{\lambda}_E = -r(c\lambda_P + 1) + \rho\lambda_E - r(\lambda_E - c\lambda_P - 1)u_r - r\lambda_E\left[1 - \frac{E}{(1-\delta)P}\right]u_s + \frac{\lambda_E Eru_s}{(1-\delta)P}\,, \tag{2}$$

$$\dot{P} = c[Er(1-u_r) - \rho P]\,, \tag{3}$$

$$\dot{E} = Er\left[u_r + u_s\left(1 - \frac{E}{(1-\delta)P}\right)\right], \tag{4}$$

and

$$u_r + u_s \leq \frac{k}{r}\,, \quad u_r \geq 0\,, \quad u_s \geq 0\,,$$

with boundary conditions

$$P(t_0) = P_0\,, \quad E(t_0) = E_0\,, \quad \lambda_P(T) = 1\,, \quad \lambda_E(T) = 0\,.$$

To facilitate the analysis, H may be written as

$$H(\lambda, P, E, u) = h(\lambda, P, E) + S_r(\lambda, P, E)u_r + S_s(\lambda, P, E)u_s\,,$$

where

$$h(\lambda, P, E) = rE + c\lambda_P(rE - \rho P)\,,$$

$$S_r(\lambda, P, E) = rE[\lambda_E - c\lambda_P - 1]\,,$$

and

$$S_s(\lambda, P, E) = rE\lambda_E\left[1 - \frac{E}{(1-\delta)P}\right].$$

The maximization of H with respect to u can be characterized by the switching functions, S_r and S_s, which are the components of the gradient of H with respect to u_r and u_s respectively.

The feasible controls lie in the triangle bounded by

$$u_r = 0\,, \quad u_s = 0 \quad \text{and} \quad u_r + u_s = \frac{k}{r}\,.$$

Thus the maximization of H with respect to u (at a particular time) is a linear programming problem which may be solved by inspection. Depending on the values of S_r and S_s, the unique solutions are on

[11] Throughout the analysis the time argument is omitted unless obvious ambiguity results.

the vertices (the bang-bang solutions) and the non-unique solutions are on the faces or in the interior (the singular solutions). The solution cases are listed below.[12]

		CONDITION
a)	$S_r > 0\,, \quad S_r > S_s$	$u_r^* = \frac{k}{r}\,, \quad u_s^* = 0$
b)	$S_s > 0\,, \quad S_s > S_r$	$u_r^* = 0\,, \quad u_s^* = \frac{k}{r}$
c)	$S_r < 0\,, \quad S_s < 0$	$u_r^* = 0\,, \quad u_s^* = 0$
d)	$S_r = 0\,, \quad S_s < 0$	$0 \leq u_r^* \leq \frac{k}{r}\,, \quad u_s^* = 0$
e)	$S_r < 0\,, \quad S_s = 0$	$u_r^* = 0\,, \quad 0 \leq u_s^* \leq \frac{k}{r}$
f)	$S_r = S_s > 0$	$u_r^* \geq 0\,, \quad u_s^* \geq 0\,, \quad u_r^* + u_s^* = \frac{k}{r}$
g)	$S_r = 0\,, \quad S_s = 0$	$u_r^* \geq 0\,, \quad u_s^* \geq 0\,, \quad u_r^* + u_s^* \leq \frac{k}{r}$

The singular solutions (conditions d through g) require further analysis for a unique determination of the control variables (see appendix).

By using the conditions on the switching function, solution synthesis can be performed by the reverse-time construction technique. The solution will be presented by demonstrating the effects on the state variables under conditions of optimal control. Displaying the state variables in terms of price-equity ratios permits a more comprehensive understanding of movement in the state space. The solution will be displayed in a plot of P/E versus time.[13]

In performing the solution synthesis it was found that the solution depended critically on the relationship between r and ρ. Five solution cases will be displayed in terms of the ratio r/ρ. These are

1. $\frac{r}{\rho} \leq 1$

2. $1 < \frac{r}{\rho} < \frac{1}{1-\delta}\,, \quad \delta > 0$

3. $\frac{r}{\rho} = \frac{1}{1-\delta}$

[12] Cases *a*, *b*, and *c* are the nonsingular solutions, while cases *d* through *g* are singular.

[13] Complete representation of the state space requires P/E, time, and either P or E itself. However, the salient features of the solution can be shown in the P/E versus time plot, which avoids the complexities of three-dimensional plots. These are, of course, price-earnings ratios adjusted by a constant.

$$4.\quad \frac{1}{1-\delta} < \frac{r}{\rho} \leq \frac{c + [c(1-\delta)+1]\frac{k}{\rho}}{c(1-\delta)}$$

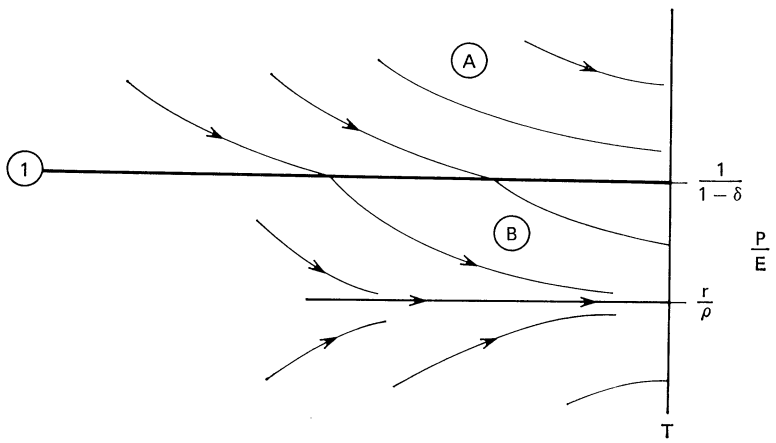
$$5.\quad \frac{r}{\rho} \geq \frac{c + [c(1-\delta)+1]\frac{k}{\rho}}{c(1-\delta)}$$

Each of these cases displays a significant change in the structure of the solution. The vector field representation of the price-equity ratio will be concentrated around the value $1/1 - \delta$. This is where the major changes in movement occur. For ratios significantly outside this range it will be obvious from the vector field how it could be smoothly extended to attain an excluded starting position.

Each solution case will be displayed as a planar vector field with time as the abscissa and price-equity ratio as the ordinate. The terminal manifold, T, bounds the field with starting positions left arbitrary. Capsule descriptions accompany each solution case. A more general discussion of the solution cases appears at the end of this section. Economic interpretation of the solution cases appears in section 7.

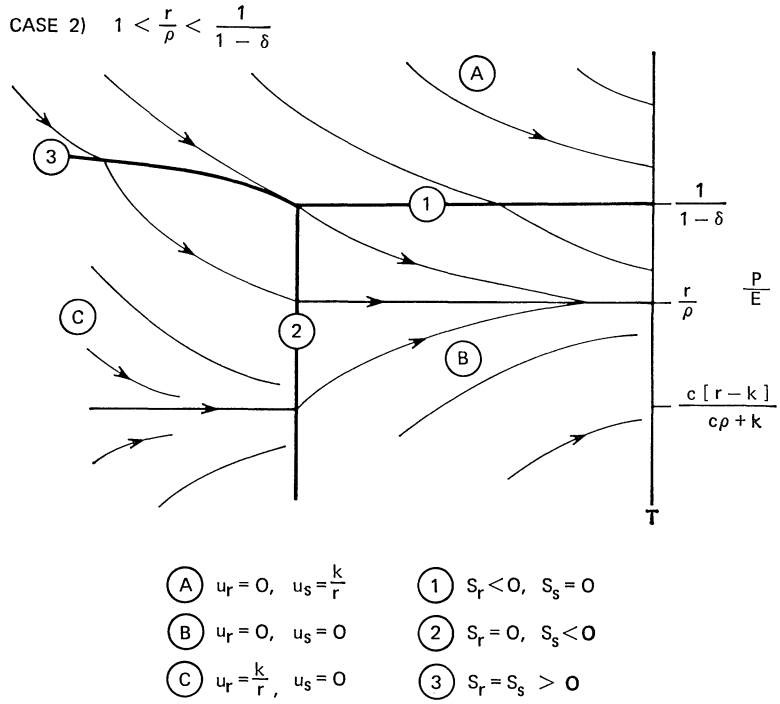
The following are the solution cases.

CASE 1) $\frac{r}{\rho} \leq 1$

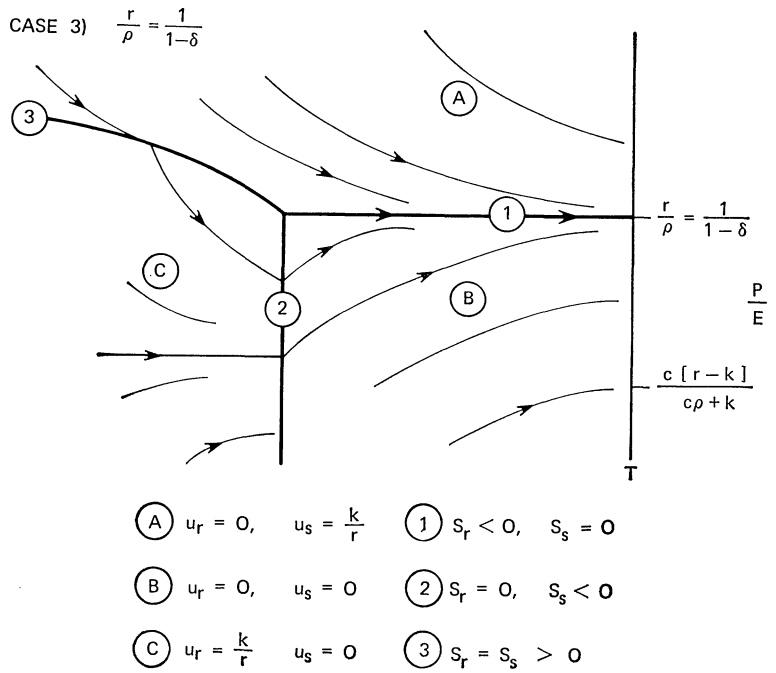


(A) $u_r = 0,\ u_s = \frac{k}{r}$ (B) $u_r = 0,\ u_s = 0$ (1) $S_r < 0,\ S_s = 0$

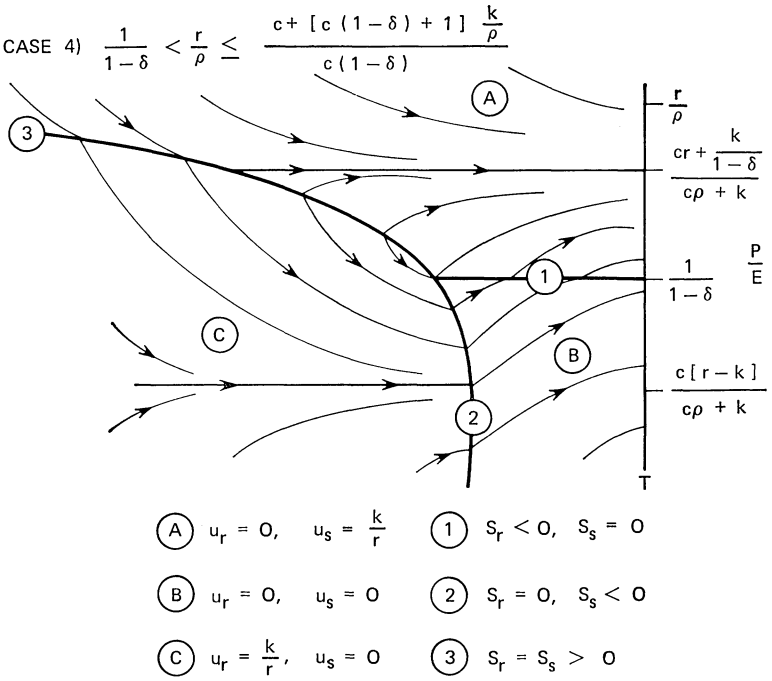
Solution case 1 has simple structure with a switching manifold at $P/E = 1/1 - \delta$. All optimal trajectories tend towards the line $P/E = r/\rho$. The trajectories in Region B are asymptotic to r/ρ; they do not intersect it. Thus the price-equity ratio tends towards r/ρ for all t. For sufficiently large T, this position will be approached yielding $\dot{P} = \dot{E} = 0$. There will be no growth, dividends will be equal to earnings and constant, and the equilibrium P/E ratio will be less than or equal to one.



The structure of solution case 2 is more complex due to the switching manifolds at $S_r = 0$, $S_s < 0$ and $S_r = S_s > 0$. It should be noted that $S_r = 0$, $S_s < 0$ will be well removed from T. For starting positions in region C retention of earnings is optimal. The optimal trajectories are again asymptotic trajectories in regions C and B. Noting that $S_r = 0$, $S_s < 0$ is well removed from T, this case is essentially the same as case 1 with essentially no growth for the bulk of the capital-budgeting program duration. Observe that in this case the manifold $S_r = S_s > 0$ appears and is a switching manifold.



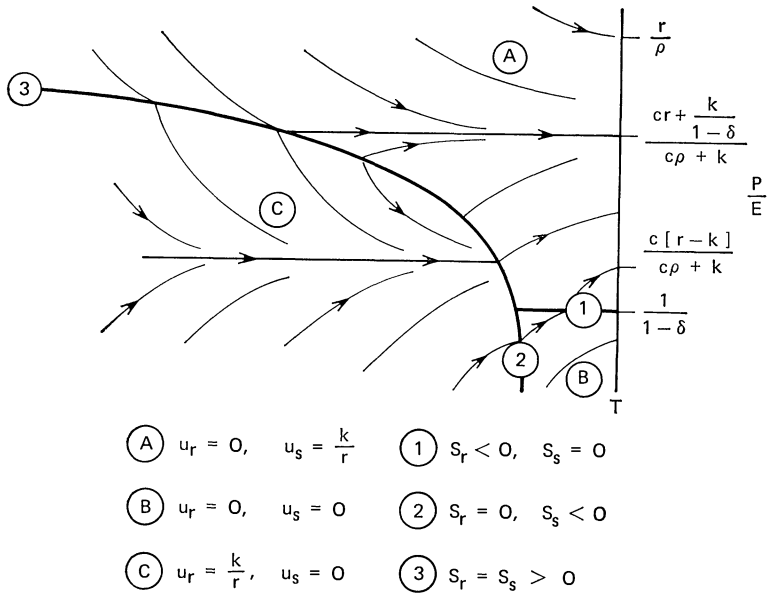
Solution case 3 is essentially the same as case 2 except that the manifold $S_r < 0$, $S_s = 0$ is singular and movement on it is optimal. The trajectories surrounding it are asymptotic to it; they do not intersect it. Again the optimal trajectories are asymptotic in region C, as was true in the other cases.



Solution case 4 shows a marked departure from the structures of cases 1, 2, and 3. In this solution, the manifold $S_r = S_s > 0$ reveals a choice in optimal strategy for a range of starting positions on it. Either choice yields the same value in terms of the objective. Movement along the surface is nonoptimal. Surfaces of this type were first demonstrated by Isaacs [11] in the context of differential games. He has named them dispersal surfaces. The optimal vector field of trajectories in this case is more complicated and clearly demonstrates the nonuniqueness of the solution for this locus of possible starting positions. Although the price-equity ratio decreases along some of the optimal trajectories, leaving the manifold $S_r = S_s > 0$, one should not automatically conclude that the price is decreasing throughout the trajectory length in region C. It can be either increasing or decreasing, depending on the starting position and the parameter values. If it is increasing, it will do so at a slower rate than the equity increase. As before, the optimal trajectories are asymptotic.

Case 5 is a direct extension of case 4 for larger values of r. The essential difference is that now the terminal price-equity ratio will normally be greater than $1/1 - \delta$ for most programs except for very low price-equity starting positions. This case demonstrates the range of r needed to guarantee this condition. Although not a part of this formulation, it is well recognized that price-equity ratios less than one are avoided in practice if possible.

CASE 5) $\frac{r}{\rho} \geq \frac{c + [\,c(1-\delta) + 1\,]\frac{k}{\rho}}{c(1-\delta)}$



The solution structures can generally be divided into two similar classes, cases 1, 2, 3, and cases 4 and 5. For rates of return in the range of $r/\rho = 1$ (cases 1, 2, 3), the solution structure is quite simple from a control theoretic point of view. The economic significance of rates of return in this range is discussed in the next section. When the rate of return is somewhat higher than $r/\rho = 1$ (cases 4 and 5), the solution structure becomes complex, especially in the region containing the manifold $S_r = S_s > 0$. An economic interpretation of this manifold is given in the next section.

7. Economic interpretation of results

■ Economic interpretation in control applications has been particularly rich in macro growth models.[14] This is generally accomplished by viewing the adjoint variables as time varying Lagrange multipliers. The results of this analysis can be interpreted in an analogous manner by identifying the adjoint variables in a similar vein.

From the investors' point of view the "product" of the utility is the value accrued from share ownership. Let $V(P, E)$ be the "product," where

$$V(P, E) = P(T)\exp[-\rho T] + \int_{t_0}^{T} [1 - u_r(t)]E(t)r\exp[-\rho t]dt\,.$$

Now

$$\frac{\partial V}{\partial P} = \text{marginal "product" with respect to price},$$

and

$$\frac{\partial V}{\partial E} = \text{marginal "product" with respect to equity}.$$

From the Hamiltonian-Jacobi partial differential equation for the

[14] See [1] and [6] as recent examples.

optimal V (see Section 5),

$$\frac{\partial V^*}{\partial P} = \lambda_P$$

and

$$\frac{\partial V^*}{\partial E} = \lambda_E .$$

Thus the marginal products at the optimal V are equal to the adjoint variables. The state variables P and E should here be considered as inputs to a structure which produces the "product," $V(P,E)$. It has been assumed that these inputs are perfectly competitive (the investor is indifferent between capital gains and dividends, with the latter accruing directly from equity). Using the economic argument that in the case of competitive inputs the proper assigned price of an input must be equal to its marginal product, the adjoint variables can be thought of as shadow prices. This gives an economic meaning to maximizing the Hamiltonian at every instant of time as a necessary condition for an optimal solution over time. By their actions via their investment policies, "management" can "give" to the shareholder at each instant of time, dividends, a change in current share price which will effect the terminal price, and a change in equity from which future earnings can accrue. The value of this at time t is simply the sum of the three weighted by their respective shadow prices or values, i.e.,

$$d(t) + \lambda_P(t)\dot{P}(t) + \lambda_E(t)\dot{E}(T) .$$

This is the implicit current value of share ownership, assuming the shadow prices are given. This is also the Hamiltonian of the control problem (see section 6). Thus by maximizing the implicit value of share ownership at every instant of time, the actual value of share ownership over time is optimal. By rearranging the above into the form used in section 6, it is now seen that the switching functions, $S_i(P, E, \lambda_P, \lambda_E)$, provide the utility management with the necessary comparison to make this maximization since their instruments of control are implicit in all the terms. The most interesting case is

$$S_r = S_s > 0 .$$

Here any comparison is inconclusive on how to maximize the implicit instantaneous value of ownership. To resolve this case the "managers" should look at the first time derivative of S_r and S_s. If they are not equal, then the condition is a transient one and in the control theoretic sense the manifold is a switching surface. This condition is encountered in solution cases 1, 2, and 3. If, however, the first time derivatives of S_r and S_s are equal, this test does not reveal the optimal strategy. It is here that the dispersal surface demonstrated in solution cases 4 and 5 can arise. From an economic point of view, either strategy leaving the manifold $S_r = S_s > 0$ is optimal as the comparison yields the same value. The resolution of the question of maintaining the condition is provided by the Generalized Legendre-Clebsch Condition.

Thus the dispersal surface is the bearer of nonuniqueness in the optimal solution to the capital-budgeting problem. This condition

has its analog in the static analyses of capital budgeting in the following sense. In most static formulations which include internal as well as external equity financing, the assumption is implicit that either form of financing is equally advantageous in that no special preference for one is obvious. This condition corresponds to, in the context of this analysis, the firm being on the manifold $S_r = S_s > 0$. Thus statically all controls are optimal and this situation yields the prevalent indeterminant solution found in static analyses. Dynamically, the situation has been reduced to nonuniqueness since there are two equally valid optimal policies to follow.

There will be no unique trajectory of capital expansion reflecting equilibrium as the planning horizon approaches infinity. However, there can be two trajectories which closely approximate this condition of constant price-equity ratio and constant dividend payout ratio. This condition yields a price appreciation equal to dividend growth, with both being constant proportions of asset growth. This condition reflects equilibrium in the capital market. Here again the nonuniqueness of the capital-budgeting firm valuation solution is reflected.

The necessary conditions for optimal control can be interpreted as providing an economic equilibrium condition in the following sense. It has been shown that maximizing the Hamiltonian can be interpreted as maximizing the implicit value of current share ownership. If the shareowner is not at an economic equilibrium point by this maximization, then he would have wanted more or less equity and/or price change than the maximization provided. To assure that this implicit value of share ownership cannot be increased by a shift in equity or price, it is necessary that the derivative of the maximal Hamiltonian with respect to each state variable be zero. Doing this operation, the adjoint variables are again interpreted in accordance with

$$\lambda_P{}^* = \frac{\partial V^*}{\partial P}$$

and

$$\lambda_E{}^* = \frac{\partial V^*}{\partial E}.$$

Under the conditions of this problem, i.e., the optimal controls are constant and assuming that the second partial of V exists, the necessary conditions for the Hamiltonian to be maximal in terms of changes in the state variables are

$$\frac{d}{dt}\left[\frac{\partial V^*}{\partial E}\right] + \frac{\partial H}{\partial E} = 0$$

$$\frac{d}{dt}\left[\frac{\partial V^*}{\partial P}\right] + \frac{\partial H}{\partial P} = 0.$$

Note, however, that these equations are the Euler-Lagrange equations of the necessary conditions for optimal control [see section 5]. Thus the necessary conditions of optimal control require the implicit value of share ownership to be maximal and the shareholder to be at an economic equilibrium point at every instant of time.

■ The main purpose of this paper has been to provide a framework to show the effect of a rate-of-return specification on the structure and composition of long-range investment decisions and the valuation of a utility. The results clearly demonstrate how dramatically this specification can affect these decisions. For relatively small changes in rate of return, totally different investment programs become possible and optimal. This impact of small changes in rate of return on capital budgeting decisions and utility valuation has long been observed empirically, and this paper provides one theoretical framework from which to analyze possible consequences quantitatively. Two observations on the solution cases warrant special attention.

Consider cases 1 through 3, where the rate of return to equity is either less than or very close to the investor discount rate. The analysis shows that, in the main, the optimal policy will be to pay out all earnings in dividends and never re-invest internally. External investment is only desirable as long as a transient favorable price-equity ratio exists. This result is consistent with economic theory, since it is well known in financial analysis that if the internal rate of return for an investment is less than the external or market rate of return for investments of comparable risk, then an alternative investment is more desirable. An alternative name for ρ, the investor discount rate, is the market rate of return for investments of comparable risk. These solution cases represent the situation of a "non-growth" return on capital. If sustained growth of the utility is to be encouraged, then the rate of return must be set above this "bare bones" level. One interpretation of this is that a "fair" rate of return must satisfy at least the inequality

$$r > \frac{\rho}{1 - \delta},$$

where the "bare bones" level would be defined as

$$r = \frac{\rho}{1 - \delta}.$$

Solution cases 4 and 5 seem to characteristically represent the central range of rates of return. It has long been recognized that a fair rate of return to equity capital must reflect in some manner the growth requirements of a utility from a viewpoint of capital attraction capability [17]. Note that this factor, k, is explicitly included in cases 4 and 5 and actually determines in part the defining range of rate of return. The nonuniqueness of the capital-budgeting solution appears in these cases. Also, any rate-of-return specification which acknowledges continuous growth considerations will be found here.

It is quite reasonable to interpret k as reflecting the rate of investment-need as supported by some aggregate demand for service. One way of ensuring a consistent growth at rate k over the planning horizon is to set

$$r = \frac{\rho}{1 - \delta} + k\left[1 + \frac{1}{c(1 - \delta)}\right].$$

This is the upper bound for solution case 4 and effectively prevents

the utility from entering region B of case 4, the "non-growth" area, for characteristic initial starting positions. It also forces a price-equity ratio close to $1/1 - \delta$, a relationship which has some empirical support.

Notice that this rate of return is a sum of the "bare bones" level plus a factor which reflects the growth requirements. Thus if no growth is warranted, i.e., $k = 0$, the "bare bones" static case results. Also, this rate of return reflects the need for rate-of-return adjustments as economic conditions change, as reflected by ρ for changes in "riskiness" and k for changes in growth requirements. This rate of return seems to satisfy the "fair" rate-of-return requirements [17]:

1. comparable earnings standard for investments of comparable risk, (ρ),
2. attraction of capital standard to reflect growth requirements, (k).

These two standards have arisen principally from a recognition that utilities must compete for capital and must do so more or less constantly. The model and analysis in this paper furnish a way of reflecting these standards explicitly in a rate-of-return relationship.

9. Conclusion

■ The model demonstrated here allows rate of return to be specified by prudent measurement of variables of recognized significance. What, if anything, can be said about the relative merits of this model and others mentioned earlier? The answer lies in comparing the merits of the underlying assumptions. Each model is a simplified representation of a very complex process. The model developed here has several appealing characteristics in terms of parameter measurability and dynamic behavior. But it also contains several strong behavioral assumptions, especially those concerning the capital market. This tradeoff is characteristic of models of rate of return under conditions of growth, since the ingredients required for determining rates can only be quantified either by imposing behavioral assumptions or by using uncertain parameters (in terms of measurability).

The results of this model and analysis suggest that there will be interesting differences between dynamic financial analyses of the firm and the familiar dynamic analyses in economics, e.g. macro growth models. The model and analysis here have indicated some of these differences.

Growth analyses characteristically use a closed macroeconomic system; the model here is a micro financial model.

The model here did not yield the familiar singular equilibrium solution (analogous to a "golden path"). This could have been remedied easily by changing the objective functional; however, the objective functional was chosen as the traditional representation of value of share ownership. A departure from this traditional representation in order to arrive at an equilibrium solution might be construed as suggesting that traditional theory precludes one.

The present model also produced a dispersal surface (in some of the solution cases), which has not been the case in any growth model. This dispersal surface has a meaningful economic interpretation, while a meaningful counterpart in a macro model is difficult to imagine.

These differences seem interesting and suggest a rich area for research, not only for applications in regulatory theory but in the general area of financial growth and cost of capital as well.

Appendix

■ Solution synthesis is a difficult part of solving optimal control problems, particularly nonlinear ones. The approach discussed in section 5 is a systematic method for attacking this step. An abbreviated sketch of this approach, drawn from [5], is given below.

Recall the statement of necessary conditions on page 257. To these are adjoined terminal conditions on the state variables in terms of a parameterization of the terminal manifold in the state space. Let

$$P(T) = s_p > 0$$
$$E(T) = s_E > 0 .$$

Define

$$\tau = T - t .$$

Thus τ represents the time needed to reach T. Since T is assumed to be a known constant, let

$$\mathring{y} = \frac{dy}{d\tau} .$$

It follows that

$$\mathring{y} = - \dot{y} ,$$

and the state and adjoint equations (equations 1, 2, 3, and 4 on page 257) in the reverse time change in sign. Initial conditions (in this reverse time sense) for these equations are provided by the tranversality condition and the parameterization of the terminal manifold, i.e.,

$$\lambda_P(0) = 1 \quad P(0) = s_P > 0$$
$$\lambda_E(0) = 0 \quad E(0) = s_E > 0 .$$

To start the construction, the Hamiltonian must be maximized at $T(\tau = 0)$. To do this, S_r and S_s must be known (recall the S_i are the switching functions defined on page 257).

At $\tau = 0$

$$S_r = Er[\lambda_E - c\lambda_P - 1] = - s_E r(c + 1) < 0$$

$$S_s = Er\lambda_E \left[1 - \frac{E}{(1 - \delta)P} \right] = 0 .$$

Thus initially $S_r < 0$, $S_s = 0$, and this singular condition must be examined.

$$S_r < 0 , \; S_s = 0 \quad \text{implies} \quad u_r^* = 0 , \; 0 \leq u_s^* \leq \frac{k}{r} .$$

$$S_s = 0 \text{ implies } Er\lambda_E \left[1 - \frac{E}{(1 - \delta)P} \right] = 0 .$$

Since $S_r < 0$, then $Er \not\equiv 0$ and this requires

$$\lambda_E \left[1 - \frac{E}{(1 - \delta)P} \right] = 0 .$$

For this equation to be satisfied for a non-zero time interval implies that all time derivatives must vanish. There are three subcases:

$$1. \quad \lambda_E = 0 , \quad \left[1 - \frac{E}{(1 - \delta)P} \right] \not\equiv 0$$

$$2. \quad \lambda_E \not\equiv 0 , \quad \left[1 - \frac{E}{(1 - \delta)P} \right] = 0$$

$$3. \quad \lambda_E = 0 , \quad \left[1 - \frac{E}{(1 - \delta)P} \right] = 0 .$$

Suppose $\lambda_E = 0$, then by equation (2) it is required that

$$\lambda_P = -\frac{1}{c} \text{ a nonzero constant} .$$

However, if $\lambda_E = 0$ there is no solution of equation (1) which satisfies the above relationship. Thus the condition $\lambda_E = 0$ cannot be satisfied for a non-zero time interval. For S_s to be zero, it is necessary that

$$1 - \frac{E}{(1 - \delta)P} = 0$$

for a non-zero time interval. Differentiation yields

$$Er = \rho P .$$

Further differentiation yields the same results. Thus it is required that

$$\frac{r}{\rho} = \frac{1}{1 - \delta} = \frac{s_P}{s_E}$$

for this condition to be sustained. Consider the case where

$$\frac{r}{\rho} > \frac{1}{1 - \delta} ,$$

hence $S_r < 0$, $S_s = 0$, holds only on the terminal manifold. It is necessary to examine the derivative of S_s at $\tau = 0$ to determine the sign of S_s at $\tau = 0^+$. The result [5] will be that

$$S_s(0^+) < 0 \quad \text{if} \quad \frac{s_P}{s_E} \leq \frac{1}{1 - \delta} ,$$

$$S_s(0^+) > 0 \quad \text{if} \quad \frac{s_P}{s_E} > \frac{1}{1 - \delta} .$$

Consider the case where $s_P/s_E \leq 1/1 - \delta$. Then $S_r(0^+) < 0$, $S_s(0^+) < 0$ and $u_r^* = u_s^* = 0$ are the extremal controls. The state

equations become

$$\mathring{P} = c[\rho P - Er]$$
$$\mathring{E} = 0\,,$$

and the adjoint variables are defined by

$$\mathring{\lambda}_P = -(c+1)\rho\lambda_P\,,$$
$$\mathring{\lambda}_E = rc\lambda_P + r - \rho\lambda_E\,.$$

Their solution yields

$$\frac{P}{E} = \left[\frac{s_P}{s_E} - \frac{r}{\rho}\right]\exp[\rho c\tau] + \frac{r}{\rho}$$

and

$$\lambda_P = \exp[-(c+1)\rho\tau]$$

$$\lambda_E = \frac{r}{\rho} - \frac{r}{\rho}\exp[-(c+1)\rho\tau]\,.$$

Since $r/\rho > 1/1 - \delta$, P/E decreases with τ. Using these solutions, S_r and S_s can be evaluated. It follows directly that S_s will remain negative as long as $S_r < 0$ and $P/E < 1/1 - \delta$. Evaluating S_r yields

$$S_r = Er\left\{\frac{r}{\rho} - 1 - \left\{\frac{r}{\rho} + c\right\}\exp[-(c+1)\rho\tau]\right\}\,.$$

This will be zero when

$$\frac{r}{\rho} - 1 = \left\{\frac{r}{\rho} + c\right\}\exp[-(c+1)\rho\tau]$$

or when

$$\hat{\tau} = \frac{1}{\rho(c+1)}\ln\frac{\rho c + r}{r - \rho}\,.$$

Now at $\hat{\tau}$, $S_r = 0$, $S_s < 0$, and since this is a singular situation it must be checked in the manner followed on the terminal manifold. By proceeding with this construction, the full solution can be gained. To do this here would be lengthy and outside the scope of this paper. Considerable additional analysis is required, particularly on the manifold $S_r = S_s > 0$, and is available in full detail in [5].

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