Backward and Forward Solutions for Economies with Rational Expectations

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In models where anticipations of future endogenous variables influence current behavior, there exists an infinity of solutions under the assumption of rational expectations. This problem has been dealt with, in the study of macro-economic models, by the implicit or explicit use of one of three additional requirements: optimality; consistency with alleged economic behavior; or conformity of the endogenous variables to an imposed stationarity condition. These requirements have coincided in existing models, leading to the choice of a unique solution, a "forward" solution. The purpose of this paper is to review the problem, characterize the solutions, and examine whether these requirements are acceptable. Section I presents a simple model and derives the set of solutions; the model makes no claim to generality, but has the major advantage that the issues are easily understood in this simple case. Section II discusses the requirement of consistency with economic behavior. Section III discusses the requirement of stationarity, and Section IV provides some conclusions.

I. The Model

Paul Samuelson's overlapping generation model with money is used. Agents live for two periods, receive one unit of perishable output in the first and can save only in the form of money. The government buys output from (sells output to) the young in exchange for money. Equilibrium is characterized by

$$(1a) \qquad (m_t - p_t)^d = -\alpha({}_tp_{t+1} - p_t)$$

$$(1b) \qquad (m_t - p_t) = (m_t - p_t)^d$$

$$(1c) \qquad {}_tp_{t+1} = E(p_{t+1}| I_t)$$

where m_t and p_t are the logarithms of the nominal money stock and of the price level, respectively; I_t is the information available to agents at time t, including current and past values of m_t and p_t; the notation ${}_{t-j}x_{t+i}$ denotes the agents' expectation of a variable x held at time $t - j$ for period $t + i$; and $E(.)$ denotes a mathematical expectation.

Equation (1a) states that the demand for money is a function of the expected rate of inflation. The sign of α is ambiguous: it depends on whether the elasticity of substitution between consumption in the two periods is less or greater than unity. If $\alpha > 0$, the equilibrium is formally equivalent to the equilibrium in the model of Philip Cagan. Equation (1b) characterizes market clearing: the demand for money is equal to the money stock inelastically supplied by the old and the government. Equation (1c) states that agents have rational expectations.

Combining these equations gives

$$(2) \qquad p_t = \frac{1}{1+\alpha} m_t + \frac{\alpha}{1+\alpha} E(p_{t+1}| I_t)$$

A solution for p_t is simply a function:

$$(3) \qquad p_t = \sum_{i=1}^{\infty} a_i m_{t-i} + b m_t + \sum_{i=1}^{\infty} c_i E(m_{t+i} | I_t)$$

with coefficients $(a_i)_i, b, (c_i)_i$ such that it satisfies (2). There are four remarks to be made:

The possibility that the price depends on other variables than money is excluded a priori. (This possibility has been examined by John Taylor and Robert Shiller.)

Variables such as past expectations of future money, formally $E(m_{t-j+i} | I_{t-j}), i, j > 0$ are omitted but not excluded a priori: if we included them, their coefficients would have to be equal to zero, for (3) to satisfy (2). This would not necessarily be the case in more general models.

Equation (3) allows past values of money

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to determine the current price level. Often the solution has been restricted by constraining $a_i = 0$ for all i, a priori.

The problem of convergence of the two infinite sums will be considered later. We may assume for the moment that there exists t_0 and t_1 such that $m_t = 0$ for $t < t_0$ and $E(m_{t+i} | I_t) = 0$ for $t + i > t_1$.

The coefficients are determined as follows: Leading equation (3) once and taking expectations on both sides, conditional on information available at time t:[1]

$$(4) \quad E(p_{t+1} | I_t) = \sum_{i=1}^{\infty} a_i m_{t-i+1} + bE(m_{t+1} | I_t) + \sum_{i=1}^{\infty} c_i E(m_{t+i+1} | I_t)$$

Replacing (4) in (2) and identifying term by term with (3), we can solve for the a_i and c_i as functions of α and b:

$$a_1 = \frac{b(1+\alpha) - 1}{\alpha}; \quad a_{i+1} = \left(\frac{1+\alpha}{\alpha}\right) a_i \qquad i = 1, 2, \ldots, \infty$$

$$c_1 = \frac{\alpha}{\alpha + 1} b; \quad c_{i+1} = \left(\frac{\alpha}{1+\alpha}\right) c_i \qquad i = 1, 2, \ldots, \infty$$

Therefore, there exists an infinity of solutions, each of them parameterized by b. Each solution can be written as a weighted average of two special solutions, a *backward solution* in which the price level depends on past values of the money stock ($b = 0$; $c_i = 0 \ \forall \ i$):

$$(4') \qquad p_t^{(B)} = -\frac{1}{\alpha} \sum_{i=0}^{\infty} \left(\frac{1+\alpha}{\alpha}\right)^i m_{t-i-1}$$

and a *forward solution*[2] in which the price depends only on current and future expected values of m ($b = 1/(1 + \alpha)$, $a_i = 0 \ \forall \ i$):

$$(4'') \quad p_t^{(F)} = \frac{1}{1+\alpha} m_t + \frac{1}{1+\alpha} \cdot \sum_{i=1}^{\infty} \left(\frac{\alpha}{\alpha+1}\right)^i E(m_{t+i} | I_t)$$

Any solution p_t can be written as

$$(4''') \quad p_t = \lambda p_t^{(B)} + (1 - \lambda) p_t^{(F)}; \qquad \lambda \equiv 1 - b(1 + \alpha)$$

Both the backward and the forward solution have been used in the literature. The backward solution is the solution traditionally used in the study of the dynamics of growth models. In the continuous time perfect foresight version of these models, it is referred to as the "myopic perfect foresight" assumption and can be obtained as the limiting case of adaptive expectations. The forward solution has been used in recent macro-economic models.

The indeterminacy does not depend on the use of discrete vs. continuous time or on certainty vs. uncertainty. The origin of the indeterminacy comes from the presence of an expected future value in the equilibrium equation. In each period both the current price and the expected future price clear the market. Over any number of periods, there is one more price (or expected price) than markets to clear. The indeterminacy will therefore be a general feature of models in which current prices depend on expected future prices (or the expected rate of change of prices). It is not present in models such as the no-speculation model of John Muth or the macro-economic model of Robert Lucas for example, in which only expected current values enter.

The indeterminacy is behaviorally significant: an unexpected increase in the nominal money stock, known to be permanent, leaves the real money stock unchanged under the forward solution, the price level unchanged this period under the backward solution. The sequence of utilities of agents will not be the same under different solutions.

We now consider the possibility of choosing between the solutions by imposing additional requirements. The first is the requirement of *optimality:* as the sequence of utilities depends on the solution chosen, it is likely that

[1]Use is made of the "law of iterated expectations":

$$E[E(m_{t+i+1} | I_{t+1}) | I_t] = E(m_{t+i+1} | I_t)$$

[2]Edwin Burmeister R. Flood, and Stephen Turnovsky have remarked that the meaning of "forward" and "backward" conflicts with the meaning of the same words in the mathematics literature. They suggest the use of "forward looking" and "backward looking" would be less confusing.

the use of a given optimality criterion will lead to the choice of a unique solution. If agents were infinitely long lived, such as in Miguel Sidrauski, they would indeed choose the solution which maximizes their utility. In the model considered here however, agents live only for two periods and it is not clear what mechanism will lead to the choice of an optimal solution, however defined. Thus, other types of requirements have to be considered.

II. The Requirement of Consistency with Economic Behavior

The first argument was presented by Thomas Sargent and Neil Wallace, using the Cagan model. In the backward solution (my terminology), the price does not move in response to current changes in money; thus "next instant's price is what adjusts to insure equality between the demand and supply of real balances at this instant . . . [whereas in the forward solution it is] the price level at each moment which adjusts instantaneously in order to insure that the real balances people hold equal the amount they would like to hold" (pp. 1044–45). Thus, they argue, the forward solution is more satisfactory.

Except for the pure backward solution however, both today's and expected next period prices move in response to a change in money: pinpointing which of the two clears the market is at best a difficult task. Furthermore, the interpretation given by Sargent and Wallace may not be the interpretation that agents have of the economy.

Consider the case where agents assume—rationally—that

$$p_t = \lambda^* p_t^{(B)} + (1 - \lambda^*) p_t^{(F)}$$

with λ^* close or equal to unity. If agents assume p_t to follow this process, they will be rational in believing that an increase in money this period affects next period's price level implying, depending on the value of α, expected inflation or deflation. This will lead them to increase their demand for real money balances; consequently, given the increase in the nominal money stock, a small change (or no change if $\lambda^* = 1$) in today's price equilibrates supply and demand. There is nothing in this description inconsistent with the way we think markets operate.

The second argument in favor of choosing the forward solution seems similarly flawed. It runs as follows: equation (2) only includes p_t, m_t, and $E(p_{t+1} | I_t)$ but no past variables. It is hard to see why the past should affect the current price at all. Only in the forward solution does the past not enter and thus this solution should be chosen. Equation (2) is however an incomplete description of the economy without an explicit expectation mechanism. If agents assume that p_t depends on the past in the way indicated by (4‴), then they will be rational and p_t will indeed depend on the past.

III. The Requirement of Stationarity

Heuristically, requiring stationarity amounts to requiring that if nominal money does not "explode," then the price level should not explode. More formally, if the logarithm of nominal money, m_t, follows a stationary process (with mean zero for convenience), so that it has a finite variance, then the equilibrium price level must also have finite variance. There are two separate questions: Why should such a requirement be imposed? Does imposing it lead to the choice of a unique solution? I consider them in turn.

Stationarity may follow from the requirements of optimality but, as indicated above, there is no reason why in this model the solution must satisfy optimality. Karl Shell and Joseph Stiglitz, and Edmund Phelps and Taylor have shown, however, that in certain models nonstationarity may violate the assumption of market clearing and of rationality of expectations. This also applies to this model and may be described—not rigorously—as follows. The model imposes implicitly a bound (upper or lower, depending on the value of α) on the expected rate of inflation. This expected rate cannot be such that it implies a demand for real money balances, or equivalently a supply of output by the young, larger than their endowment.

If p_t is nonstationary, then such an expected rate of inflation may be reached with some positive probability. In this case either the market will not clear or the price will follow another solution, making expectations irrational. This may be a good reason therefore to require stationarity, without reference to optimality. But will this requirement lead to the choice of a unique solution? The answer depends on whether the elasticity of the current price with respect to the price expected next period is less or greater than unity in absolute value:[3]

A. If this elasticity is less than unity, i.e., if $|\alpha/(\alpha+1)| < 1$ (or equivalently $\alpha > -1/2$), then only the forward solution is stationary. This follows directly from inspection of equations (4′) and (4″).

B. If this elasticity is greater than unity, i.e., if $|\alpha/(\alpha+1)| > 1$, then clearly the backward solution is stationary. The forward solution may however also be stationary. Consider the following example:

$$m_t = \rho m_{t-1} + \eta_t, |\rho| < 1, \quad \eta_t \; IID$$

so that

$$E(m_{t+i} | I_t) = \rho^i m_t$$

If ρ is "small enough," i.e., if $|\alpha\rho/(\alpha+1)| < 1$, then $p_t^{(F)}$ is stationary:

$$p_t^{(F)} = \frac{1}{1+\alpha} \sum_{i=0}^{\infty} \left(\frac{\alpha\rho}{\alpha+1}\right)^i m_t$$

$$= \frac{1}{1+\alpha(1-\rho)} m_t$$

Therefore, if m_t is expected to return to its mean "fast enough," then the forward solution may be stationary. In this case all solutions are stationary. Requiring stationarity does not yield a unique solution. There are two ways in which the requirement of stationarity may be strengthened so as to yield a unique solution.

The first one is suggested by Taylor: it is to require that the price level not only have finite variance but also minimum variance. This criterion indeed allows us to choose a unique solution; one of its characteristics is that the choice is not independent of the stationary process followed by m_t.

The second one is suggested by the above example. If $|\alpha/(\alpha+1)| > 1$, then the forward solution cannot be stationary for all stationary processes followed by m_t. In the above example, ρ can always be chosen so that $|\alpha\rho/(\alpha+1)| > 1$. Thus, requiring that p_t be stationary for *any* stationary process generating m_t leads in this case to the choice of the backward solution.

Both of these strengthened criteria thus allow us to choose a unique (but possibly different) solution. However, they both lack the justification of the original stationarity criterion and it is hard to see why this decentralized economy will be led to apply them.

IV. Conclusion

Section I has characterized solutions as linear combinations of a backward and a forward solution. Section II has shown the requirement of consistency with alleged economic behavior to be unacceptable. Section III has shown that the requirement of stationarity may be justified but may not ensure the choice of a unique solution. This suggests two directions of research.

In this model, if the elasticity of the current price with respect to next period's expected price is less than unity, imposing stationarity leads to the choice of a unique solution—the forward solution. This raises two questions: how does this condition translate in more general models? Is such a condition likely to hold? Both questions are addressed in another paper (see the author). The answer is that the generalized condition is indeed likely to hold.

In models in which optimality cannot be invoked, and where imposing stationarity is not justified or does not lead to the choice of a unique solution, a new criterion must be found. A possible direction of research is the study of how the economy converges to a rational expectation solution and how "his-

[3]In this model, whether a solution is stationary or not depends on the value of a utility parameter α. Another approach would have been possible: allowing m_t to follow a simple feedback rule on p_t would have made the stationarity of a solution depend on the rule. The simplest example is $m_t = \gamma p_t$. Which solution is stationary depends on the value of γ. This is the approach followed by Fisher Black.

tory" may determine the value of λ. The difficulty lies in defining plausible revision rules. Using the revision rules suggested by Stephen Decanio, preliminary results indicate that, in the space of solutions, only the forward solution may be stable. If this result is robust, it might be the strongest argument for the choice of the forward solution.

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