4

GROUPS AND CATEGORIES

This chapter is devoted to some of the various connections between groups and categories. If you already know the basic group theory covered here, then this will give you some insight into the categorical constructions we have learned so far; and if you do not know it yet, then you will learn it now as an application of category theory. We will focus on three different aspects of the relationship between categories and groups:

1. groups in a category,
2. the category of groups,
3. groups as categories.

4.1 Groups in a category

As we have already seen, the notion of a group arises as an abstraction of the automorphisms of an object. In a specific, concrete case, a group G may thus consist of certain arrows $g : X \to X$ for some object X in a category $\mathbf{C}$,

$$G \subseteq \mathrm{Hom}_{\mathbf{C}}(X, X)$$

But the abstract group concept can also be described directly as an object in a category, equipped with a certain structure. This more subtle notion of a "group in a category" also proves to be quite useful.

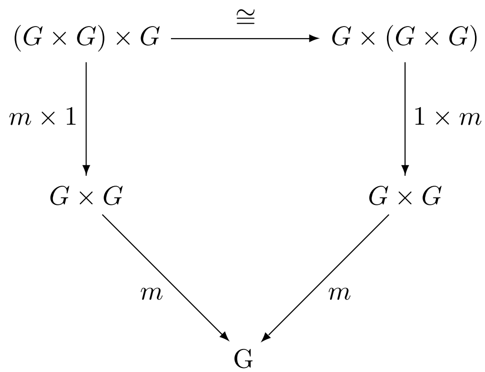
Let $\mathbf{C}$ be a category with finite products. The notion of a group in $\mathbf{C}$ essentially generalizes the usual notion of a group in **Sets**.

Definition 4.1. A *group* in $\mathbf{C}$ consists of objects and arrows as so:

$$\begin{array}{ccccc} G \times G & \xrightarrow{m} & G & \xleftarrow{i} & G \\ & & \uparrow{\scriptstyle u} & & \\ & & 1 & & \end{array}$$

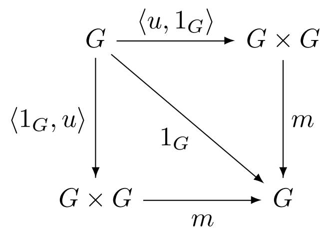
satisfying the following conditions:

1. m is associative, that is, the following commutes:



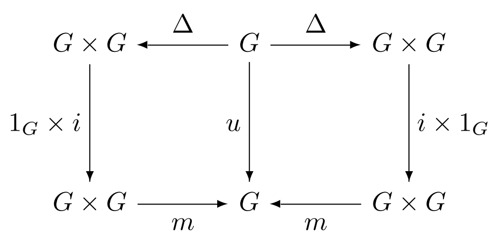
   where $\cong$ is the canonical associativity isomorphism for products.

2. u is a unit for m, that is, both triangles in the following commute:



   where we write u for the "constant arrow" $u! : G \xrightarrow{!} 1 \xrightarrow{u} G$.

3. i is an inverse with respect to m, that is, both sides of this commute:



   where $\Delta = \langle 1_G, 1_G \rangle$.

Note that the requirement that these diagrams commute is equivalent to the more familiar condition that, for all (generalized) elements,

$$x, y, z\colon\ Z \to\ G$$

the following equations hold:

$$m(m(x, y), z) = m(x, m(y, z))$$

$$m(x, u) = x = m(u, x)$$

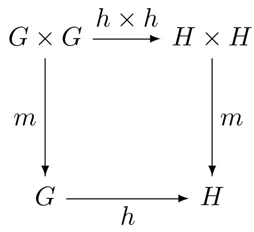
$$m(x, ix) = u = m(ix, x)$$

Definition 4.2. A *homomorphism* $h : G \to H$ of groups in **C** consists of an arrow in **C**,

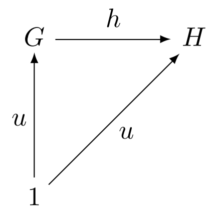
$$h : G \to H$$

such that

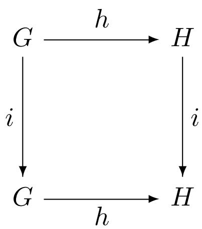
1. h preserves m:



2. h preserves u:



3. h preserves i:



With the evident identities and composites, we thus have a category of groups in **C**, denoted:

$$\text{Group}(\mathbf{C})$$

Example 4.3. The idea of an internal group in a category captures the familiar notion of a group with additional structure.

- A group in the usual sense is a group in the category **Sets**.
- A topological group is a group in **Top**, the category of topological spaces.
- A (partially) ordered group is a group in the category **Pos** of posets (in this case the inverse operation is usually required to be order-*reversing*, that is, of the form $i : G^{\text{op}} \to G$).

For example, the real numbers $\mathbb{R}$ under addition are a topological and an ordered group, since the operations of addition $x+y$ and additive inverse $-x$ are continuous and order-preserving (resp. reversing). They are a topological "semigroup" under multiplication $x \cdot y$ as well, but the multiplicative inverse operation $1/x$ is not continuous (or even defined!) at 0.

Example 4.4. Suppose we have a group G in the category **Groups** of groups. So G is a group equipped with group homomorphisms $m : G \times G \to G$, etc. as in definition 4.1. Let us take this apart in more elementary terms. Write the multiplication of the group G, i.e. on the underlying set $|G|$, as $x \circ y$ and write the homomorphic multiplication m as $x \star y$. That the latter *is* a homomorphism from the product group $G \times G$ to G says in particular that, for all $g, h \in G \times G$ we have $m(g \circ h) = m(g) \circ m(h)$. Recalling that $g = (g_1, g_2)$, $h = (h_1, h_2)$ and multiplication $\circ$ on $G \times G$ is pointwise, this then comes to the following:

$$(g_1 \circ h_1) \star (g_2 \circ h_2) = (g_1 \star g_2) \circ (h_1 \star h_2) \tag{4.1}$$

Write 1° for the unit with respect to $\circ$ and $1^\star$ for the unit of $\star$. The following proposition is called the "Eckmann-Hilton argument," and was first used in homotopy theory.

Proposition 4.5. *Given any set G equipped with two binary operations $\circ, \star : G \times G \to G$ with units 1° and $1^\star$ respectively and satisfying* (4.1)*, the following hold.*

1. $1^\circ = 1^\star$.
2. $\circ = \star$.
3. *The operation $\circ = \star$ is commutative.*

Proof. First, we have,

$$\begin{aligned} 1^\circ &= 1^\circ \circ 1^\circ \\ &= (1^\circ \star 1^\star) \circ (1^\star \star 1^\circ) \\ &= (1^\circ \circ 1^\star) \star (1^\star \circ 1^\circ) \\ &= 1^\star \star 1^\star \\ &= 1^\star. \end{aligned}$$

Thus let us write $1^\circ = 1 = 1^\star$. Next, we have,

$$x \circ y = (x \star 1) \circ (1 \star y) = (x \circ 1) \star (1 \circ y) = x \star y.$$

Thus let us write $x \circ y = x \cdot y = x \star y$. Finally, we have,

$$x \cdot y = (1 \cdot x) \cdot (y \cdot 1) = (1 \cdot y) \cdot (x \cdot 1) = y \cdot x.$$

□

We therefore have the following.

Corollary 4.6. *The groups in the category of groups are exactly the abelian groups.*

Proof. We have just shown that a group in **Groups** is necessarily abelian, so it just remains to see that any abelian group admits homomorphic group operations. We leave this as an easy exercise. □

Remark 4.7. Note that we did not really need the full group structure in this argument. Indeed, the same result holds for monoids in the category of monoids: these are exactly the commutative monoids.

Example 4.8. A further example of an internal algebraic structure in a category is provided by the notion of a (strict) *monoidal category.*

Definition 4.9. A *strict monoidal category* is a category $\mathbf{C}$ equipped with a binary operation $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ which is functorial and associative,

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C, \tag{4.2}$$

together with a distinguished object I that acts as a unit,

$$I \otimes C = C = C \otimes I. \tag{4.3}$$

A strict monoidal category is exactly the same thing as a monoid in **Cat**. Examples where the underlying category is a poset P include both the meet $x \wedge y$ and join $x \vee y$ operations, with terminal object 1 and initial object 0 as units, respectively (assuming P has these structures), as well as the poset $\mathrm{End}(P)$ of monotone maps $f : P \to P$, ordered pointwise, with composition $g \circ f$ as $\otimes$ and 1_P as unit. A discrete monoidal category, i.e. one with a discrete underlying category, is obviously just a regular monoid (in **Sets**), while a monoidal category with only one object is a monoidal monoid, and thus exactly a commutative monoid, by the foregoing remark 4.7.

More general strict monoidal categories, i.e. ones having a proper category with many objects and arrows, are less common—not for a paucity of such structures, but because the required equations (4.2) and (4.3) typically hold only "up to isomorphism." This is so e.g. for products $A \times B$ and coproducts $A + B$, as well as many other operations like tensor products $A \otimes B$ of vector spaces, modules, algebras over a ring, etc. (the category of proofs in linear logic provides more examples). We will return to this more general notion of a (not necessarily strict) monoidal category once we have the required notion of a "natural isomorphism" (in chapter 7 below), which is required to make the above notion of "up to isomorphism" precise.

In logical terms, the concept of an internal group corresponds to the observation that one can "model the theory of groups" in *any* category with finite products, not just **Sets**. Thus, for instance, one can also define the notion of a *group in the lambda-calculus*, since the category of types of the lambda-calculus also has finite products. Of course the same is true for other algebraic theories, like monoids and rings, given by operations and equations. Theories involving other logical operations like negations, implication, or quantifiers can be modeled in categories having more structure than just finite products. Here we have a glimpse of so-called *categorical semantics*. Such semantics can be useful for theories that are not complete with respect to models in **Sets**, such as certain theories in intuitionistic logic.

4.2 The category of groups

Let G and H be groups (in **Sets**), and let

$$h : G \to H$$

be a group homomorphism. The *kernel* of h is defined by the equalizer

$$\ker(h) = \{g \in G \mid h(g) = u\} \longrightarrow G \underset{u}{\overset{h}{\rightrightarrows}} H$$

where, again, we write $u : G \to H$ for the constant homomorphism

$$u! = G \xrightarrow{!} 1 \xrightarrow{u} H.$$

We have already seen that this specification makes the above an equalizer diagram.

Observe that $\ker(h)$ is a *subgroup*. Indeed, it is a *normal subgroup*, in the sense that for any $k \in \ker(h)$, we have (using multiplicative notation)

$$g \cdot k \cdot g^{-1} \in \ker(h) \quad \text{for all } g \in G.$$

Now if $N \overset{i}{\rightarrowtail} G$ is *any* normal subgroup, we can construct the *coequalizer*

$$N \underset{u}{\overset{i}{\rightrightarrows}} G \xrightarrow{\pi} G/N$$

sending $g \in G$ to u iff $g \in N$ ("killing off N"), as follows: the elements of G/N are the "cosets of N," that is, equivalence classes of the form $[g]$ for all $g \in G$, where we define

$$g \sim h \quad \text{iff} \quad g \cdot h^{-1} \in N.$$

(Prove that this is an equivalence relation!) The multiplication on the *factor group* G/N is then given by

$$[x] \cdot [y] = [x \cdot y]$$

which is well defined since N is normal: given any u, v with $x \sim u$ and $y \sim v$, we have

$$x \cdot y \sim u \cdot v \iff (x \cdot y) \cdot (u \cdot v)^{-1} \in N$$

but

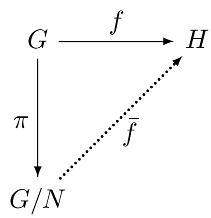
$$\begin{aligned}(x \cdot y) \cdot (u \cdot v)^{-1} &= x \cdot y \cdot v^{-1} \cdot u^{-1} \\ &= x \cdot (u^{-1} \cdot u) \cdot y \cdot v^{-1} \cdot u^{-1} \\ &= (x \cdot u^{-1}) \cdot (u \cdot (y \cdot v^{-1}) \cdot u^{-1}),\end{aligned}$$

the last of which is evidently in N.

Let us show that the diagram above really is a coequalizer. First, it is clear that

$$\pi \circ i = \pi \circ u!$$

since $n \cdot u = n$ implies $[n] = [u]$. Suppose we have $f : G \to H$ killing N, that is, $f(n) = u$ for all $n \in N$. We then propose a "factorization" $\bar{f}$, as indicated in



to be defined by

$$\bar{f}[g] = f(g).$$

This will be well defined if $x \sim y$ implies $f(x) = f(y)$. But, since $x \sim y$ implies $f(x \cdot y^{-1}) = u$, we have

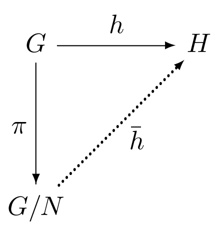
$$f(x) = f(x \cdot y^{-1} \cdot y) = f(x \cdot y^{-1}) \cdot f(y) = u \cdot f(y) = f(y).$$

Moreover, $\bar{f}$ is unique with $\pi\bar{f} = f$, since π is epic. Thus, we've shown most of the following classical *Homomorphism Theorem for Groups.*

Theorem 4.10. *Every group homomorphism* $h : G \to H$ *has a kernel* $\ker(h) = h^{-1}(u)$, *which is a normal subgroup of* G *with the property that, for any normal subgroup* $N \subseteq G$

$$N \subseteq \ker(h)$$

iff there is a (necessarily unique) homomorphism $\bar{h} : G/N \to H$ *with* $\bar{h} \circ \pi = h$, *as indicated in:*



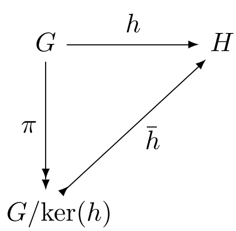
Proof. It only remains to show that if such a factorization $\bar{h}$ exists, then $N \subseteq \ker(h)$. But this is clear, since $\pi(N) = \{[u_G]\}$. So $h(n) = \bar{h}\pi(n) = \bar{h}([n]) = u_H$. □

Finally, putting $N = \ker(h)$ in the theorem, and taking any $[x], [y] \in G/\ker(h)$, we have

$$\begin{aligned} \bar{h}[x] = \bar{h}[y] &\Rightarrow h(x) = h(y) \\ &\Rightarrow h(xy^{-1}) = u \\ &\Rightarrow xy^{-1} \in \ker(h) \\ &\Rightarrow x \sim y \\ &\Rightarrow [x] = [y]. \end{aligned}$$

Thus, $\bar{h}$ is injective, and we conclude:

Corollary 4.11. *Every group homomorphism $h : G \to H$ factors as a quotient followed by an injective homomorphism,*



Thus $\bar{h} : G/\ker(h) \xrightarrow{\sim} \mathrm{im}(h) \subseteq H$ is an isomorphism onto the subgroup $\mathrm{im}(h)$ *that is the image of h.*

In particular, therefore, a homomorphism h is injective if and only if its kernel is "trivial," in the sense that $\ker(h) = \{u\}$.

There is a dual to the notion of a kernel of a homomorphism $h : G \to H$, namely a cokernel $c : H \to C$, which is the universal way of "killing off h" in the sense that $c \circ h = u$. Cokernels are special coequalizers, in just the way that kernels are special equalizers. We leave the details as an exercise.

4.3 Groups as categories

First, let us recall that a group is a category. In particular, a group is a category with one object, in which every arrow is an iso. If G and H are groups, regarded as categories, then we can consider arbitrary functors between them

$$f : G \to H.$$

It is obvious that a functor between groups is exactly the same thing as a group homomorphism.

What is a functor $R : G \to \mathbf{C}$ from a group G to another category $\mathbf{C}$ that is not necessarily a group? If $\mathbf{C}$ is the category of (finite-dimensional) vector spaces and linear transformations, then such a functor is just what the group theorist calls a "linear representation" of G; such a representation permits the description of the group elements as matrices, and the group operation as matrix multiplication. In general, any functor $R : G \to \mathbf{C}$ may be regarded as a representation of G in the category $\mathbf{C}$: the elements of G become automorphisms of some object in $\mathbf{C}$. A permutation representation, for instance, is simply a functor into **Sets**.

We now want to generalize the notions of kernel of a homomorphism, and quotient or factor group by a normal subgroup, from groups to arbitrary categories, and then give the analogous homomorphism theorem for categories.

Definition 4.12. A *congruence* on a category $\mathbf{C}$ is an equivalence relation $f \sim g$ on arrows such that:

1. $f \sim g$ implies $\mathrm{dom}(f) = \mathrm{dom}(g)$ and $\mathrm{cod}(f) = \mathrm{cod}(g)$,

$$\bullet \underset{g}{\overset{f}{\rightrightarrows}} \bullet$$

2. $f \sim g$ implies $bfa \sim bga$ for all arrows $a : A \to X$ and $b : Y \to B$, where $\mathrm{dom}(f) = X = \mathrm{dom}(g)$ and $\mathrm{cod}(f) = Y = \mathrm{cod}(g)$,

$$\bullet \xrightarrow{a} \bullet \underset{g}{\overset{f}{\rightrightarrows}} \bullet \xrightarrow{b} \bullet$$

Let $\sim$ be a congruence on the category $\mathbf{C}$, and define the *congruence category* $\mathbf{C}^{\sim}$ by:

$$\begin{aligned} (\mathbf{C}^{\sim})_0 &= \mathbf{C}_0 \\ (\mathbf{C}^{\sim})_1 &= \{\langle f, g\rangle | f \sim g\} \\ \tilde{1}_C &= \langle 1_C, 1_C\rangle \\ \langle f', g'\rangle \circ \langle f, g\rangle &= \langle f'f, g'g\rangle \end{aligned}$$

One easily checks that this composition is well defined, using the congruence conditions.

There are two evident projection functors:

$$\mathbf{C}^{\sim} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathbf{C}$$

We build the *quotient category* $\mathbf{C}/{\sim}$ as follows:

$$\begin{aligned} (\mathbf{C}/{\sim})_0 &= \mathbf{C}_0 \\ (\mathbf{C}/{\sim})_1 &= (\mathbf{C}_1)/{\sim} \end{aligned}$$

The arrows have the form $[f]$ where $f \in \mathbf{C}_1$, and we can put $1_{[C]} = [1_C]$, and $[g] \circ [f] = [g \circ f]$, as is easily checked, again using the congruence conditions.

There is an evident quotient functor $\pi : \mathbf{C} \to \mathbf{C}/\sim$, making the following a coequalizer of categories:

$$\mathbf{C}^{\sim} \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathbf{C} \xrightarrow{\pi} \mathbf{C}/\sim$$

This is proved much as for groups.

An exercise shows how to use this construction to make coequalizers for certain functors. Let us show how to use it to prove an analogous "homomorphism theorem for categories." Suppose we have categories $\mathbf{C}$ and $\mathbf{D}$ and a functor

$$F : \mathbf{C} \to \mathbf{D}.$$

Then F determines a congruence $\sim_F$ on $\mathbf{C}$ by setting:

$$f \sim_F g \quad \text{iff} \quad \mathrm{dom}(f) = \mathrm{dom}(g),\ \mathrm{cod}(f) = \mathrm{cod}(g),\ F(f) = F(g)$$

That this *is* a congruence is easily checked.

Let us write

$$\ker(F) = \mathbf{C}^{\sim_F} \rightrightarrows \mathbf{C}$$

for the congruence category, and call this the *kernel category* of F.

The quotient category

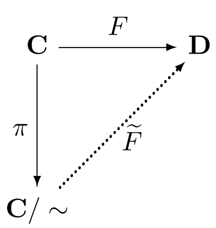
$$\mathbf{C}/\sim_F$$

then has the following UMP:

Theorem 4.13. *Every functor $F : \mathbf{C} \to \mathbf{D}$ has a kernel category $\ker(F)$, determined by a congruence $\sim_F$ on $\mathbf{C}$ such that given any congruence $\sim$ on $\mathbf{C}$ one has:*

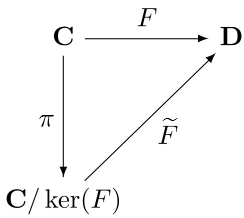
$$f \sim g \Rightarrow f \sim_F g$$

if and only if there is a factorization $\widetilde{F} : \mathbf{C}/\sim \longrightarrow \mathbf{D}$, as indicated in:



Just as in the case of groups, applying the theorem to the case $\mathbf{C}^{\sim} = \ker(F)$ gives a factorization theorem:

Corollary 4.14. *Every functor $F : \mathbf{C} \to \mathbf{D}$ factors as $F = \widetilde{F} \circ \pi$,*



where π is bijective on objects and surjective on Hom-sets, and $\widetilde{F}$ is injective on Hom-sets (i.e. "faithful"*):*

$$\widetilde{F}_{A,B} : \mathrm{Hom}(A, B) \rightarrowtail \mathrm{Hom}(FA, FB) \quad \textit{for all } A, B \in \mathbf{C}/\ker(F)$$

4.4 Finitely presented categories

Finally, let us consider categories presented by generators and relations.

We begin with the free category $\mathbf{C}(G)$ on some finite graph G, and then consider a finite set Σ of relations of the form

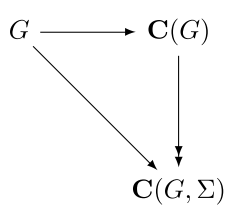
$$(g_1 \circ \ldots \circ g_n) = (g'_1 \circ \ldots \circ g'_m)$$

with all $g_i \in G$, and $\mathrm{dom}(g_n) = \mathrm{dom}(g'_m)$ and $\mathrm{cod}(g_1) = \mathrm{cod}(g'_1)$. Such a relation identifies two "paths" in $\mathbf{C}(G)$ with the same "endpoints" and "direction." Next, let $\sim_\Sigma$ be the smallest congruence $\sim$ on $\mathbf{C}$ such that $g \sim g'$ for each equation $g = g'$ in Σ. Such a congruence exists simply because the intersection of a family of congruences is again a congruence. Taking the quotient by this congruence we have a notion of a *finitely presented category*:

$$\mathbf{C}(G, \Sigma) = \mathbf{C}(G)/\sim_\Sigma$$

This is completely analogous to the notion of a finite presentation for groups, and indeed specializes to that notion in the case of a graph with only one vertex. The UMP of $\mathbf{C}(G, \Sigma)$ is then an obvious variant of that already given for groups.

Specifically, in $\mathbf{C}(G, \Sigma)$ there is a "diagram of type G," that is, a graph homomorphism $i : G \to |\mathbf{C}(G, \Sigma)|$, satisfying all the conditions $i(g) = i(g')$, for all $g = g' \in \Sigma$. Moreover, given any category $\mathbf{D}$ with a diagram of type G, say $h : G \to |\mathbf{D}|$, that satisfies all the conditions $h(g) = h(g')$, for all $g = g' \in \Sigma$, there is a unique functor $\bar{h} : \mathbf{C}(G, \Sigma) \to \mathbf{D}$ with $|\bar{h}| \circ i = h$.



Just as in the case of presentations of groups, one can describe the construction of $\mathbf{C}(G, \Sigma)$ as a coequalizer for two functors. Indeed, suppose we have arrows $f, f' \in \mathbf{C}$. Take the least congruence $\sim$ on $\mathbf{C}$ with $f \sim f'$. Consider the diagram

$$\mathbf{C}(\mathbf{2}) \underset{f'}{\overset{f}{\rightrightarrows}} \mathbf{C} \xrightarrow{\quad q \quad} \mathbf{C}/\sim$$

where $\mathbf{2}$ is the graph with two vertices and an edge between them, f and f' are the unique functors taking the generating edge to the arrows by the same names, and q is the canonical functor to the quotient category. Then q is a coequalizer of f and f'. To show this, take any $d : \mathbf{C} \to \mathbf{D}$ with

$$df = df'.$$

Since $\mathbf{C}(\mathbf{2})$ is free on the graph $\cdot \xrightarrow{x} \cdot$, and $f(x) = f$ and $f'(x) = f'$, we have

$$d(f) = d(f(x)) = d(f'(x)) = d(f').$$

Thus, $\langle f, f' \rangle \in \ker(d)$, so $\sim \subseteq \ker(d)$ (since $\sim$ is minimal with $f \sim f'$). So there is a functor $\bar{d} : \mathbf{C}/\sim \to \mathbf{D}$ such that $d = \bar{d} \circ q$ by the homomorphism theorem.

For the case of several equations rather than just one, in analogy with the case of finitely presented algebras (example **??**), one replaces $\mathbf{2}$ by the graph $n \times \mathbf{2}$, and thus the free category $\mathbf{C}(\mathbf{2})$ by

$$\mathbf{C}(n \times \mathbf{2}) = n \times \mathbf{C}(\mathbf{2}) = \mathbf{C}(\mathbf{2}) + \cdots + \mathbf{C}(\mathbf{2}).$$

Example 4.15. The category with two uniquely isomorphic objects is not free on *any* graph, since it's finite, but has "loops" (cycles). But it *is* finitely presented with graph

$$A \underset{g}{\overset{f}{\rightleftarrows}} B$$

and relations

$$gf = 1_A, \qquad fg = 1_B.$$

Similarly, there are finitely presented categories with just one non-identity arrow $f : \cdot \to \cdot$ and either

$$f \circ f = 1 \quad \text{or} \quad f \circ f = f.$$

In the first case we have the group $\mathbb{Z}/2\mathbb{Z}$. In the second case an "idempotent" (but not a group) Indeed, any of the cyclic groups

$$\mathbb{Z}_n \cong \mathbb{Z}/\mathbb{Z}n$$

occur in this way, with the graph $f : \star \to \star$ and the relation $f^n = 1$.

Of course, there are finitely presented categories with many objects as well. These are always given by a finite graph, the vertices of which are the objects and the edges of which generate the arrows, together with finitely many equations among paths of edges.

4.5 Exercises

1. Regarding a group G as a category with one object and every arrow an isomorphism, show that a categorical congruence $\sim$ on G is the same thing as (the equivalence relation on G determined by) a normal subgroup $N \subseteq G$, that is, show that the two kinds of things are in isomorphic correspondence.

   Show further that the quotient category $G/\sim$ and the factor group G/N coincide. Conclude that the homomorphism theorem for groups is a special case of the one for categories.

2. Consider the definition of a group in a category as applied to the category **Sets**$/I$ of sets sliced over a set I. Show that such a group G determines an I-indexed family of (ordinary) groups G_i by setting $G_i = G^{-1}(i)$ for each $i \in I$. Show that this determines a functor $\mathbf{Groups}(\mathbf{Sets}/I) \to \mathbf{Groups}^I$ into the category of I-indexed families of groups and I-indexed families of homomorphisms.

3. Complete the proof that the groups in the category of groups are exactly the abelian groups by showing that any abelian group admits homomorphic group operations.

4. Use the Eckmann-Hilton argument to prove that every monoid in the category of groups is an internal group.

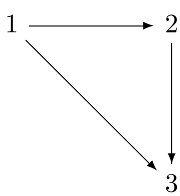
5. Given a homomorphism of abelian groups $f : A \to B$, define the cokernel $c : B \to C$ to be the quotient of B by the subgroup $\mathrm{im}(f) \subseteq B$.

   (a) Show that the cokernel has the following UMP: $c \circ f = 0$, and if $g : B \to G$ is any homomorphism with $g \circ f = 0$ then g factors uniquely through c as $g = u \circ c$.

   (b) Show that the cokernel is a particular kind of coequalizer, and use cokernels to construct arbitrary coequalizers.

   (c) Take the kernel of the cokernel, and show that $f : A \to B$ factors through it. Show moreover that this kernel is (isomorphic to) the image of $f : A \to B$. Infer that the factorization of $f : A \to B$ determined by cokernels agrees with that determined by taking the kernels.

6. Give four different presentations by generators and relations of the category **3**, pictured:



   Is **3** free?

7. Given a congruence $\sim$ on a category $\mathbf{C}$ and arrows in $\mathbf{C}$ as follows,

$$A \underset{f'}{\overset{f}{\rightrightarrows}} B \underset{g'}{\overset{g}{\rightrightarrows}} C$$

show that $f \sim f'$ and $g \sim g'$ implies $g \circ f \sim g' \circ f'$.

8. Given functors $F, G : \mathbf{C} \to \mathbf{D}$ such that for all $C \in \mathbf{C}$, $FC = GC$, define a congruence on $\mathbf{D}$ by the condition:

$$\begin{aligned} f \sim g \quad \text{iff} \quad & \mathrm{dom}(f) = \mathrm{dom}(g) \;\&\; \mathrm{cod}(f) = \mathrm{cod}(g) \\ & \&\; \forall\, \mathbf{E}\; \forall\, H : \mathbf{D} \to \mathbf{E} : \;\; HF = HG \Rightarrow H(f) = H(g) \end{aligned}$$

Prove that this is indeed a congruence. Prove, moreover, that $\mathbf{D}/\sim$ is the coequalizer of F and G.