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SELECTION AND THE EVOLUTION OF INDUSTRY

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Recent evidence shows that within an industry, smaller firms grow faster and are more likely to fail than large firms. This paper provides a theory of selection with incomplete information that is consistent with these and other findings. Firms learn about their efficiency as they operate in the industry. The efficient grow and survive; the inefficient decline and fail. A perfect foresight equilibrium is proved by means of showing that it is a unique maximum to discounted net surplus. The maximization problem is not standard, and some mathematical results might be of independent interest.

1. THEORY AND EVIDENCE ON THE GROWTH AND SURVIVAL OF FIRMS

DO SMALL FIRMS grow faster than large firms? Are they less likely to survive? Early studies found no relation between the size of firms and their growth rates [**8**, **14**, **16**]. The growth of firms seemed to be proportional to their size. In later work, adjustment costs with constant returns to scale were shown to imply that firms *should* grow in proportion to their size [**10**, **11**].

Recent evidence from larger samples tells a different story. Mansfield [**13**] finds that smaller firms have *higher* and *more variable* growth rates. Du Rietz [**6**], in a sample of Swedish firms, again finds that smaller firms grow faster, and that they *are* less likely to survive [**6**, **8**, **13**]. These findings conflict with the adjustment costs theory in which all firms grow at the same rate, and in which failure does not happen.

To explain these deviations from the proportional growth law, I propose a theory of "noisy" selection. Efficient firms grow and survive; inefficient firms decline and fail. Firms differ in size not because of the fixity of capital, but because some discover that they are more efficient than others. The model gives rise to entry, growth, and exit behavior that agrees, broadly, with the evidence.

The model also agrees with some more tentative findings. First, firm size and concentration seem to be positively related to rates of return.[2] Second, the correlation *over time* of rates of return is higher for larger firms and in the concentrated industries [**15**, **17**]. Third, the variability of rates of return *at a point in time* is higher in the concentrated industries [**17**]. Finally, higher concentration is associated with higher profits for the larger firms, but *not* for the smaller firms [**4**].

Enduring differences in size and in growth are no doubt caused in part by the fixity of capital. This paper shows, I think, that selection matters too.

[1] The original draft is dated December, 1979. I would like to thank Carl Futia, Roy Radner, and Ed Green for helpful discussion.

[2] Weiss [**18**] summarizes a number of studies that report this finding. Two exceptions are Stigler [**17**] who found no relation and Samuels and Smyth [**15**] who found a negative relation.

2. A BRIEF DESCRIPTION OF THE MODEL

The model deals with a small industry to which factors are supplied at a constant price. The product is homogeneous and the time-path of the demand for the product is deterministic and known.

Costs are random, and different among firms. For each firm, the mean of its costs may be thought of as the firm's "true cost." The *distribution* of true costs among the potential firms is known to all, but no firm knows what *its* true cost is. All firms have the same prior beliefs, and each firm regards itself as a random draw from the population distribution of true costs. This "prior" distribution is then updated as evidence comes in.

If the firm has low true costs, it is likely that the evidence will be favorable, and the firm will survive. If its costs are high and the evidence adverse, the firm may not wait too long before withdrawing from the industry.

The number of firms in the industry is always infinite—each firm is of measure zero so that it is too small to affect price. With uncertainty at the individual level but with no *aggregate* uncertainty, the path of output prices is deterministic and is assumed to be self-fulfilling in equilibrium.

Firms and potential entrants know the entire equilibrium price sequence, and based on it, they make entry, production, and exit decisions. A one-time entry cost is borne at the time of entry. Thereafter, only production costs are incurred. In equilibrium, the net present value of entry cannot be positive, for if it were, more firms would enter.

In the next section, the model is presented, and the firm's optimization problem is defined. Some of the properties of the model then become clear. Figure 1 portrays them concisely: efficient firms grow and survive; the inefficient decline and fail. Toward the end of the section, results are described which are obtained in the later, mathematical sections of the paper. The implications are

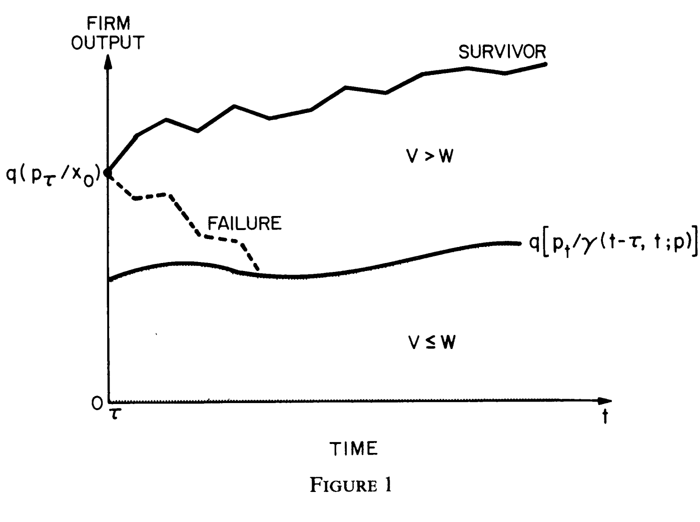

FIGURE 1

compared to the empirical evidence. The perfect foresight equilibrium is defined in Section 4. Section 5 is devoted to proving that equilibrium exists, is unique, and is, in a sense, socially optimal. The proof involves showing that equilibrium coincides with the unique maximum of the discounted sum of consumer plus producer surplus. Two theorems that describe the equilibrium appear in Section 6, and some of the longer proofs are contained in the Appendix.

A curious feature of the paper is that proofs of "obvious" results are complicated. The reader may be eager—on the first reading at least—to take such results on trust. So, the remainder of the paper is arranged into two distinct parts. The next section contains a discussion of all the results. If the reader is not interested in the mathematics, he can stop there, for the discussion is self-contained. Sections 4 and beyond are devoted to the formal development, which is interesting in its own right. Proving that equilibrium is a maximum is not new, of course. But the problem of maximizing discounted surplus here does not fit a regular mold. Similar problems will no doubt arise again and results proved here could then be useful. A description of the nature of the problem being solved and of how it differs from other work on stochastic optimization is contained at the beginning of Section 5.

3. THE MODEL

In an industry with a homogeneous output, firms differ in efficiency. Some are more efficient than others at *all* levels of output. Let q be the output of a firm, and $c(q)$ a cost function which satisfies

$$c(0)=0, \quad c'(0)=0, \quad c'(q)>0, \quad c''(q)>0 \quad \text{and} \quad \lim_{q\to\infty} c'(q)=\infty.$$

Total costs are $c(q_t)x_t$ where x_t is a random variable independent across firms. For the firm of type θ, let $x_t=\xi(\eta_t)$ where $\xi(\cdot)$ is a positive, strictly increasing, and continuous function with $\lim_{\eta_t\to-\infty}\xi(\eta_t)=\alpha_1>0$ and $\lim_{\eta_t\to\infty}\xi(\eta_t)=\alpha_2 \leq \infty$, and where

$$\eta_t=\theta+\epsilon_t, \qquad \epsilon_t \sim N(0,\sigma^2) \text{ iid.}$$

Firms with large values of θ will generate larger x_t's, and be less efficient at all levels of output. The ϵ_t are firm-specific shocks, independent over time and across firms.

Among potential firms, θ is normally distributed with mean $\bar{\theta}$ and variance σ_θ^2. An entrant does not know his own θ, but he knows that he is a random draw from $N(\bar{\theta},\sigma_\theta^2)$. He also knows the variance of ϵ_t, as well as the exact form of $\xi(\cdot)$ so that observing his own costs at t allows him to infer η_t.

The firm is too small to affect price. It chooses q_t so as to maximize expected profits:

$$\max_{q_t}\left[p_t q_t - c(q_t)x_t^*\right]$$

where x_t^* is the expectation of x_t conditional upon information received prior to t. The output decision is made *before* x_t is observed, and is denoted by $q(p_t/x_t^*)$. As one would expect, it is decreasing in x:

$$(3.1)\qquad \frac{\partial q}{\partial x_t^*} = \frac{-c'}{x_t^* c''} < 0 \quad \text{and} \quad \frac{\partial^2 q}{\partial x_t^{*2}} = \frac{1}{x_t^*}\left[\frac{c'c'''}{(c'')^2} - 2\right]\frac{\partial q}{\partial x_t^*}.$$

THE EXIT DECISION: Let $W > 0$ be the expected present value of the firms's fixed factor (its "managerial ability" or "advantageous location") if it is employed in a different activity. The value of W is the same for all firms in the industry *regardless of how successful they are in that industry*. In other words, if the firm learns that it is efficient in this industry, this does not increase its estimated efficiency anywhere else. This assumption may seem restrictive, but it could be relaxed to allow for correlation in firms' efficiencies in different industries, without changing the nature of the results. What really matters is that if favorable information about a firm's costs in an industry raises its expected earnings *in that industry* by one dollar, its expected earnings elsewhere increase by *less* than a dollar. Here it is assumed that new information about θ leaves expected alternative earnings unchanged.

A cost of entry, k, is borne by the firm when it enters—the cost of establishing a particular location for example. And θ might be the degree of suitability of the location. The firm learns about θ with the passage of time.

The firm has an infinite horizon and a constant discount rate, r. At time t, if the firm is in the industry, it will have a pair of sufficient statistics $(\bar{\eta}_n, n)$ which characterize its beliefs about its parameter θ. Here n is the number of periods that the firm has been in the market (the age of the firm) and $\bar{\eta}_n = \sum^n \eta_i / n$. These two statistics are sufficient for the posterior distribution on θ, as this distribution is normal [**20**, p. 15].

In spite of the infinite horizon and the constant discount rate, the present value of earnings will depend on t too, because the price path, $\{p_t\}_0^\infty$, treated as given by the firm, is in general not constant over time. Therefore, once $\{p_t\}_0^\infty$ is given, t determines where one is along the price sequence.

Since $x_t^* = \int \xi(\eta) P^0(d\eta \mid \bar{\eta}_n, n)$, (where $P^0(\cdot \mid \bar{\eta}_n, n)$ is the normal posterior distribution of η_t with variance which depends only on n), and since $\xi(\eta)$ is strictly increasing, x_t^* is strictly increasing in $\bar{\eta}_n$ for each n. Therefore the pair (x_t^*, n) is also a sufficient statistic.

Let $\pi(p_t, x) \equiv p_t q(p_t/x) - c[q(p_t/x)]x$ be the expected value of profits maximized with respect to q when $x_t^* = x$. For a bounded price sequence $p \equiv \{p_t\}_0^\infty$, let $V(x,n,t;\, p)$ be the value, at t, of staying in the industry for one period and then behaving optimally, when the information is (x,n) and when the price sequence is p. Then V satisfies[3]

$$(3.2)\qquad V(x,n,t;\, p) = \pi(p_t, x) + \beta \int \max\left[W, V(z, n+1, t+1;\, p)\right] P(dz \mid x, n).$$

[3] In equation (3.2), $P(z \mid x, n)$ is the probability that $x_{t+1}^* \leq z$ given that $x_t^* = x$, and given that the firm has been in the industry for n periods.

At entry, when the firm has only its prior information, $x = x_0 \equiv$ prior mean of x_t, and $V(x_0, 0, t; p) - k$ is the net value of entry at t. The following simple result is proved in the Appendix.

THEOREM 1: (i) *A unique, bounded and continuous solution for V in equation* (3.2) *exists and* (ii) *V is strictly decreasing in x.*

Thus firms with higher expected costs have a lower value of staying in the industry. Let $\gamma(n,t; p)$ be the level of x_t^* at which the firm is *indifferent* between staying in the industry and leaving it. Then $\gamma(\cdot)$ is the solution for x to the equation[4]

$$V(x,n,t; p) = W. \tag{3.3}$$

As V is strictly decreasing in x, $\gamma(\cdot)$ is uniquely defined. Consequently, the level of output below which the firm will not produce (but will exit instead) is $q[p_t/\gamma(n,t; p)]$. It is drawn in Figure 1. Here τ is the time at which the firm enters the industry, so $n = t - \tau$. For any price sequence, the boundary defines an "exit" region in which $V \leq W$ (shaded area) and a "continuation" region, in which $V > W$.

The firm's output sequence, $q(p_t/x_t^*)$, is a random process which starts from $q(p_\tau/x_0)$. The x_t^* sequence is a Martingale: $E_t x_{t+k}^* = x_t^*$ for any $k \geq 0$ [7, p. 212]. If output remains above the shaded area, the firm stays in the industry. Therefore the firms that survive are larger than the firms that fail—at the point of failure, the firm is smaller than all surviving members of its cohort.

The x_t^* sequences are independent across firms. They tend to *diverge*, as do the output sequences. Each firm is of measure zero, so the number of firms is always infinite, and the concentration *ratio* always zero. But a popular measure of concentration is the Gini coefficient. The greater the dispersion of x_t^* across firms, the greater the dispersion of firm size. At $t = 0$, all firms are of the same size, and the Gini coefficient is zero. As the x_t^* diverge, so do the outputs, and the Gini coefficient increases over time. But the increase need not be monotonic. And this is *exactly* the type of increase the Gini coefficient has exhibited at the economy-wide level in the U.K. **[8]**.

Average profits also rise as the industry matures—at least they do if the equilibrium price sequence is constant (the latter possibility is the subject of Theorem 3 and is discussed below as well). The reason is simple. Since the unprofitable firms leave while profitable firms stay, the profits of the survivors as a group will increase so long as the price of the product does not fall. If it does not, profits increase with the age of the industry, as does concentration. In this sense then, the model predicts a positive relation between profits and concentration. But if the equilibrium price sequence falls over time, it may offset the upward "selection effect" on profits, and nothing can then be said about the time path of profits.

[4] If $V < W$ for all $x \in [\alpha_1, \alpha_2]$, we set $\gamma = \alpha_1$, while if $V > W$ for all $x \in [\alpha_1, \alpha_2]$, we set $\gamma = \alpha_2$.

Why does concentration increase the profits of the large firms but not of the small firms? A high Gini coefficient results from high inequality in firm size in the industry. And the latter is caused by more inequality in efficiency. This means that some firms—large firms—earn higher rents. But the marginal firms—small firms—do not earn rents; there is no reason to expect a positive relation between concentration and the profits of these firms.

As for the positive relation between concentration and the *variability* of profits, take the case where $\sigma_t^2 = 0$—all firms are equally efficient, and all are equal in size. The variability of profits is zero, as is the Gini coefficient. Since both profits and output decrease with x, there is a *one-to-one* relation between the relative dispersion in profits and in size.

Another implication is that unusually high profits today lead to unusually high growth between today and tomorrow. The reason is that the firm's *revision* of x_t^* depends on *realized* profits. Since $\pi_t = p_t q_t - c(q_t)x_t$,

$$\pi_t - E_t\pi_t = -c(q_t)(x_t - x_t^*).$$

So if profits at t are large compared to average profits *for that size of firm*, it means that x_t is unusually low. And this leads to a downward revision of expected marginal costs: $x_{t+1}^* < x_t^*$. But then next period's output—and growth—will be higher than usual. So, high profits are transformed into high growth. Of course, the standard "explanation" has relied on imperfect capital markets: constraints on borrowing lead to higher growth for firms that can finance it internally.

Fluctuations in output occur in any industry. What fraction is due to changes in the output of existing firms as opposed to changes—through entry or exit—in the number of firms? If demand changes are erratic and unforeseen, one might expect the existing firms to meet a large proportion of such changes. But in this paper, all the demand changes are foreseen. Holding constant the behavior of demand, a lot hinges on whether $q(\cdot)$ is concave in x or not (see (3.1) for the concavity condition). Why should this matter? Suppose for the moment that price is constant, say at $\bar{p}$. Since x_t^* is a Martingale, concavity and Jensen's inequality imply that $E_t q(\bar{p}/x_{t+1}^*) < q(\bar{p}/x_t^*)$. So when the output price does not increase, the existing firms would produce *less* in each successive period. This is only re-inforced by the reduction of output due to exit of some of the existing firms. So, if $q(\cdot)$ is concave in x, increases in demand should be met by new entrants. Theorem 3 proves that when demand is non-decreasing and $q(\cdot)$ concave in x, the unique equilibrium is one in which price is constant over time and in which entry and exit occur in every period.

No technological progress takes place in this model. Yet it seems possible (but I cannot prove it) that even if demand is constantly shifting to the right, equilibrium price will constantly decline. Since the efficient survive while the inefficient fail, the average efficiency of the survivors improves from period to period. Convexity of $q(\cdot)$ in x is a necessary condition for such an equilibrium to occur. On the other hand, prices cannot monotonically increase, as the net value to entry would become positive.

What can be proved is that equilibrium *always* coincides with the unique maximum to the discounted consumer surplus (Theorem 2). This is not surprising—firms take prices as given and there are no externalities. Still, as background to the current work on oligopolistic entry-deterrence, it is useful to know that at least when numbers are large, entry—and exit—occurs neither too early nor too late.

Some specialized results can also be proved. The variability of growth rates will be largest among the young (and therefore smaller) firms. But for mature firms that have survived for a long time, x_t^* converges to a constant. Therefore, if growth rates are to be equal among *mature* firms (and there is some empirical basis for this: samples of *large* firms were used in [**14** and **16**]), one must have, for each x,

$$\frac{1}{q}\frac{dq}{dp_t} = k(p_t)$$

where $k(p_t)$ is some function which does not depend on firm size and is therefore independent of x.

Solving this differential equation under the constraint that q is a function only of the ratio p_t/x,

$$q(p_t/x) = \delta_1\left[\frac{p_t}{x}\right]^{\delta}, \tag{3.4}$$

where δ_1 and δ are positive constants. This can only happen if $c(q)$ assumes the Cobb-Douglas form

$$c(q) = \beta_1 q^{\beta_2} \qquad \text{with} \qquad \beta_2 = 1/\delta + 1$$

and with

$$\beta_1 = \delta_1^{-1/\delta}\left[\frac{\delta}{1+\delta}\right].$$

Since the restriction in (3.4) applies to *all* firms, the growth rate of any given firm is

$$\left[\frac{p_{t+1}}{p_t}\right]^{\delta}\left[\frac{x_t^*}{x_{t+1}^*}\right]^{\delta} - 1. \tag{3.5}$$

In the sections that follow, we shall *not* restrict the cost function to be Cobb-Douglas. But it is instructive to pursue further the implications of the existence of approximately equal expected growth rates of firms of a given vintage. Let $z_t \equiv x_t^*/x_{t+1}^*$. If (3.5) holds, a weak form of the proportional growth law requires that $E_t z_t^{\delta}$ be the same for all x_t^*. A strong form requires that the entire distribution of z_t (conditional on information at t) be the same for all firms. The strong form of the law has been empirically rejected, and we now show that it cannot hold within the framework of this model: since the Martingale x_t^* has decreasing incremental variance as the precision on θ grows, two firms with the same x_t^* but different precisions could not have the same distribution for z_t. However, z_t can have the same distribution for firms in the same age cohort. Since one may write $x_{t+1}^* = x_t^*(1 + u_t)$ where u_t has mean zero due to the Martingale property, $z_t = 1/(1 + u_t)$ will be identically distributed for each firm in the cohort so long as the distribution of u_t does not depend on x_t^*. Then $\xi(\eta)$ may be chosen so that this property holds. Let $x_t - \alpha_1$ be log-normally distributed so that $\xi(\eta) = \alpha_1 + e^{\eta}$. Let θ_n and ν_n denote the posterior mean and variance on θ when n observations $\{\eta_i\}_{i=0}^{n-1}$ are available. Then,

$$x_t^* = \alpha_1 + \exp\left\{\theta_{t-\tau} + \tfrac{1}{2}\left(\nu_{t-\tau} + \sigma^2\right)\right\}.$$

As $\alpha_1 \to 0$,

$$z_t \to \frac{1}{1 + u_t} = \frac{x_t^*}{x_{t+1}^*} = \exp\left\{(\theta_{t-\tau} - \theta_{t+1-\tau}) + \frac{1}{2}(\nu_{t-\tau} - \nu_{t+1-\tau})\right\}$$

and since the distribution of $\theta_{t-\tau} - \theta_{t+1-\tau}$ (conditional on information available at t) is normal with mean zero and variance that does *not* depend on $\bar{\eta}_n$, only on $t - \tau$, the variable $1/(1 + u_t)$ does not depend on x_t^*.

Since the variance of u_t declines as the firm becomes more mature, younger firms have more variability in their growth rates. They will also grow faster than the older firms. This follows from the convexity of x_t^*/x_{t+1}^* and the application of Jensen's inequality: $E_t(x_t^*/x_{t+1}^*) > x_t^*/E_t x_{t+1}^* = 1$. So even the weak form of the law cannot hold except within a single age cohort.

The implication that smaller firms should have higher and more variable growth rates is in accord with evidence. But there is a selection bias in the data. Smaller firms are more likely to fail and the model implies (see Figure 1) that the firms which fail are exactly those which otherwise would have grown more slowly. If, as is done in practice, all failures are omitted from the sample, one overestimates the growth rate of small firms relative to that of large firms. The model implies that even if one were to eliminate this bias by an appropriate choice of statistical technique, the results should show a higher growth-rate for smaller firms.

4. EQUILIBRIUM

Before defining equilibrium, the industry supply and demand functions are defined. Let

$$\text{(4.1)} \quad \Psi(x \mid t,\tau; p) = \text{Prob}\big[x_s^* < \gamma(s-\tau, s; p),\ s = \tau+1, \dots, t-1, \text{ and } x_t^* < \min\left[x, \gamma(t-\tau, t; p)\right] \text{ given } x_\tau^* = x_0 \text{ and given that entry occurred at } \tau\ (\tau < t)\big].$$

Ψ is the probability that the firm which enters at τ, and follows its optimal stopping policy, is still in the industry at t, at which time its $x_t^* \leq x$. (Note that since s runs from $\tau + 1$, *immediate* exist is not feasible—it is not feasible for the firm to exit without having spent at least one period in the industry.) Then,

$$\phi(t,\tau;\, p) \equiv \int q(p_t/x)\Psi(dx \mid t,\tau;\, p) \tag{4.2}$$

is the expected output at t of a firm of vintage τ. If y_τ is the measure of entrants at time τ, the output of these firms at t is $y_\tau\phi(t,\tau;\, p)$. This output is deterministic, as each firm is of measure zero. Let Q_t be the aggregate industry output at t. Then, the industry *supply function* is

$$Q_t = \sum_{\tau=0}^{t} y_\tau\phi(t,\tau;\, p) \equiv Q_t(p, y) \tag{4.3}$$

where $y \equiv \{y_\tau\}_0^\infty$ is a sequence of entry. So $Q_t(p, y)$ is the industry output at t which results if the firms are faced with an *arbitrary* pair of price and entry sequences (p, y), and if they make optimal output and exit decisions in response to the price sequence p.

A deterministic *demand function* $D[Q_t, t]$ is given for each t. For each t, $D(\cdot)$ is strictly decreasing in Q_t.

A perfect foresight equilibrium has the property that if firms and prospective entrants behave on the assumption that a particular price sequence will occur, then their behavior does in fact give rise to this price sequence. In other words, the equilibrium price sequence is self-fulfilling.

Definition of Equilibrium: Equilibrium is of a pair of functions $q(\cdot)$ and $\Psi(\cdot)$ that characterize optimal output and exit behavior of firms, and a pair of nonnegative sequences (p, y) such that for all $t = 0, 1, \ldots,$

$$p_t = D\{Q_t(p, y), t\}, \tag{D.1}$$

$$\begin{aligned} V(x_0, 0, t;\, p) - k &= W \qquad \text{if } y_t > 0, \\ V(x_0, 0, t;\, p) - k &\leq W \qquad \text{if } y_t = 0, \end{aligned} \tag{D.2}$$

where $Q_t(p, y)$ is defined in (4.3).

Condition D.1 expresses the self-fulfilling property of the equilibrium price sequence. Condition D.2 states that at each t, the net present value to entry cannot be positive, for if it were, more firms would enter, so that this could not be an equilibrium. The value to entry may be negative in some of the periods, in which case no firm would enter.

5. EXISTENCE, UNIQUENESS, AND OPTIMALITY OF EQUILIBRIUM

We prove the existence and uniqueness of the equilibrium by showing that it is a unique maximum to a particular functional—the discounted consumer plus producer surplus. The equilibrium is an optimum in this sense.

Before plunging into the algebra, let us review the elements of the argument. A benevolent planner chooses an industry output sequence $\{Q_t\} \equiv Q$ so as to maximize the discounted sum of surplus. He has exactly the same amount of information as is collectively available to firms. He assigns the entry and exit in each time period, as well as the output of each firm. The benevolent planner is concerned only with one industry and shadow prices in other industries are held constant both through time and with respect to the industry output level.

One possible approach could have been to use dynamic programming. But the state space is too large here: a firm is characterized by its expected cost and its age (this is the basis for (3.1)) while the industry is described by a *measure* over (x_t^*, n)—the measure of firms, at t, with those characteristics. Because of the dimensionality of the state space, we take the route of direct optimization subject to constraints. There are problems with this approach too, because the horizon is infinite. The constraint space is ℓ_∞, and linear functionals in the dual of ℓ_∞ do not assume a simple form (see (5.8)). The key result is in Lemma 5 (which is probably of independent interest) where it is shown that—essentially because of discounting—the class of linear functionals in the dual of ℓ_∞ can be reduced to a "manageable" subclass.

The cheapest way (for the planner) of producing an aggregate output sequence Q, is defined as $K(Q)$. The hard part of the proof is showing that $K(Q)$ is well-defined (Lemma 2), convex, and differentiable (Lemma 7).

Having established the relevant properties of $K(Q)$, we proceed to compare the necessary conditions for $S(Q)$ to be at a maximum, with the necessary conditions for the actions of firms to be optimal in the perfect foresight equilibrium. This comparison turns out to be easier if an alternative representation for the value of entry is used. This representation is derived (at the outset) in equation (5.2). We now proceed with the analysis.

In view of equation (4.1), another representation for the value of entry at τ, $V(x_0, 0, \tau; p)$ is

$$V(x_0, 0, \tau; p) = \sum_{t=\tau}^{\infty} \beta^{t-\tau} \int_{\alpha_1}^{\alpha_2} \pi(p_t, x) \Psi(dx \mid t, \tau; p) + \sum_{t=\tau}^{\infty} \beta^{t-\tau} W \{\Psi(\alpha_2 \mid t-1, \tau; p) - \Psi(\alpha_2 \mid t, \tau; p)\} \tag{5.1}$$

because the expression in curly brackets is just the probability of exit exactly at t. (Here we define $\Psi(\alpha_2 \mid \tau - 1, \tau; p) = 1$.) Writing $\psi_t \equiv \Psi(\alpha_2 \mid t, \tau; p)$ (since α_2, τ, and p are fixed), the second part of equation (5.1) may be written as

$$W \sum_{t=\tau}^{\infty} \beta^{t-\tau}(\psi_t - \psi_{t-1}) = W\left[\psi_{\tau-1} + \beta \sum_{t=\tau}^{\infty} \beta^{t-\tau}\psi_t - \sum_{t=\tau}^{\infty} \beta^{t-\tau}\psi_t\right]$$

$$= -(1-\beta) W \sum_{t=\tau}^{\infty} \beta^{t-\tau}\psi_t + W$$

(because $\psi_{\tau-1} = 1$). Therefore,

$$V(x_0, 0, \tau; p) = W + \sum_{t=\tau}^{\infty} \beta^{t-\tau} \int_{\alpha_1}^{\alpha_2} [\pi(p_t, x) - (1-\beta)W] \Psi(dx \mid t, \tau; p). \tag{5.2}$$

(The intuition here is that $(1-\beta)W$ is the per-unit time expected foregone income for the firm while it is in the industry.)

Now let $\gamma \equiv \{\gamma_t\}_{t=1}^{\infty}$ be a particular sequence bounded above by α_2 and below by α_1. Any such sequence γ represents a *feasible* exit policy. Let Γ be the set of all such sequences. That is,

$$\Gamma = \{\gamma : \gamma_t \in [\alpha_1, \alpha_2], t = 1, \dots\}.$$

Then, analogously to equation (4.1), for any $\gamma \in \Gamma$, let

$$\hat{\Psi}_\gamma(x \mid t, \tau) = \text{Prob}\big[x_s^* < \gamma_{s-\tau}, s = \tau + 1, \dots, t, \text{ and } x_t^* < \min(x, \gamma_{t-\tau}) \text{ given } x_\tau = x_0 \text{ and entry occurred at } \tau\big].$$

In words, $\hat{\Psi}_\gamma(\cdot)$ is the distribution of x at t (for vintage $-\ \tau$ firms) if they follow the feasible policy characterized by γ.

LEMMA 1: *The density $\hat{\psi}_\gamma(x \mid t, \tau) = \partial \hat{\Psi}_\gamma(x \mid t, \tau)/\partial x$ exists for all γ, and $t > \tau$, and any $x \in (\alpha_1, \alpha_2)$. Furthermore, $\hat{\psi}_\gamma$ is differentiable in each element, γ_j, of the sequence γ, for all $j \geq 1$.*

PROOF: Contained in the Appendix.

Now let $\gamma(p, \tau) \equiv \{\gamma(t-\tau, t; p)\}_{t=\tau}^{t=\infty}$ [the latter is defined in (3.3)]. Then $\gamma(p, \tau) \in \Gamma$. Therefore $\Psi(x \mid t, \tau; p)$ [defined in (4.1)] has density $\psi(x \mid t, \tau; p)$. Then $\psi(x \mid t, \tau; p)$ is the solution to the problem

$$\max_{\gamma \in \Gamma} \left\{ W + \sum_{t=\tau}^{\infty} \beta^{t-\tau} \int_{\alpha_1}^{\alpha_2} [\pi(p_t, x) - (1-\beta)W] \hat{\psi}_\gamma(x \mid t, \tau) \, dx \right\} \tag{5.3}$$

because the optimal "cutoff" sequence $\gamma(p, \tau)$ is contained in Γ. Of course, $\psi(x \mid t, \tau; p) = \hat{\psi}_{\gamma(p,\tau)}(x \mid t, \tau)$. Writing $\gamma_j(p, \tau)$ for the jth member of the optimal sequence, since $\gamma_j(p, \tau) \in [\alpha_1, \alpha_2]$, the necessary conditions for optimality may be written for $j = 1, \dots,$

$$\left\{ \sum_{t=\tau}^{\infty} \beta^{t-\tau} \int_{\alpha_1}^{\alpha_2} [\pi(p_t, x) - (1-\beta)W] \frac{\partial \psi}{\partial \gamma_j}(x \mid t, \tau; p) \, dx \right\} \times [\gamma_j - \gamma_j(p, \tau)] \leq 0 \tag{5.4}$$

for all $\gamma_j \in [\alpha_1, \alpha_2]$.

Let $Q \equiv \{Q_t\}_0^\infty$ be a bounded output sequence. Let ℓ_∞ denote the set of bounded infinite sequences. We now define the social benefit functional $S:\ell_\infty \to R$ as

$$(5.5) \qquad S(Q) = \sum_{t=0}^{\infty} \beta^t \int_0^{Q_t} D(z,t)\,dz - K(Q)$$

where $K:\ell_\infty \to R$ is the present value of the minimized cost of producing the output sequence Q.

The problem of attaining $K(Q)$ will be referred to as the "planner's cost minimization problem." We shall assume that the planner possesses all of the information available to each firm. He chooses an entry sequence[5] $y \equiv \{y_t\}_0^\infty \in A_1$, where $A_1 = \{y : 0 \le y_t \le \bar{y}_t$, where $\sum_{t=0}^\infty \bar{y}_t < \infty\}$, and for firms of vintage τ he chooses an exit policy $\gamma(\tau) \equiv \{\gamma_j(\tau)\}_{j=1}^\infty \in \Gamma$. Finally he assigns a nonnegative output level $\hat{q}(x,t)$ to a firm which at t has $x_t^* = x$. Let $A_2 = \{\hat{q}(x,t)$: each t and $x \in [\alpha_1, \alpha_2]$, $0 \le \hat{q}(x,t) \le b(x)$, with $b(x)$ Lebesgue integrable$\}$. By Tychonoff's theorem on product spaces, A_1, Γ, and A_2 are compact sets in their product topology. Let $\Gamma^\infty = \Gamma \times \Gamma \times \cdots$, and let $\gamma \equiv \{\gamma(\tau)\}_{\tau=0}^\infty$, so that $\gamma \in \Gamma^\infty$. Similarly, let $A_2^\infty = A \times A \times \cdots$, and let $\hat{q} \equiv \{\hat{q}(\cdot, t)\}_{t=0}^\infty$, so that $\hat{q} \in A_2^\infty$. Finally let $s \equiv (y, \gamma, \hat{q}) \in A_1 \times \Gamma^\infty \times A_2^\infty \equiv \Omega$. Again, Ω is a product of compact spaces and is therefore compact in its product topology. Then one may write

$$K(Q) = \inf_{s \in \Omega} f(s) \qquad \text{subject to} \qquad G_t(s, Q_t) \le 0$$

for $t = 0, 1, \ldots$, where

$$f(s) = \sum_{t=0}^{\infty} \beta^t \left\{ \sum_{\tau=0}^{t} y_\tau \int_{\alpha_1}^{\alpha_2} (c[\hat{q}(x,t)]x + (1-\beta)W)\hat{\psi}_{\gamma(\tau)}(x \mid t, \tau)\,dx + k y_t \right\}$$

[5] The restriction that $\sum y_t$ be bounded is stated for analytical convenience. The optimal entry sequence is bounded because at the optimum one must have $G_t = 0$ for all t. For if $G_t < 0$, the planner could reduce $\hat{q}(x,t)$ for some firms, thereby reducing f without violating any of the constraints. Rewriting G_t as $-\sum_{\tau=0}^t y_\tau \hat{\phi}(t,\tau) + Q_t \le 0$, one then has that $\sum_{\tau=0}^t y_\tau \hat{\phi}(t,\tau)$ is bounded because Q_t is bounded. Then, if $\sum^t y_\tau$ did not converge, one would have output per entrant

$$\lim_{\substack{t \to \infty \\ (t > T)}} \frac{\sum_{\tau=T}^t \hat{\phi}(t,\tau) y_\tau}{\sum_{\tau=T}^t y_\tau} = 0$$

for every $T < \infty$, which cannot be optimal. It is of relevance, however, that one may have entry at each point in time ($y_t > 0$ each t) while $\sum^\infty y_t < \infty$. In fact, this may well happen in equilibrium (Theorem 3).

and where

$$G_t(s, Q_t) = -\sum_{\tau=0}^{t} y_\tau \int_{\alpha_1}^{\alpha_2} \hat{q}(x,t)\hat{\psi}_{\gamma(\tau)}(x \mid t,\tau)\,dx + Q_t$$

so that $f:\Omega \to R^1$ and $G_t:\Omega \times \ell_\infty \to R^1$. The meaning of the constraint $G_t(s, Q_t) \leq 0$ is that the planner is required to produce *no less* than the target output Q_t at t. This constraint holds at each t.

Let $G(s, Q) \equiv \{G_t(s, Q_t)\}_{t=0}^{\infty}$. Since G_t is bounded (uniformly in t), $G:\Omega \times \ell_\infty \to \ell_\infty$. The constraint may then be written as $G(s, Q) \leq 0$.

LEMMA 2: *There exists a constrained minimum for f on Ω.*

PROOF: Contained in the Appendix.

Simplifying the notation slightly, we write $G(s) \leq 0$ for the constraint, dropping Q from the notation. We now define the Gateaux differentials $\delta f(s_0, s_1) = df[s_0 + \alpha(s_1 - s_0)]/d\alpha$ and $\delta G(s_0, s_1) = dG[s_0 + \alpha(s_1 - s_0)]/d\alpha$ for $s_0, s_1, \in \Omega$. They are differentials at the point s_0, with increment $(s_1 - s_0)$. We have $\delta f:\Omega \to R$ and $\delta G:\Omega \to \ell_\infty$. Since Ω is convex, the point $s_0 + \alpha(s_0 - s_0)(= (1-\alpha)s_0 + \alpha s_1) \in \Omega$ for all $\alpha \in [0, 1]$.

DEFINITION: A point $s_0 \in \Omega$ is said to be a regular point of the inequality $G(s) \leq 0$ if $G(s_0) \leq 0$ and there exists a point $s_1 \in \Omega$ such that $G(s_0) + \delta G(s_0, s_1) < 0$ [**12**, p. 248]. The inequality is strict at each coordinate, i.e., $G_t(s_0) + \delta G_t \cdot (s_0, s_1) < 0$ for each t.

LEMMA 3: *Let $f(s_0) = \min f(s)$ subject to $G(s) \leq 0$. Then, if $Q_t > 0$ (each t), s_0 is a regular point.*

PROOF: Contained in the Appendix. Of course, the condition that the maximum occurs at a regular point is the analogue of the Kuhn-Tucker constraint qualification in the finite dimensional case.

Let P be the positive cone of ℓ_∞. Then P contains an interior point (any bounded, strictly positive sequence which is bounded away from zero is an interior point of P). Let ℓ_∞^* be the dual space of ℓ_∞, that is, the space of bounded linear functionals on ℓ_∞. Let $\lambda \in \ell_\infty^*$ be a particular functional. Its value at the point $z \in \ell_\infty$ will be denoted by $\lambda(z)$. We then have the following lemma.

LEMMA 4: *If s_0 minimizes f on Ω subject to $G(s) \leq 0$, then there exists a functional $\lambda^* \in \ell_\infty^*$ such that if $Q > 0$,*

$$\delta f(s_0; s) + \lambda^*[\delta G(s_0; s)] \geq 0 \qquad \textit{for all } s \in \Omega \tag{5.6}$$

and such that $\lambda^* > 0$ *and*

(5.7) $\lambda^*\left[G(s_0)\right] = 0.$

PROOF: [**12**, pp. 249–250]: (His proof relies on s_0 being regular (and this we have established in Lemma 3) and on the interior point property of P.)

We now proceed to characterize λ^*. Since $\lambda^* \in \ell_\infty^*$, it is representable in the following way: for any sequence $\eta \in \ell_\infty$:

(5.8) $$\lambda^*(\eta) = \sum_{t=0}^{\infty} \lambda_t^* \eta_t + \int \eta \, d\hat{\lambda}^*$$

where $\hat{\lambda}^*$ is a bounded and *purely finitely additive* measure on the set of all possible subsets of the positive integers [**9**, p. 870; **19**, p. 52] and where $\sum|\lambda_t^*| < \infty$. The purely finitely additive property implies $\int \eta \, d\hat{\lambda}^* = 0$ for all $\eta \in \ell_\infty$ such that only finitely many elements of the sequence η are nonzero. We now prove that $\lambda_t^* > 0$ (each t) and that $\int \eta \, d\hat{\lambda}^* = 0$, for all $\eta \in \ell_\infty$.

At the optimal solution, $s_0, G_t(s_0) = 0$. For if $G_t(s_0) < 0$, one could reduce f by reducing $\hat{q}^0(x,t)$ (for t only) on a set of positive measure without violating the constraint at t, and leaving all the other contraints unaffected. If s_1 differs from s_0 only in as much as $\hat{q}^1(x,t) \neq \hat{q}^0(x,t)$ for some x and for *fixed* t, only the constraint at t is affected. More precisely $\delta G_i(s_0; s_1)$ is zero except for $i = t$. Hence, for such variations, $\lambda^*[\delta G(s_0; s_1)] = \lambda_t^* \delta G_t(s_0; s_1)$ (in view of equation (5.8)). Equation (5.6) then implies

(5.9) $$\sum_{\tau=0}^{t} y_\tau^0 \int \left\{ \beta^t c'\left[\hat{q}^0(x,t)\right]x - \lambda_t^* \right\}\left[\hat{q}^1(x,t) - \hat{q}^0(x,t)\right]\hat{\psi}_{Y^0(\tau)}(x \mid t,\tau)\, dx \geq 0$$

for all $\hat{q}^1$ which are integrable. (We shall refer to "sets on which

$$\sum_{\tau=0}^{t} y_\tau^0 \hat{\psi}_{Y^0(\tau)}(x \mid t,\tau)$$

is of positive Lebesgue measure" as sets of positive measure. Clearly, variations in $\hat{q}$ on sets of measure zero in this sense are immaterial since they affect neither f nor G.) Since $Q_t > 0$, $\hat{q}^0(x,t) > 0$ on a set of positive measure, so that $c'[\hat{q}^0(\cdot)] > 0$ on these sets. But then $\lambda_t^* > 0$, or else there would exist an admissible variation which would violate equation (5.9). Also, if $\hat{q}^0(x,t) = 0$ on some set of positive measure, it is readily seen that since $c'(0) = 0$, equation (5.9) would again be violated by an admissible variation. Therefore $\hat{q}^0 > 0$, which implies that except on sets of measure zero, for each t,

(5.10) $\lambda_t^* = \beta^t c'\left[\hat{q}^0(x,t)\right]x > 0.$

We are now able to prove the following lemma.

LEMMA 5: $\int \eta \, d\hat{\lambda}^* = 0$ *for all* $\eta \in \ell_\infty$.

PROOF: Contained in the Appendix.

Lemma 5 establishes that the second expression on the right-hand side of equation (5.8) is equal to zero. Now let $s_1 = s_0$ except that $y_t^1 \neq y_t^0$ for some t. Then equation (5.6) reads (in view of Lemma 5)

$$\left\{ \frac{\partial f}{\partial y_t} + \sum_{j=t}^{\infty} \lambda_j^* \frac{\partial G_j}{\partial y_t} \right\} (y_t^1 - y_t^0) \geq 0$$

for all y_t^1. Therefore, if $y_t^0 > 0$, $\partial f / \partial y_t + \sum_{j=t}^{\infty} \lambda_j^* \partial G_j / \partial y_t = 0$ while if $y_t^0 = 0$, then $\partial f / \partial y_t + \sum_{j=t}^{\infty} \lambda_j^* \partial G_j / \partial y_t \geq 0$. This implies that

$$\text{(5.11)} \qquad \sum_{j=t}^{\infty} \int \left\{ \beta^j \left[c(\hat{q}^0)x + W(1-\beta) \right] - \lambda_j^* \hat{q}^0 \right\} \hat{\psi}_{\gamma^0(t)}(x \mid j,t)\, dx + \beta^t k \overset{\geq}{=} 0 \qquad \text{as} \quad y_t^0 \overset{=}{>} 0.$$

Finally we consider the variation for which $s_1 = s_0$, except that $\gamma_i^1(\tau) \neq \gamma_i^0(\tau)$ for some $i \geq 1$ and some $\tau \geq 0$. Applying Lemma 5, equation (5.6) reads

$$\text{(5.12)} \qquad \left\{ \frac{\partial f}{\partial \gamma_i(\tau)} + \sum_{j=\tau+i}^{\infty} \lambda_j^* \frac{\partial G_j}{\partial \gamma_i(\tau)} \right\} \left[\gamma_i^1(\tau) - \gamma_i^0(t) \right] \geq 0$$

(note that a change in $\gamma_i(\tau)$ does not affect G_t for $t < \tau + i$). Then equation (5.12) implies that for all $\gamma_i^1(\tau) \in [\alpha_1, \alpha_2]$,

$$\text{(5.13)} \qquad \left\{ \sum_{j=\tau+i}^{\infty} \int \left\{ \beta^j \left[c(\hat{q}^0)x + W(1-\beta) \right] - \lambda_j^* q^0 \right\} \frac{\partial \hat{\psi}_{\gamma^0(\tau)}}{\partial \gamma_i(\tau)} (x \mid t, \tau)\, dx \right\} \times y_\tau^0 \left[\gamma_i^1(\tau) - \gamma_i^0(\tau) \right] \geq 0.$$

If $y_\tau^0 = 0$, this inequality is automatically satisfied—there are no firms of vintage τ.

LEMMA 6: *The Lagrangean* $L(s, \lambda^*) = f(s) + \lambda^*[G(s)]$ *is minimized over* s *at* (s_0, λ^*) *satisfying equations* (5.10), (5.11), *and* (5.13). *The point* $f(s_0)$ *is a global constrained minimum.*

PROOF: Contained in the Appendix. Lemma 6 ensures that the minimum attained at s_0 is indeed the global minimum.

LEMMA 7: $K(Q)$ *is convex in* Q, *and is differentiable in the elements of* Q, *with* $\partial K / \partial Q_t = \lambda_t^*$.

PROOF: Contained in the Appendix.

The social marginal cost is λ_t^*, in present value terms. From equation (5.10) this is also the discounted expected marginal cost of each firm.

We recall that the derived properties of $K(Q)$ hold only for sequences Q which are bounded away from zero by some positive ϵ. However, for each such ϵ, λ_t^* is bounded uniformly in $Q \in \ell_\infty$. Let $\bar{\lambda}$ be the upper bound on λ. The following assumption on demand ensures that marginal benefits exceed marginal costs at some sufficiently small level of output at each t: we assume that for some $\epsilon > 0$ sufficiently small,

$$D(\epsilon, t) > \bar{\lambda} \qquad \text{all } t. \tag{5.14}$$

Secondly, in order to ensure that the socially optimal output sequence is bounded, we assume that for each t

$$\int_0^\infty D(z,t)\,dz < A \tag{5.15}$$

for some A sufficiently large and independent of t.

PROPOSITION 1: *There is exactly one bounded sequence* $\{Q_t^*\}$ *which maximizes* $S(Q)$ *in equation* (5.5), *and it satisfies*

$$\beta^t D[Q_t^*, t] = \lambda_t^*, \qquad t = 0, 1, \ldots . \tag{5.16}$$

PROOF: Equation (5.15) implies that Q_t^* is bounded. Equation (5.14) ensures that Q_t^* is bounded away from zero. Each decision Q_t then belongs to a compact set, and $S(Q)$ is strictly concave in Q, because $D(Q_t, t)$ is downward sloping. Therefore the maximum exists and is unique. Since $S(Q)$ is also differentiable, one has $\partial S/\partial Q_t = 0$ at the optimum, which implies that equation (5.16) holds, and the proof is complete.

Let $\hat{q}^*(x,t)$ and $\hat{\psi}_{Y^*(\tau)}(x \mid t, \tau)$ denote the optimal $\hat{q}$ and $\hat{\psi}$ for minimizing f when $Q = Q^*$. We may now state:

THEOREM 2 (Existence, Uniqueness, Optimality): *There is exactly one equilibrium price sequence* $\tilde{p}$ *and entry sequence* $\tilde{y}$ *satisfying* D.1. *and* D.2. *in Section* 4. *The price sequence is given by* $\tilde{p}_t = \beta^{-t}\lambda_t^*$. *The social benefit functional* $S(Q)$ *is at a maximum with* $Q_t^* = Q_t(\tilde{p}, \tilde{y})$, *where* $\hat{q}^*(x,t) = q(\tilde{p}_t/x)$ *and where* $\hat{\psi}_{Y^*(\tau)}(x \mid t, \tau) = \psi_{Y(\tilde{p},\tau)}(x \mid t, \tau)$.

PROOF: From equation (5.10) it is seen that $\hat{q}^*(x,t) = q(\tilde{p}_t/x)$. Substituting this for $\hat{q}^*$ into equation (5.13), this condition becomes identical to equation (5.4) when the latter is evaluated at $\psi(x \mid t, \tau; \tilde{p})$. Therefore $\hat{\psi}_{Y^*(t)} = \psi(x \mid t, \tau; \tilde{p})$ as asserted. Substituting for $\hat{q}^*$ and $\hat{\psi}_{Y^*(t)}$ in equation (5.11), this condition is identical to Condition D.2. Finally, from Lemma 8, $D[Q_t(\tilde{p}, \tilde{y}), t] = p_t^*$ so that Condition D.1. is satisfied. Therefore, the maximum is necessarily an equilibrium. This establishes existence. For uniqueness and optimality, note that since any candidate equilibrium price and entry sequence, $(\tilde{p}, \tilde{y})$ satisfies $\tilde{p}_t = \beta^{-t}(\partial K/\partial Q_t)[Q_t(\tilde{p}, \tilde{y})]$, Condition (D.1) requires that $\tilde{p}_t = D[Q_t(\tilde{p}, \tilde{y}), t]$. This requires that $S(Q)$ be maximized by $Q_t(\tilde{p}, \tilde{y})$. By lemma 7 this maximum is unique. Therefore $Q(\tilde{p}, \tilde{y}) = Q^*$, and $\tilde{p}_t = \beta^{-t}\lambda_t^*$, so that the equilibrium $(\tilde{p}, \tilde{y})$ is necessarily a maximum. This completes the proof of the theorem.

6. TWO CHARACTERIZATION THEOREMS

This section gives some characterization of the behavior of entry and prices in equilibrium. Much depends on whether $q(p_t/x)$ is convex in x. This is made precise in the following theorem.

THEOREM 3: *Assume that $D[Q_t, x]$ is nondecreasing in t (demand grows monotonically). If $q(p/x)$ is a strictly concave function of x [see equation (3.1)], then the equilibrium price sequence is constant, with $p_t^* = \bar{p}$ for each t, and entry occurs at each t ($y_t^* > 0$) while $\sum_0^\infty y_t^* < \infty$, and $V(x_0, 0, t; \bar{p}) - k = W$ for each t.*

PROOF: Since (D.2) must hold, profits to entry must be zero at each t. Suppose the optimal exit policies of firms are such that no firm exits under any circumstance. Since x_t^* is a martingale, strict concavity of $q(\cdot)$ in x implies (by Jensen's inequality) $E_t q(\bar{p}/x_{t+1}^*) < q(\bar{p}/x_t^*)$. This implies that $\phi(t, \tau; p^*) > \phi(t+1, \tau; p^*)$ for each $t \geq \tau$. This inequality is only reinforced by firms making an exit under some contingencies. In other words, if the price is constant at $\bar{p}$, the output of the survivors (of any given vintage) would decline with time *even if none were to exit*. It follows that it will decline even more if some do exit. Define $A_t(\bar{p})$ by $\bar{p} = D[A_t(\bar{p}), t]$. In other words, $A_t(\bar{p})$ is the amount of output which must be forthcoming at t if price is to be kept constant at $\bar{p}$. Then since demand is nondecreasing in t, the sequence $\{A_t(\bar{p})\}$ is also nondecreasing in t. Since for each t, ϕ declines in t, the output of the survivors is declining. Therefore $y_t^* > 0$ for each t because if $y_t^* = 0$ for some t, price at t would be strictly greater than in the previous period. Finally, since $A_t(\bar{p})$ is bounded, one has $\sum_0^\infty y_t^* < \infty$ if $\inf_{t,\tau}\phi(t, \tau; p^*) > 0$. For this is sufficient to show that for each τ, $\lim_{t\to\infty}\Psi(\alpha_2 \mid t, \tau; p^*) > 0$, that is, the probability of permanent survival is positive for firms of each vintage. To prove that this is so, consider the policy: "stay in for one more period, then exit no matter what." The expected reward from this policy is $\pi(\bar{p}, x) + \beta W$. Therefore the firm will not exit if this reward is greater than W, or, equivalently, if $\pi(\bar{p}, x) > (1-\beta)W$. Let $\hat{x}$ be such that $\pi(\bar{p}, \hat{x}) = (1-\beta)W$, so that the firm stays in if $x_t^* < \hat{x}$. Of course, $\hat{x} > \alpha_1$, otherwise the net value of entry at all t would be negative. For each firm of age n (see Section 3), $x_t^* = E(x \mid \bar{\eta}_n, n)$. Let $\bar{\eta}_n$ be such that $\hat{x} = E(x \mid \bar{\eta}_n, n)$. Since $\hat{x} > \alpha_1, \hat{\eta}_n$ is bounded from below. Therefore exit does not occur so long as $\bar{\eta}_n < \hat{\eta}_n$. But the probability that a normal posterior mean does not ever reach a boundary that is *bounded* is strictly positive (Chernoff [3]). Therefore $\lim_{t\to\infty}\Psi(\alpha_2 \mid t, \tau; p^*) > 0$ for all τ, and $\sum_0^\infty y_t^* < \infty$. This completes the proof of the theorem.

The concavity of $q(\cdot)$ in x is (under conditions of increasing demand) sufficient to imply a constant-price equilibrium. But it is not necessary. The concavity of q in x was, in the proof of Theorem 3, used only to prove that $\phi(t+1,\tau;\, p^*) < \phi(t,\tau;\, p^*)$. If q is convex in x, this means that the collective output of the *surviving* firms is larger at $t+1$ than it was at t. But the subtraction from the total output due to firms' exit might still be larger than this increase, resulting, on net, in a decrease in ϕ.

It seems possible that after withdrawals have occurred, the surviving firms will increase their output by *more* than the increase in the quantity demanded (at an unchanged price), thereby making it necessary for price to fall over time. In view of Theorem 3, convexity of q in x is a *necessary* condition for such an equilibrium to occur.

THEOREM 4: *$K(Q)$ is homogeneous to degree one: for any $\alpha > 0$, $\alpha K(Q) = K(\alpha Q)$. An equiproportional shift to the right of demand (for all t) brings about an increase in Q_t in the same proportion for all t. Equilibrium prices remain unchanged.*

PROOF: If $(y^0, \hat{q}^0, \gamma^0)$ minimizes f when the constraint is Q_0, then $(\alpha y^0, \hat{q}^0, \gamma^0)$ minimizes it when the constraint is αQ_0: Since f is linear homogeneous in y, this proves $\alpha K(Q) = K(\alpha Q)$ for all Q. Now introduce the demand-shift-parameter μ, such that $D[Q_t, t, u] = D[\mu Q_t, t]$. Equilibrium requires that $\beta^t D[\mu Q_t(\mu), t] = \partial K[Q(\mu)]/\partial Q_t$, where $\{Q_t(\mu)\}$ is the equilibrium quantity sequence indexed by μ. We need to show that $Q(\mu) = Q(1)/\mu$. By homogeneity of K, $\alpha \partial K(Q)/\partial Q_t = \alpha \partial K(\alpha Q)/\partial Q_t$ for all $\alpha > 0$ and Q, so that $\partial K(Q)/\partial Q_t = \partial K(\alpha Q)/\partial Q_t$. Letting $Q = Q(1)$ and $\alpha = 1/\mu$, the result follows.

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APPENDIX

PROOF OF THEOREM 1: Part (i): The proof of this part consists of an application of Theorem 5 of [2]. Let T denote the operator which defines V as the fixed point of the equation (3.2) $V = TV$. First it needs to be shown that T transforms continuous, bounded functions into other continuous, bounded functions. Boundedness follows if $\pi(p_t, x_t^*)$ is bounded—the latter is true because (a) p_t is bounded, (b) $x_t^* \geq \alpha_1 > 0$, and (c) $\lim_{q\to\infty} c'(q) = \infty$, so that the necessary condition $p_t - c'(q_t)x_t^*$ is always satisfied at a bounded output level. To prove that T preserves continuity, note firstly that $\pi(\cdot)$ is continuous in x_t^*, as is P. Therefore T preserves boundedness and continuity.

Secondly, T is a monotone (increasing) operator. That is, for two functions f_1 and f_2, if $f_1 \geq f_2$ everywhere, then $Tf_1 \geq Tf_2$ everywhere.

Finally, for any function f and any constant $c > 0$, $T(f + c) \leq Tf + \beta c$.

Therefore T is a contraction operator with modulus $\beta < 1$, and the Banach fixed-point theorem may be applied to yield assertion (i) of the theorem.

Part (ii): The proof is in two steps. We first prove that V is nonincreasing in x and then use this to prove that it is strictly decreasing.

Let T denote the operator in equation (3.2) so that V is the unique solution to the equation $V = TV$. Since T is a contraction operator, we have $V = \lim_{n\to\infty} T^n g$ for *any* bounded, continuous function g [where $T^n g \equiv T(T^{n-1}g)$]. Any monotonicity properties of *each* member of the sequence $T^n g$ are preserved (weakly) by the limit function V. We shall now show that if g is decreasing in x, so is Tg. If $g(z)$ is decreasing in z, then so is the function $\max[W, g(z)]$. Then, since $P(z \mid x, n)$ is increasing in x, $\int \max[W, g(z)]P(dz \mid x, n)$ is decreasing in x. By induction, these properties carry over to the limit function V which is therefore nonincreasing in x. But then $\int \max[W, V(z, n+1, t+1; p)]P(dz \mid x, n)$ is nonincreasing in x. But π is strictly decreasing in x and therefore so is V.

PROOF OF LEMMA 1: One has

$$\hat{\Psi}_\gamma(x \mid t, \tau) = \int_{\alpha_1}^{\alpha_2} H[\min(x, \gamma_{t-\tau}) \mid z, t-1-\tau]\hat{\Psi}_\gamma(dz \mid t-1, \tau)$$

where $H(z' \mid z, n)$ is the (one-step) transition probability that $x_t^* \leq z'$ given the pair of sufficient statistics $x_{t-1}^* = z$ and $n = t - 1 - \tau$. But H is a continuous transform of the normal CDF, and is differentiable in z' for $z' \in (\alpha_1, \alpha_2)$. Therefore if $\hat{\Psi}$ is continuous at t, then it is also differentiable at $t+1$. Thus if $\hat{\Psi}$ is differentiable when $t = \tau + 1$, it is differentiable for all $t > \tau$. But $\hat{\Psi}(x \mid \tau+1, \tau) = H[\min(x, \gamma_1 \mid x_0, 0]$ if $x_0 < \gamma_0$, while if $x_0 \geq \gamma_0$, $\hat{\Psi}(x \mid \tau+1, \tau) = 0$, so that $\hat{\Psi}$ is differentiable when $t = \tau + 1$. Furthermore since H is continuously differentiable, so is $\hat{\Psi}$, except at the point $x = \gamma_{t-\tau}$, (where the derivative is in general discontinuous as it becomes zero for $x > \gamma_{t-\tau}$) so that the density $\hat{\psi}$ exists and is piecewise continuous. Turning to differentiability of $\hat{\psi}$ in γ_j, note that $\hat{\psi}_\gamma(x \mid t, \tau)$ does not depend on γ_j if $j > t - \tau$. For any $j < t - \tau$, $H[\cdot]$ does not depend on γ_j, and therefore if $\hat{\psi}$ is differentiable at t, it is differentiable at $t+1$. Since this is true for any t, it is sufficient to show that $\hat{\psi}$ is differentiable at $t = \tau + j$, where

$$\hat{\Psi}_\gamma(x \mid \tau + j, \tau) = \int_{\alpha_1}^{\alpha_2} H[\min(x, \gamma_j) \mid z, j + \tau - 1]\hat{\psi}_\gamma(z \mid \tau + j - 1, \tau)\,dz$$

and where $\hat{\psi}_\gamma(z \mid \tau + j - 1, \tau)$ does not depend on γ_j. Then the cross-derivative $\partial^2\hat{\Psi}/\partial x\,\partial\gamma_j$ exists, and this completes the proof of the lemma.

PROOF OF LEMMA 2: Let $\Omega(Q) = \{s : G(s, Q) \leq 0\} \subseteq \Omega$. We need to show that f is continuous on $\Omega(Q)$ and that $\Omega(Q)$ is compact [**1**, p. 69]. In the product topology, sequences $\{s_t\}_0^\infty$ $(s_t \in \Omega)$ converge to s_0 *weakly* [**5**, p. 32]. That is, $s_1 \to s_0$ if and only if $y_t^1 \to y_t^0$ each t, $\hat{q}^1(x, t) \to \hat{q}^0(x, t)$ for each t except on a set of Lebesgue measure zero, and $\gamma_j^1(\tau) \to \gamma_j^0(\tau)$ each j. Suppose then that $|f(s_1) - f(s_0)| \to \delta > 0$ as $s_1 \to s_0$, so that f is not continuous at s_0. Write $f(s)$ as $\sum_{t=0}^\infty \beta^t f_t(s)$. Since $f_t(s)$ is bounded on Ω, one may choose $T(\delta) < \infty$ such that $\sum_{t=T(\delta)}^\infty \beta^t |f_t(s_1) - f_t(s_0)| < (\delta/2)$ $\forall s_1 \in \Omega$. Then

$$|f(s_1) - f(s_0)| \leq \sum_{t=0}^{T(\delta)-1} \beta^t |f_t(s_1) - f_t(s_0)| + \frac{\delta}{2}$$

$$\leq [T(\delta) - 1] \max_{0 \leq t \leq T(\delta)-1} |f_t(s_1) - f_t(s_0)| + \frac{\delta}{2}.$$

But for any $t < \infty$, $\hat{\psi}_{\gamma(\tau)}(x \mid t, \tau)$ depends only upon $\gamma_j(\tau)$ for $j = 1, \cdots t - \tau$ and is continuous in γ_j, so that for each $t > \tau$, and each x, $\hat{\psi}_{\gamma^1(\tau)} \to \hat{\psi}_{\gamma^0(\tau)}$ as $s_1 \to s_0$ so that $|f_t(s_1) - f_t(s_0)| \to 0$ as $s_1 \to s_0$. But then $|f(s_1) - f(s_0)| \to (\delta/2) < \delta$, a contradiction. Therefore f is continuous on Ω.

Compactness of $\Omega(Q)$ is assured if $\Omega(Q)$ is closed, because a closed subset of a compact space is compact [**1**, p. 68]. $\Omega(Q)$ is closed if its complement is open. Consider then a point $s_0 \in \Omega$ such that $G_t(s_0, Q_t) > 0$ for each t. One must show that there exists an open neighborhood of s_0, $N(s_0)$, such that $G(s, Q) > 0$ for all $s \in N(s_0)$. Let $G_t(s, Q_t) = -\sum_0^t y_i z_i(\hat{q}, \gamma) + Q_t$. Then let $G(s_0, Q) > 0$. For any $s \in \Omega$, since y_t and z_t are bounded,

$$G_t(s, Q_t) - G_t(s_0, Q_t) = -\sum_0^t y_i(z_i - z_i^0) - \sum_0^t (y_i - y_i^0)z_i^0.$$

Since $\sum_0^t y_i < \infty$, and $y_i \geq 0$, one has $\sum_T^\infty y_t \to 0$ as $T \to \infty$. Then one may choose $T = T(\delta)$ sufficiently large such that $\sum_{T(\delta)}^t y_i|z_i - z_i^0| + \sum_{T(\delta)}^t |y_i - y_i^0| z_i^0 < (\delta/2)$ for all $t \geq T(\delta)$. Then for any $t \geq T(\delta)$,

$$|G_t(s, Q_t) - G_t(s_0, Q_t)| \leq \sum_{i=0}^{T(\delta)-1} (y_i|z_i - z_i^0| + |y_i - y_i^0| z_i^0) + \frac{\delta}{2}$$

$$\leq [T(\delta) - 1]\Big(c_1 \max_{0 \leq i \leq T(\delta)-1} |z_i - z_i^0|$$

$$+ c_2 \max_{0 \leq i \leq T(\delta)-1} |y_i - y_i^0|\Big) + \frac{\delta}{2}$$

where the second inequality follows as both y_t and z_t are bounded. But for any $t < \infty$, $|z_t - z_t^0| \to 0$ and $|y_t - y_t^0| \to 0$ as $s \to s_0$. Since this is true for any $\delta > 0$, one has, for any t, that $G_t(s, Q_t) \to G_t(s_0, Q_t)$ as $s \to s_0$. This is true for each t; therefore $G(s, Q)$ converges to $G(s_0, Q)$, so that G is a continuous mapping. Therefore the set $\{s : G(s, Q) > 0\}$ is open. But s_0 is a member of this set (since by assumption $G(s_0, Q) > 0$). Therefore $\Omega(Q)$ is closed and the proof of Lemma 2 is complete.

Proof of Lemma 3: The $s_1 = (y^1, \gamma^0, \hat{q}^0)$ and $s_0 = (y^0, \gamma^0, \hat{q}^0)$. That is s_1 differs from s_0 only in the entry sequence y. Then,

$$\delta G_t(s_0; s_1) = -\sum_{\tau=0}^{t} (y_\tau^1 - y_\tau^0) \int \hat{q}^0(x, t) \hat{\psi}_{\gamma^0(\tau)}(x \mid t, \tau)\, dx.$$

Since $G_t(s_0) \leq 0$, one need only show that $\delta G_t < 0$ (each t) for some $s_1 \in \Omega$. Since $Q_t > 0$, one has $\sum_0^t y_\tau^0 \int q^0 \hat{\psi}_{\gamma^0}\, dx > 0$ for each t. But then one may set $y_\tau^1 - y_\tau^0 = \alpha y_\tau^0$ $(\alpha > 0)$ in which case $\delta G_t(s_0; s_1) < 0$ for each t. Since $\sum^\infty y_\tau^0$ is finite, so is $(1 + \alpha)\sum^\infty y_\tau^0$ so that $s_1 \in \Omega$ and Lemma 3 has been proved.

Proof of Lemma 5: Suppose $\exists \eta \in \ell_\infty$ such that $|\int \eta\, d\hat{\lambda}^*| = \epsilon_1 > 0$. Then for any ϵ_2 no matter how small, $|\int \epsilon_2 \eta\, d\hat{\lambda}^*| = \epsilon_2 \epsilon_1 > 0$. Let $T > 0$ be given, and let

$$\eta_t^T = \begin{cases} \eta_t & \text{if } t \geq T, \\ 0 & \text{if } 0 \leq t < T, \end{cases} \quad \text{and} \quad \eta_t^1 = \begin{cases} 0 & \text{if } t \geq T, \\ \eta_t & \text{if } 0 \leq t < T. \end{cases}$$

By linearity, $\int \epsilon_2 \eta\, d\hat{\lambda}^* = \epsilon_2 \int (\eta^1 + \eta)\, d\hat{\lambda}^* = \epsilon_2 \int \eta^T d\hat{\lambda}^*$, where the second equality is due to the purely finitely additive property of $\hat{\lambda}^*$. Next we show that $\exists s_1, s_2 \in \Omega$ such that $\delta G(s_0; s_1) = \epsilon_2 \eta^T$ and $\delta G(s_0; s_2) = -\epsilon_2 \eta^T$. For the first equality one may choose s_1 the same as s_0 except that $\hat{q}^1 = \hat{q}^0(1 + \epsilon_2 \eta_t^T / Q_t)$ for each x, t. This is a feasible variation, since $\exists \epsilon > 0$ such that $Q_t \geq \epsilon \forall t$ so that one may set ϵ_2 sufficiently small such that $\epsilon_2 \bar{\eta}/\epsilon \leq 1$ (where $\bar{\eta}$ is such that $\eta_t \leq \bar{\eta}$ $\forall \eta \in \ell_\infty$). Similarly, to obtain the second equality one sets $\hat{q}^1 = \hat{q}^0(1 - \epsilon_2 \eta_t^T / Q_t)$. For each η, $\|\eta^T\| \not\to 0$ unless $\lim_{t\to\infty} \eta_t = 0$. But if $\lim_{t\to\infty} \eta_t = 0$, then the lemma is true [**9**, p. 872]. Suppose then that $\|\eta^T\| \to \epsilon_3 > 0$. (The limit must exist because $\|\eta^T\|$ is nonincreasing in T and positive.) Clearly, $\lim_{T\to\infty} \delta f(s_0; s_0 + \epsilon_2 \eta^T) = 0$, and $\lim_{T\to\infty} \sum_{t=0}^\infty \hat{\lambda}_t^* \delta G_t(s_0; s_0 + \epsilon_2 \eta^t) = 0$, while for T no matter how large, $|\int \epsilon_2 \eta^T d\hat{\lambda}^*| = \epsilon_2 \epsilon_1 = |\int -\epsilon_2 \eta^T d\hat{\lambda}^*| > 0$. By linearity, therefore, there is a feasible variation for which $\delta f + \lambda^*(\delta G) < 0$, a contradiction to equation (5.6). This completes the proof of the lemma.

Proof of Lemma 6: If the Lagrangean is minimized at s_0, then the global minimum property follows from [**12**, p. 220, Theorem 1], and from the fact that $G(s_0) = 0$. It then remains to be shown that $L(s, \lambda^*) \geq L(s_0, \lambda^*)$ for $s \in \Omega$. Since $c(\cdot)$ is convex, one has $c(\hat{q}) \geq c(\hat{q}^0) + c'(\hat{q}^0)(\hat{q} - \hat{q}^0)$

$= c(\hat{q}^0) + x^{-1}\beta^{-t}\lambda_t^*(\hat{q} - \hat{q}^0)$. Therefore, for $s \in \Omega$,

$$f(s) + \lambda^*[G(s)]$$

$$\geq \sum_{t=0}^{\infty}\left[\int\{\beta^t[c(\hat{q}^0)x + \beta^{-t}\lambda_t^*(\hat{q} - \hat{q}^0) + (1-\beta)W] - \lambda_t^*\hat{q}\}\right.$$

$$\left.\times \sum_{\tau=0}^{t}\psi_{\gamma(\tau)}y_\tau\,dx + \beta^t k y_t + \lambda_t^* Q_t\right]$$

$$= \sum_{t=0}^{\infty}\int \beta^t[c(\hat{q}^0)x - \beta^{-t}\lambda_t^*\hat{q}^0 + (1-\beta)W]\sum_{\tau=0}^{t}\hat{\psi}_{\gamma(\tau)}y_\tau\,dx + \beta^t k y_t + \lambda_t^* Q_t$$

$$= \sum_{\tau=0}^{\infty} y_\tau\left\{\sum_{t=\tau}^{\infty}\beta^t\int[c(\hat{q}^0)x - \beta^{-t}\lambda_t^*\hat{q}^0 + (1-\beta)W]\hat{\psi}_{\gamma(\tau)}\,dx\right\} + \beta^\tau k y_\tau + \lambda_\tau^* Q_\tau.$$

However, $c(\hat{q}^0)x - \beta^{-t}\lambda_t^*\hat{q}^0 = \pi(\beta^{-t}\lambda_t^*, x)$. Therefore, in view of equation (5.3), the expression in curly brackets is minimized by $\gamma(\{\beta^{-t}\lambda_t^*\}_0^\infty, \tau)$ (see the first line following equation (5.3)). But the necessary conditions for this to occur, in equation (5.4) are identical to those of equation (5.13) (when one sets $p_t = \beta^{-t}\lambda_t^*$ in equation (5.4)). Therefore, the expression in curly brackets is minimized for each τ, by $\hat{\psi}_{\gamma^0(\tau)}$. Therefore,

$$f(s) + \lambda^*[G(s)] \geq \sum_0^\infty y_\tau r_\tau + \sum_0^\infty \lambda_\tau^* Q_\tau,$$

where

$$r_\tau = \sum_{t=\tau}^{\infty}\beta^t\int[c(\hat{q}^0)x - \beta^{-t}\lambda_t^*\hat{q}^0 + (1-\beta)W]\hat{\psi}_{\gamma^0(\tau)}\,dx + \beta^t k.$$

Therefore,

$$f(s) + \lambda^*[G(s)] \geq \sum y_t^0 r_t + \sum(y_t - y_t^0)r_t + \sum\lambda_t^* Q_t$$

$$= f(s_0) + \lambda^*[G(s_0)] + \sum(y_t - y_t^0)r_t.$$

But equality implies that $r_t(y_t - y_t^0) \geq 0$ $\forall t$, implying that $f(s) + \lambda^*[G(s)] \geq f(s_0) + \lambda^*[G(s_0)]$, for $s \in \Omega$. The lemma has been proved.

Proof of Lemma 7: Let s_0 be optimal when $Q = Q_0$, and let s_1 be optimal when $Q = Q_1$. $f(s_0) + \lambda^*[G(s_0, Q_0)] \leq f(s_1) + \lambda^*[G(s_1, Q_0)]$ (using Lemma 6). Therefore $f(s_1) - f(s_0) \geq \lambda^*[G(s_0, Q_0) - G(s_1, Q_0)]$. But $G(s_0, Q_0) = 0$. Also, $G(s_1, Q_1) = 0$. Therefore $G(s_1, Q_0) = -Q_1 + Q_0$. Therefore $K(Q_1) - K(Q_0) = f(x_1) - f(x_0) \geq \lambda^*[Q_1 - Q_0]$. Next let Q_1, Q_2 be given and let $Q_\rho \equiv \rho Q_1 + (1-\rho)Q_2$. Then

$$K(Q_1) - K(Q_\rho) \geq \lambda^*[Q_1 - Q_\rho], \quad \text{and} \quad K(Q_2) - K(Q_\rho) \geq \lambda^*[Q_2 - Q_\rho].$$

Multiplying these expressions by ρ and by $1-\rho$ respectively, and adding them together, one obtains

$$\rho K(Q_1) + (1-\rho)K(Q_2) - K(Q_\rho) \geq \lambda^*[\rho(Q_1 - Q_\rho) + (1-\rho)(Q_2 - Q_\rho)]$$

$$= \lambda^*[\rho Q_1 + (1-\rho)Q_2 - Q_\rho] = 0$$

so that K is convex in Q. Turning to differentiability, let $Q_1 = Q_0$ except that $Q_t^1 = (1+\epsilon)Q_t$ where $\epsilon > 0$. Let s_1 be optimal when $Q = Q_1$. Then $f(s_1) \leq f(\tilde{s}_1)$, where $\tilde{s}_1 = s_0$ except that $\hat{q}^1(x, t) = (1+\epsilon)$

$\hat{q}^0(x,t)$. Then $\tilde{s}_1$ meets all the constraints, and

$$\text{(A.1)}\qquad f(s_1)-f(s_0)\le f(\tilde{s}_1)-f(s_0)$$

$$=\beta'\sum_{\tau=0}^{t}y_\tau\int\{c'[\hat{q}^0(x,t)]x\epsilon\hat{q}^0(x,t)\}\hat{\psi}_{\gamma^0(\tau)}(x\mid t,\tau)\,dx+o(\epsilon)$$

$$=Q_t^0\lambda_t^*\epsilon+o(\epsilon)$$

where the last equality follows from equation (5.10) and the definition of $G_t(s_0, Q)$. But $K(Q_1)-K(Q_0)\ge\lambda^*[Q_1-Q_0]=\lambda_t^*\epsilon Q_t^0$. Together with equation (A.1) this implies $Q_t^0\lambda_t^*\epsilon+o(\epsilon)\ge K(Q_1)-K(Q_0)=\lambda_t^*\epsilon Q_t^0$. Dividing by ϵ and taking the limit as $\epsilon\to 0$ establishes that the right-derivative $\partial K/\partial Q_t$ exists and is equal to λ_t^*. A parallel argument taking $\epsilon<0$ establishes the same property for the left-derivative. The proof is complete.

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