3

DUALITY

We have seen a few examples of definitions and statements which exhibit a kind of "duality," like initial and terminal object and epimorphisms and monomorphisms. We now want to consider this duality more systematically. Despite its rather trivial first impression, it is indeed a deep and powerful aspect of the categorical approach to mathematical structures.

3.1 The duality principle

First, let us look again at the formal definition of a category: There are two kinds of things, objects A, B, C and..., arrows $f, g, h, \ldots$; four operations $\mathrm{dom}(f)$, $\mathrm{cod}(f)$, 1_A, $g \circ f$; and these satisfy the following seven axioms:

$$\begin{array}{cc} \mathrm{dom}(1_A) = A & \mathrm{cod}(1_A) = A \\ f \circ 1_{\mathrm{dom}(f)} = f & 1_{\mathrm{cod}(f)} \circ f = f \\ \mathrm{dom}(g \circ f) = \mathrm{dom}(f) & \mathrm{cod}(g \circ f) = \mathrm{cod}(g) \\ \multicolumn{2}{c}{h \circ (g \circ f) \ = \ (h \circ g) \circ f} \end{array} \tag{3.1}$$

The operation "$g \circ f$" is only defined where

$$\mathrm{dom}(g) = \mathrm{cod}(f),$$

so a suitable form of this should occur as a condition on each equation containing $\circ$, as in $\mathrm{dom}(g) = \mathrm{cod}(f) \Rightarrow \mathrm{dom}(g \circ f) = \mathrm{dom}(f)$.

Now, given any sentence Σ in the elementary language of category theory, we can form the "dual statement" Σ^* by making the following replacements:

$$\begin{array}{ccc} f \circ g & \text{for} & g \circ f \\ \mathrm{cod} & \text{for} & \mathrm{dom} \\ \mathrm{dom} & \text{for} & \mathrm{cod}. \end{array}$$

It is easy to see that then Σ^* will also be a well-formed sentence. Next, suppose we have shown a sentence Σ to entail one Δ, i.e. $\Sigma \Rightarrow \Delta$, without using any of the category axioms, then clearly $\Sigma^* \Rightarrow \Delta^*$, since the substituted terms are treated as mere undefined constants. But now observe that the axioms (3.1) for category theory CT are themselves "self-dual," in the sense that we have,

$$\mathrm{CT}^* = \mathrm{CT}.$$

We therefore have the following *duality principle.*

Proposition 3.1 (Formal duality). *For any sentence Σ in the language of category theory, if Σ follows from the axioms for categories, then so does its dual Σ^*:*

$$\mathrm{CT} \Rightarrow \Sigma \quad \textit{implies} \quad \mathrm{CT} \Rightarrow \Sigma^*$$

Taking a more conceptual point of view, note that if a statement Σ involves some diagram of objects and arrows,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g \circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

then the dual statement Σ^* involves the diagram obtained from it by reversing the direction and the order of compositions of arrows.

$$\begin{tikzcd} A & B \arrow[l, "f"'] \\ & C \arrow[u, "g"'] \arrow[ul, "f \circ g"] \end{tikzcd}$$

Recalling the opposite category $\mathbf{C}^{\mathrm{op}}$ of a category $\mathbf{C}$, we see that an interpretation of a statement Σ in $\mathbf{C}$ automatically gives an interpretation of Σ^* in $\mathbf{C}^{\mathrm{op}}$.

Now suppose that a statement Σ holds for all categories $\mathbf{C}$. Then it also holds in all categories $\mathbf{C}^{\mathrm{op}}$, and so Σ^* holds in all categories $(\mathbf{C}^{\mathrm{op}})^{\mathrm{op}}$. But since for every category $\mathbf{C}$,

$$(\mathbf{C}^{\mathrm{op}})^{\mathrm{op}} = \mathbf{C}, \tag{3.2}$$

we see that Σ^* also holds in all categories $\mathbf{C}$. We therefore have the following conceptual form of the duality principle:

Proposition 3.2 (Conceptual duality). *For any statement Σ about categories, if Σ holds for all categories, then so does the dual statement Σ^*.*

It may seem that only very simple or trivial statements such as "terminal objects are unique up to isomorphism" are going to be subject to this sort of duality, but in fact this is far from being so. Categorical duality turns out to be a very powerful and far-reaching phenomenon, as we shall see. Like the duality between points and lines in projective geometry, it effectively doubles ones "bang for the buck," yielding two theorems for every proof.

One way this occurs is that, rather than considering statements about all categories, we can also consider the dual of an abstract definition of a structure or property of objects and arrows, like "being a product diagram." The dual structure or property is arrived at by reversing the order of composition and the words "domain" and "codomain." (Equivalently, it results from interpreting the original property in the opposite category.) The next section provides an example of this kind.

3.2 Coproducts

Let us consider the example of products and see what the dual notion must be. First, recall the definition of a product.

Definition 3.3. A diagram $A \xleftarrow{p_1} P \xrightarrow{p_2} B$ is a *product* of A and B, if for any Z and $A \xleftarrow{z_1} Z \xrightarrow{z_2} B$ there is a unique $u : Z \to P$ with $p_i \circ u = z_i$, all as indicated in

$$\begin{tikzcd} & Z \arrow[dl, "z_1"'] \arrow[d, dotted, "u"] \arrow[dr, "z_2"] & \\ A & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & B \end{tikzcd}$$

Now what is the dual statement?

A diagram $A \xrightarrow{q_1} Q \xleftarrow{q_2} B$ is a "dual-product" of A and B if for any Z and $A \xrightarrow{z_1} Z \xleftarrow{z_2} B$ there is a unique $u : Q \to Z$ with $u \circ q_i = z_i$, all as indicated in

$$\begin{tikzcd} & Z & \\ A \arrow[ur, "z_1"] \arrow[r, "q_1"'] & Q \arrow[u, dotted, "u"] & B \arrow[ul, "z_2"'] \arrow[l, "q_2"] \end{tikzcd}$$

Actually, these are called *coproducts*; the convention is to use the prefix "co-" to indicate the dual notion. We usually write $A \xrightarrow{i_1} A+B \xleftarrow{i_2} B$ for the coproduct and $[f, g]$ for the uniquely determined arrow $u : A+B \to Z$. The "coprojections" $i_1 : A \to A+B$ and $i_2 : B \to A+B$ are usually called *injections*, even though they need not be "injective" in any sense.

A coproduct of two objects is therefore exactly their product in the opposite category. Of course, this immediately gives lots of examples of coproducts. But what about some more familiar ones?

Example 3.4. In **Sets**, the coproduct $A + B$ of two sets is their disjoint union, which can be constructed e.g. as

$$A + B = \{(a, 1) \mid a \in A\} \cup \{(b, 2) \mid b \in B\}$$

with evident coproduct injections

$$i_1(a) = (a, 1), \qquad i_2(b) = (b, 2).$$

Given any functions f and g as in:

$$\begin{tikzcd} & Z & \\ A \arrow[ur, "f"] \arrow[r, "i_1"'] & A + B \arrow[u, dotted, "{[f,g]}"'] & B \arrow[ul, "g"'] \arrow[l, "i_2"] \end{tikzcd}$$

we define

$$[f, g](x, \delta) = \begin{cases} f(x) & \delta = 1 \\ g(x) & \delta = 2. \end{cases}$$

Then if we have an h with $h \circ i_1 = f$ and $h \circ i_2 = g$, then for any $(x, \delta) \in A + B$, we must have

$$h(x, \delta) = [f, g](x, \delta)$$

as can be easily calculated.

Note that in **Sets**, every finite set A is a coproduct:

$$A \cong 1 + 1 + \cdots + 1 \ \ (n\text{-times})$$

for $n = \mathrm{card}(A)$. This is because a function $f : A \to Z$ is uniquely determined by its values $f(a)$ for all $a \in A$. So we have

$$\begin{aligned} A &\cong \{a_1\} + \{a_n\} + \cdots + \{a_n\} \\ &\cong 1 + 1 + \cdots + 1 \ \ (n\text{-times}). \end{aligned}$$

In this spirit, we often write simply $2 = 1 + 1, 3 = 1 + 1 + 1$, etc.

Example 3.5. If $M(A)$ and $M(B)$ are *free* monoids on sets A and B, then in **Mon** we can construct their coproduct as

$$M(A) + M(B) \cong M(A + B).$$

One can see this directly by considering words over $A + B$, but it also follows abstractly by using the diagram

$$\begin{tikzcd}
& N & \\
\mathrm{M(A)} \arrow[ur] \arrow[r] & M(A+B) \arrow[u, dotted] & \mathrm{M(B)} \arrow[ul] \arrow[l] \\
A \arrow[u, "\eta_A"] \arrow[r] & A+B \arrow[u, "\eta_{A+B}"'] & B \arrow[u, "\eta_B"'] \arrow[l]
\end{tikzcd}$$

in which the η's are the respective insertions of generators. The UMPs of $M(A)$, $M(B)$, $A+B$, and $M(A+B)$ then imply that the last of these has the required UMP of $M(A)+M(B)$. Note that the set of elements of the coproduct $M(A)+M(B)$ of $M(A)$ and $M(B)$ is *not* the coproduct of the underlying sets, but is only *generated by* the coproduct of their generators, $A + B$. We shall consider coproducts of arbitrary, i.e. not necessarily free, monoids presently.

The foregoing says that the free monoid functor $M : \mathbf{Sets} \to \mathbf{Mon}$ preserves coproducts. This is an instance of a much more general phenomenon, which we will consider later, related to the fact we have already seen that the forgetful functor $U : \mathbf{Mon} \to \mathbf{Sets}$ is representable and so preserves products.

Example 3.6. In **Top** the coproduct of two spaces

$$X + Y$$

is their disjoint union with the topology $O(X + Y) \cong O(X) \times O(Y)$. Note that this follows the pattern of discrete spaces, for which $O(X) = P(X) \cong 2^X$. Thus, for discrete spaces we indeed have

$$O(X + Y) \cong 2^{X+Y} \cong 2^X \times 2^Y \cong O(X) \times O(Y).$$

A related fact is that the product of two powerset boolean algebras $\mathcal{P}(A)$ and $\mathcal{P}(B)$ is also a powerset, namely of the coproduct of the sets A and B,

$$\mathcal{P}(A) \times \mathcal{P}(B) \cong \mathcal{P}(A+B).$$

We leave the verification as an exercise.

Coproducts of posets are similarly constructed from the coproducts of the underlying sets, by "putting them side by side." What about "rooted" posets, that is, posets with a distinguished initial element 0? In the category Pos_0 of such posets and monotone maps that preserve 0, one constructs the coproduct of two such posets A and B from the coproduct $A + B$ in the category Pos of posets, by "identifying" the two different 0s,

$$A +_{\mathrm{Pos}_0} B = (A +_{\mathrm{Pos}} B)/\text{“}0_A = 0_B\text{”}.$$

We shall soon see how to describe such identifications (quotients of equivalence relations) as "coequalizers."

Example 3.7. In a fixed poset P, what is a coproduct of two elements $p, q \in P$? We have

$$p \leq p + q \quad \text{and} \quad q \leq p + q$$

and if

$$p \leq z \quad \text{and} \quad q \leq z$$

then

$$p + q \leq z.$$

So $p + q = p \vee q$ is the *join*, or "least upper bound," of p and q.

Example 3.8. In the *category of proofs* of a deductive system of logic of example 10, section **??**, the usual natural deduction rules of disjunction introduction and elimination give rise to coproducts. Specifically, the introduction rules,

$$\frac{\varphi}{\varphi \vee \psi} \qquad \frac{\psi}{\varphi \vee \psi}$$

determine arrows $i_1 : \varphi \to \varphi \vee \psi$ and $i_2 : \psi \to \varphi \vee \psi$, and the elimination rule,

$$\frac{\varphi \vee \psi \qquad \begin{matrix} [\varphi] \\ \vdots \\ \vartheta \end{matrix} \qquad \begin{matrix} [\psi] \\ \vdots \\ \vartheta \end{matrix}}{\vartheta}$$

turns a pair of arrows $p : \varphi \to \vartheta$ and $q : \psi \to \vartheta$ into an arrow $[p, q] : \varphi \vee \psi \to \vartheta$. The required equations,

$$[p, q] \circ i_1 = p \qquad [p, q] \circ i_2 = q \tag{3.3}$$

will evidently not hold, however, since we are taking identity of proofs as identity of arrows. In order to get coproducts, then, we need to "force" these equations to hold by passing to equivalence classes of proofs, under the equivalence relation generated by these equations, together with the complementary one,

$$[r \circ i_1, r \circ i_2] = r \tag{3.4}$$

for any $r : A + B \to C$. (The intuition behind these identifications is that one should equate proofs which become the same when one omits such "detours.") In the new category with equivalence classes of proofs as arrows, the arrow $[p, q]$ will also be the *unique* one satisfying (3.3), so that $\varphi \vee \psi$ indeed becomes a coproduct.

Closely related to this example (via the Curry-Howard correspondence of remark **??**) are the sum types in the λ-calculus, as usually formulated using `case` terms; these are coproducts in the *category of types* defined in subsection 2.5.

Example 3.9. Two monoids A, B have a coproduct of the form

$$A + B = M(|A| + |B|)/\sim$$

where, as before, the free monoid $M(|A|+|B|)$ is strings (words) over the disjoint union $|A| + |B|$ of the underlying sets—i.e. the elements of A and B—and the equivalence relation $v \sim w$ is the least one containing all instances of the following equations:

$$\begin{aligned}
(\dots x\, u_A\, y \dots) &= (\dots x\, y \dots) \\
(\dots x\, u_B\, y \dots) &= (\dots x\, y \dots) \\
(\dots a\, a' \dots) &= (\dots a \cdot_A a' \dots) \\
(\dots b\, b' \dots) &= (\dots b \cdot_B b' \dots).
\end{aligned}$$

(If you need a refresher on quotienting a set by an equivalence relation, skip ahead and read the beginning of Section 3.4 now.) The unit is of course the equivalence class $[-]$ of the empty word (which is the same as $[u_A]$ and $[u_B]$). Multiplication of equivalence classes is also as expected, namely

$$[x \dots y] \cdot [x' \dots y'] = [x \dots y x' \dots y'].$$

The coproduct injections $i_A : A \to A + B$ and $i_B : B \to A + B$ are simply

$$i_A(a) = [a], \qquad i_B(b) = [b],$$

which are now easily seen to be homomorphisms. Given any homomorphisms $f : A \to M$ and $g : B \to M$ into a monoid M, the unique homomorphism

$$[f, g] : A + B \longrightarrow M$$

is defined by first extending the function $[|f|, |g|] : |A| + |B| \to |M|$ to one $[f, g]'$ on the free monoid $M(|A| + |B|)$,

$$\begin{tikzcd}
{|A| + |B|} \arrow[r, "{[|f|, |g|]}"] & {|M|} \\
{M(|A| + |B|)} \arrow[r, "{[f,g]'}"] \arrow[d] & M \\
{M(|A| + |B|)/\sim} \arrow[ur, dotted, "{[f,g]}"'] &
\end{tikzcd}$$

and then observing that $[f, g]'$ "respects the equivalence relation $\sim$," in the sense that if $v \sim w$ in $M(|A| + |B|)$, then $[f, g]'(v) = [f, g]'(w)$. Thus the map $[f, g]'$

extends to the quotient to yield the desired map $[f,g] : M(|A|+|B|)/\sim \longrightarrow M$. (Why is this homomorphism the *unique* one $h : M(|A|+|B|)/\sim \longrightarrow M$ with $hi_A = f$ and $hi_B = g$?) Summarizing, we thus have:

$$A+B \;\cong\; M(|A|+|B|)/\sim .$$

This construction also works to give coproducts in **Groups**, where it is usually called the *free product* of A and B and written $A \oplus B$, as well as many other categories of "algebras," i.e. sets equipped with operations. Again, as in the free case, the underlying set of $A+B$ is *not* the coproduct of A and B as sets (the forgetful functor **Mon** $\to$ **Sets** does not preserve coproducts).

Example 3.10. For *abelian groups* A, B, the free product $A \oplus B$ need not be abelian. One could, of course, take a further quotient of $A \oplus B$ to get a coproduct in the category Ab of abelian groups, but there is a more convenient (and important) presentation, which we now consider.

Since the words in the free product $A \oplus B$ must be forced to satisfy the further commutativity conditions

$$(a_1 b_1 b_2 a_2 \ldots) \sim (a_1 a_2 \ldots b_1 b_2 \ldots)$$

we can shuffle all the a's to the front, and the b's to the back, of the words. But, furthermore, we already have

$$(a_1 a_2 \ldots b_1 b_2 \ldots) \sim (a_1 + a_2 + \cdots + b_1 + b_2 + \cdots).$$

Thus, we in effect have pairs of elements (a, b). So we can take the *product* set as the underlying set of the coproduct

$$|A+B| = |A \times B|.$$

As inclusions, we use the homomorphisms

$$i_A(a) = (a, 0_B)$$
$$i_B(b) = (0_A, b).$$

Then given any homomorphisms $A \xrightarrow{f} X \xleftarrow{g} B$, we let $[f,g] : A+B \to X$ be defined by

$$[f,g](a,b) = f(a) +_X g(b)$$

which can easily be seen to do the trick (exercise!).

Moreover, not only can the underlying *sets* be the same, the product and coproduct of abelian groups are actually isomorphic as *groups*:

Proposition 3.11. *In the category* **Ab** *of abelian groups, there is a canonical isomorphism between the binary coproduct and product,*

$$A+B \cong A \times B.$$

Proof. To define an arrow $\vartheta : A+B \to A \times B$ we need one $A \to A \times B$ (and one $B \to A \times B$), so we need arrows $A \to A$ and $A \to B$ (and $B \to A$ and $B \to B$). For these we take $1_A : A \to A$ and the zero homomorphism $0_B : A \to B$ (and $0_A : B \to A$ and $1_B : B \to B$). Thus, all together we get

$$\vartheta = [\langle 1_A, 0_B \rangle, \langle 0_A, 1_B \rangle] : A + B \to A \times B.$$

Then given any $(a, b) \in A + B$, we have

$$\begin{aligned} \vartheta(a,b) &= [\langle 1_A, 0_B \rangle, \langle 0_A, 1_B \rangle](a,b) \\ &= \langle 1_A, 0_B \rangle(a) + \langle 0_A, 1_B \rangle(b) \\ &= (1_A(a), 0_B(a)) + (0_A(b), 1_B(b)) \\ &= (a, 0_B) + (0_A, b) \\ &= (a + 0_A, 0_B + b) \\ &= (a, b). \end{aligned}$$

□

This fact was first observed by Mac Lane, and it was shown to lead to a binary operation of addition on parallel arrows $f, g : A \to B$ between abelian groups (and related structures like modules and vector spaces). In fact, the group structure of a particular abelian group A can be recovered from this operation on arrows into A. More generally, the existence of such an addition operation on arrows can be used as the basis of an abstract description of categories like **Ab**, called "abelian categories," which are suitable for axiomatic homology theory.

Just as with products, one can consider the empty coproduct, which is an initial object 0, as well as coproducts of several factors, and the coproduct of two arrows,

$$f + f' : A + A' \to B + B'$$

which leads to a coproduct functor $+ : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ on categories $\mathbf{C}$ with binary coproducts. All of these facts follows simply by duality; that is, by considering the dual notions in the opposite category. Similarly, we have the following proposition.

Proposition 3.12. *Coproducts are unique up to isomorphism.*

Proof. Use duality and the fact that the dual of "isomorphism" is "isomorphism."

□

In just the same way, one also shows that binary coproducts are associative up to isomorphism, $(A + B) + C \cong A + (B + C)$.

Thus is general, in the future it will suffice to introduce new notions once and then simply observe that the dual notions have analogous (but dual) properties. The next two sections give another example of this sort.

3.3 Equalizers

In this section, we consider another abstract characterization; this time a common generalization of the kernel of a homomorphism and an equationally defined "variety," like the set of zeros of a real-valued function—as well as set theory's axiom of separation.

Definition 3.13. In any category $\mathbf{C}$, given parallel arrows

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

an *equalizer* of f and g consists of an object E and an arrow $e : E \to A$, universal such that

$$f \circ e = g \circ e.$$

That is, given any $z : Z \to A$ with $f \circ z = g \circ z$ there is a *unique* $u : Z \to E$ with $e \circ u = z$, all as in the diagram

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Z \arrow[u, dotted, "u"] \arrow[ur, "z"'] & & \end{tikzcd}$$

Let us consider some simple examples.

Example 3.14. Suppose we have the functions $f, g : \mathbb{R}^2 \rightrightarrows \mathbb{R}$ where

$$f(x, y) = x^2 + y^2$$
$$g(x, y) = 1$$

and we take the equalizer, say in **Top**. This is the subspace,

$$S = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \hookrightarrow \mathbb{R}^2,$$

i.e. the unit circle in the plane. For, given any "generalized element" $z : Z \to \mathbb{R}^2$, we get a pair of such "elements" $z_1, z_2 : Z \to \mathbb{R}$ just by composing with the two projections, $z = \langle z_1, z_2 \rangle$, and for these we then have:

$$\begin{aligned} f(z) = g(z) &\text{ iff } {z_1}^2 + {z_2}^2 = 1 \\ &\text{ iff } \text{“}\langle z_1, z_2 \rangle = z \in S\text{”}, \end{aligned}$$

where the last line really means that there is a factorization $z = \bar{z} \circ i$ of z through the inclusion $i : S \hookrightarrow \mathbb{R}^2$, as indicated in the following diagram.

$$\begin{tikzcd} S \arrow[r, hook, "i"] & \mathbb{R}^2 \arrow[r, shift left, "x^2+y^2"] \arrow[r, shift right, "1"'] & \mathbb{R} \\ Z \arrow[u, dotted, "\bar{z}"] \arrow[ur, "z"'] & & \end{tikzcd}$$

Since the inclusion i is monic, such a factorization, if it exists, is necessarily unique, and thus $S \hookrightarrow \mathbb{R}^2$ is indeed the equalizer of f and g.

Example 3.15. Similarly, in **Sets**, given any functions $f, g : A \rightrightarrows B$, their equalizer is the inclusion into A of the equationally defined subset

$$\{x \in A \mid f(x) = g(x)\} \hookrightarrow A.$$

The argument is essentially the same as the one just given.

Let us pause here to note that in fact, every subset $U \subseteq A$ is of this "equational" form, that is, every subset is an equalizer for some pair of functions. Indeed, one can do this in a very canonical way. First, let us put

$$2 = \{\top, \bot\},$$

thinking of it as the set of "truth values." Then consider the *characteristic function*

$$\chi_U : A \to 2,$$

defined for $x \in A$ by

$$\chi_U(x) = \begin{cases} \top & x \in U \\ \bot & x \notin U. \end{cases}$$

Thus we have

$$U = \{x \in A \mid \chi_U(x) = \top\}.$$

So the following is an equalizer

$$\begin{tikzcd} U \arrow[r] & A \arrow[r, shift left, "\top!"] \arrow[r, shift right, "\chi_U"'] & 2 \end{tikzcd}$$

where $\top ! = \top \circ ! : U \xrightarrow{!} 1 \xrightarrow{\top} 2$.

Moreover, for every function,

$$\varphi : A \to 2$$

we can form the "variety" (i.e. equational subset)

$$V_\varphi = \{x \in A \mid \varphi(x) = \top\}$$

as an equalizer, in the same way. (Thinking of φ as a "propositional function" defined on A, the subset $V_\varphi \subseteq A$ is the "extension" of φ provided by the axiom of separation.)

Now, it is easy to see that these operations χ_U and V_φ are mutually inverse:

$$\begin{aligned} V_{\chi_U} &= \{x \in A \mid \chi_U(x) = \top\} \\ &= \{x \in A \mid x \in U\} \\ &= U \end{aligned}$$

for any $U \subseteq A$, and given any $\varphi : A \to 2$,

$$\begin{aligned} \chi_{V_\varphi}(x) &= \begin{cases} \top & x \in V_\varphi \\ \bot & x \notin V_\varphi \end{cases} \\ &= \begin{cases} \top & \varphi(x) = \top \\ \bot & \varphi(x) = \bot \end{cases} \\ &= \varphi(x). \end{aligned}$$

Thus we have the familiar isomorphism

$$\mathrm{Hom}(A, 2) \cong P(A),$$

mediated by taking equalizers.

The fact that equalizers of functions can be taken to be subsets is a special case of a more general phenomenon:

Proposition 3.16. *In any category, if $e : E \to A$ is an equalizer of some pair of arrows, then e is monic.*

Proof. Consider the diagram:

$$\begin{tikzcd} E \arrow[r, "e"] & \mathrm{A} \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ Z \arrow[u, shift right, "x"] \arrow[u, shift left, "y"'] \arrow[ur, "z"'] & & \end{tikzcd}$$

in which we assume e is the equalizer of f and g. Supposing $ex = ey$, we want to show $x = y$. Put $z = ex = ey$. Then $fz = fex = gex = gz$, so there is a *unique* $u : Z \to E$ such that $eu = z$. So from $ex = z$ and $ey = z$ it follows that $x = u = y$. □

Example 3.17. In many other categories, such as posets and monoids, the equalizer of a parallel pair of arrows $f, g : A \rightrightarrows B$ can be constructed by taking the equalizer of the underlying functions as above, that is, the subset $A(f = g) \subseteq A$ of elements $x \in A$ where f and g agree, $f(x) = g(x)$, and then restricting the structure of A to $A(f = g)$. For instance, in posets one takes the ordering from A restricted to this subset $A(f = g)$, and in topological spaces one takes the subspace topology.

In monoids, the subset $A(f = g)$ is then also a monoid with the operations from A, and the inclusion is therefore a homomorphism. This is so because $f(u_A) = u_B = g(u_A)$, and if $f(a) = g(a)$ and $f(a') = g(a')$, then $f(a \cdot a') = f(a) \cdot f(a') = g(a) \cdot g(a') = g(a \cdot a')$. Thus $A(f = g)$ contains the unit and is closed under the product operation.

In abelian groups, for instance, one has an alternate description of the equalizer, using the fact that,

$$f(x) = g(x) \quad \text{iff} \quad (f - g)(x) = 0.$$

Thus the equalizer of f and g is the same as that of the homomorphism $(f - g)$ and the zero homomorphism $0 : A \to B$, so it suffices to consider equalizers of the special form $A(h, 0) \rightarrowtail A$ for arbitrary homomorphisms $h : A \to B$. This subgroup of A is called the *kernel* of h, written $\ker(h)$. Thus we have the equalizer:

$$\begin{tikzcd} \ker(f-g) \arrow[r, hook] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

The kernel of a homomorphism is of fundamental importance in the study of groups, as we shall consider further in the next chapter.

3.4 Coequalizers

A coequalizer is a generalization of a quotient by an equivalence relation, so let us begin by reviewing that notion, which we have already made use of several times. Recall first that an *equivalence relation* on a set X is a binary relation $x \sim y$ which is

reflexive: $x \sim x$,
symmetric: $x \sim y$ implies $y \sim x$,
transitive: $x \sim y$ and $y \sim z$ implies $x \sim z$.

Given such a relation, define the *equivalence class* $[x]$ of an element $x \in X$ by

$$[x] = \{y \in X \mid x \sim y\}.$$

The various different equivalence classes $[x]$ then form a *partition* of X, in the sense that every element y is in exactly one of them, namely $[y]$ (prove this!).

One sometimes thinks of an equivalence relation as arising from the equivalent elements having some property in common (like being the same color). One can then regard the equivalence classes $[x]$ as the properties and in that sense as "abstract objects" (the colors red, blue, etc., themselves). This is sometimes known as "definition by abstraction," and it describes e.g. the way that the real numbers can be constructed from Cauchy sequences of rationals or the finite cardinal numbers from finite sets.

The set of all equivalence classes

$$X/\sim \;=\; \{[x] \,|\, x \in X\}$$

may be called the *quotient* of X by $\sim$. It is used in place of X when one wants to "abstract away" the difference between equivalent elements $x \sim y$, in the sense that in $X/\sim$ such elements (and only such) are identified, since

$$[x] = [y] \quad \text{iff} \quad x \sim y.$$

Observe that the *quotient mapping*,

$$q : X \longrightarrow X/\sim$$

taking x to $[x]$ has the property that a map $f : X \to Y$ extends along q,

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr, "f"'] & X/\sim \arrow[d, dotted] \\ & Y \end{tikzcd}$$

just in case f respects the equivalence relation, in the sense that $x \sim y$ implies $f(x) = f(y)$.

Now let us consider the notion dual to that of equalizer, namely that of a coequalizer.

Definition 3.18. For any parallel arrows $f, g : A \to B$ in a category $\mathbf{C}$, a *coequalizer* consists of Q and $q : B \to Q$, universal with the property $qf = qg$, as in:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "z"'] & Q \arrow[d, dotted, "u"] \\ & & Z \end{tikzcd}$$

That is, given any Z and $z : B \to Z$, if $zf = zg$, then there exists a unique $u : Q \to Z$ such that $uq = z$.

First, observe that by duality, we know that such a coequalizer q in a category $\mathbf{C}$ is an equalizer in $\mathbf{C}^{\mathrm{op}}$, hence monic by proposition 3.16, and so epic in $\mathbf{C}$.

Proposition 3.19. *If $q : B \to Q$ is a coequalizer of some pair of arrows, then q is epic.*

We can therefore think of a coequalizer $q : B \twoheadrightarrow Q$ as a "collapse" of B by "identifying" all pairs $f(a) = g(a)$ (speaking as if there were such "elements" $a \in A$). Moreover, we do this in the "minimal" way, i.e. disturbing B as little as possible, in that one can always map Q to anything else Z in which all such identifications hold.

Example 3.20. Let $R \subseteq X \times X$ be an equivalence relation on a set X, and consider the diagram

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \end{tikzcd}$$

where the r's are the two projections of the inclusion $R \subseteq X \times X$,

$$\begin{tikzcd} & R \arrow[dl, "r_1"'] \arrow[d, hook] \arrow[dr, "r_2"] & \\ X & X \times X \arrow[l, "p_1"] \arrow[r, "p_2"'] & X \end{tikzcd}$$

The quotient projection

$$\pi : X \longrightarrow X/R$$

defined by $x \mapsto [x]$ is then a coequalizer of r_1 and r_2. For given an $f : X \to Y$ as in:

$$\begin{tikzcd} R \arrow[r, shift left, "r_1"] \arrow[r, shift right, "r_2"'] & X \arrow[r, "\pi"] \arrow[dr, "f"'] & X/R \arrow[d, dotted, "\bar{f}"] \\ & & \mathrm{Y} \end{tikzcd}$$

there exists a function $\bar{f}$ such that

$$\bar{f}\pi(x) = f(x)$$

whenever f respects R in the sense that $(x, x') \in R$ implies $f(x) = f(x')$, as already noted. But this condition just says that $f \circ r_1 = f \circ r_2$, since $f \circ r_1(x, x') = f(x)$ and $f \circ r_2(x, x') = f(x')$ for all $(x, x') \in R$. Moreover, if it exists, such a function $\bar{f}$, is then necessarily unique, since π is an epimorphism.

The coequalizer in **Sets** of an arbitrary parallel pair of functions $f, g : A \rightrightarrows B$ can be constructed by quotienting B by the equivalence relation generated by the equations $f(x) = g(x)$ for all $x \in A$. We leave the details as an exercise.

Example 3.21. In example 3.6, we considered the coproduct of *rooted* posets P and Q by first making $P + Q$ in posets and then "identifying" the resulting two different 0-elements 0_P and 0_Q (i.e. the images of these under the respective coproduct inclusions. We can now describe this "identification" as a coequalizer, taken in posets,

$$1 \underset{0_Q}{\overset{0_P}{\rightrightarrows}} P + Q \longrightarrow P + Q/(0_P = 0_Q)$$

This clearly has the right UMP to be the coproduct in *rooted* posets.

In topology one also often makes "identifications" of points (as in making the circle out of the interval by identifying the endpoints), of subspaces (making the sphere from the disk), etc. These and many similar "gluing" constructions can be described as coequalizers. In **Top**, the coequalizer of a parallel pair of maps $f, g : X \to Y$ can be constructed as a *quotient space* of Y (see the exercises).

Example 3.22. *Presentations of algebras*
Consider any category of "algebras," i.e. sets equipped with operations (of finite arity), such as monoids or groups. We shall show later that such a category has free algebras for all sets and coequalizers for all parallel pairs of arrows (see the exercises for a proof that monoids have coequalizers). We can use these to determine the notion of a *presentation* of an algebra by *generators* and *relations*. For example, suppose we are given:

$$\begin{aligned} &\text{Generators:} \quad x,\ y,\ z \\ &\text{Relations:} \quad xy = z,\ y^2 = 1 \end{aligned} \tag{3.5}$$

To build an algebra on these generators and satisfying these relations, start with the free algebra,

$$F(3) = F(x, y, z),$$

and then "force" the relation $xy = z$ to hold by taking a coequalizer of the maps

$$F(1) \underset{z}{\overset{xy}{\rightrightarrows}} F(3) \xrightarrow{\quad q \quad} Q$$

We use the fact that maps $F(1) \to A$ correspond to elements $a \in A$ by $v \mapsto a$, where v is the single generator of $F(1)$. Now similarly, for the equation $y^2 = 1$, take the coequalizer:

$$F(1) \underset{q(1)}{\overset{q(y^2)}{\rightrightarrows}} Q \longrightarrow Q'$$

These two steps can actually be done simultaneously; let

$$F(2) = F(1) + F(1)$$

$$\begin{tikzcd} F(2) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & F(3) \end{tikzcd}$$

where $f = [xy, y^2]$ and $g = [z, 1]$. The coequalizer $q : F(3) \to Q$ of f and g then "forces" both equations to hold, in the sense that in Q we have

$$q(x)q(y) = q(z), \quad q(y)^2 = 1.$$

Moreover, no other relations among the generators hold in Q except those required to hold by the stipulated equations. To make the last statement precise, observe that given any algebra A and any three elements $a, b, c \in A$ such that $ab = c$ and $b^2 = 1$, by the UMP of Q there is a unique homomorphism $u : Q \to A$ such that

$$u(x) = a, \qquad u(y) = b, \qquad u(z) = c.$$

Thus any other equation that holds among the generators in Q will also hold in any other algebra in which the stipulated equations (3.5) hold, since the homomorphism u also preserves equations. In this sense, Q is the "universal" algebra with three generators satisfying the stipulated equations; as may be written suggestively in the form

$$Q \cong F(x, y, z)/(xy = z,\ y^2 = 1).$$

Generally, given a finite presentation:

$$\begin{aligned} &\text{Generators:} \quad g_1, \ldots, g_n \\ &\text{Relations:} \quad l_1 = r_1,\ \ldots,\ l_m = r_m \end{aligned} \tag{3.6}$$

(where the l_i and r_i are arbitrary terms built from the generators and the operations) the algebra determined by that presentation is the coequalizer

$$\begin{tikzcd} F(m) \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & F(n) \arrow[r] & Q = F(n)/(l = r) \end{tikzcd}$$

where $l = [l_1, \ldots, l_m]$ and $r = [r_1, \ldots, r_m]$. Moreover, any such coequalizer between (finite) free algebras can clearly be regarded as a (finite) presentation by generators and relations. Algebras that can be given in this way are said to be *finitely presented.*

Warning 3.23. Presentations are not unique. One may well have two different presentations $F(n)/(l = r)$ and $F(n')/(l' = r')$ by generators and relations of the same algebra,

$$F(n)/(l = r) \ \cong\ F(n')/(l' = r').$$

For instance, given $F(n)/(l = r)$ just add a new generator g_{n+1} and the new relation $g_n = g_{n+1}$. In general, there are many different ways of presenting a given algebra, just like there are many ways of axiomatizing a logical theory.

We did not really make use of the finiteness condition in the foregoing considerations. Indeed, *any* sets of generators G and relations R give rise to an algebra in the same way, by taking the coequalizer

$$F(R) \underset{r_2}{\overset{r_1}{\rightrightarrows}} F(G) \longrightarrow F(G)/(r_1 = r_2).$$

In fact, every algebra can be "presented" by generators and relations in this sense, i.e. as a coequalizer of maps between free algebras. Specifically, we have the following proposition for monoids, an analogous version of which also holds for groups and other algebras.

Proposition 3.24. *For every monoid M there are sets R and G and a coequalizer diagram,*

$$F(R) \underset{r_2}{\overset{r_1}{\rightrightarrows}} F(G) \longrightarrow M$$

with $F(R)$ and $F(G)$ free; thus $M \cong F(G)/(r_1 = r_2)$.

Proof. For any monoid N, let us write $TN = M(|N|)$ for the free monoid on the set of elements of N (and note that T is therefore a functor). There is a homomorphism,

$$\pi : TN \to N$$
$$\pi(x_1, \ldots, x_n) = x_1 \cdot \ldots \cdot x_n$$

induced by the identity $1_{|N|} : |N| \to |N|$ on the generators. (Here we are writing the elements of TN as tuples $(x_1, \ldots, x_n)$ rather than strings $x_1 \ldots x_n$ for clarity.) Applying this construction twice to a monoid M results in the arrows π and ε in the following diagram,

$$T^2M \underset{\mu}{\overset{\varepsilon}{\rightrightarrows}} TM \xrightarrow[\pi]{} M \tag{3.7}$$

where $T^2M = TTM$ and $\mu = T\pi$. Explicitly, the elements of T^2M are tuples of tuples of elements of M, say $((x_1, \ldots, x_n), \ldots, (z_1, \ldots, z_m))$, and the homomorphisms ε and μ have the effect:

$$\varepsilon((x_1, \ldots, x_n), \ldots, (z_1, \ldots, z_m)) = (x_1, \ldots, x_n, \ldots, z_1, \ldots, z_m)$$
$$\mu((x_1, \ldots, x_n), \ldots, (z_1, \ldots, z_m)) = (x_1 \cdot \ldots \cdot x_n, \ldots, z_1 \cdot \ldots \cdot z_m)$$

Briefly, ε uses the multiplication in TM and μ uses that in M.

Now clearly $\pi \circ \varepsilon = \pi \circ \mu$. We claim that (3.7) is a coequalizer of monoids. To that end, suppose we have a monoid N and a homomorphism $h : TM \to N$ with $h\varepsilon = h\mu$. Then for any tuple $(x, \dots, z)$ we have

$$\begin{aligned} h(x, \dots, z) &= h\varepsilon((x, \dots, z)) \\ &= h\mu((x, \dots, z)) \\ &= h(x \cdot \dots \cdot z). \end{aligned} \tag{3.8}$$

Now define $\bar{h} = h \circ i$, where $i : |M| \to |TM|$ is the insertion of generators, as indicated in the following:

$$\begin{tikzcd} T^2 \arrow[r, shift left, "\varepsilon"] \arrow[r, shift right, "\mu"'] & \mathrm{TM} \arrow[r, shift left, "\pi"] \arrow[dr, "h"'] & M \arrow[l, shift left, dotted, "i"] \arrow[d, "h \circ i"] \\ & & \mathrm{N} \end{tikzcd}$$

We then have:

$$\begin{aligned} \bar{h}\pi(x, \dots, z) &= hi\pi(x, \dots, z) \\ &= h(x \cdot \dots \cdot z) \\ &= h(x, \dots, z) \qquad \text{by (3.8)} \end{aligned}$$

We leave it as an easy exercise for the reader to show that $\bar{h}$ is a homomorphism. $\square$

3.5 Exercises

1. In any category $\mathbf{C}$, show that

$$A \xrightarrow{c_1} C \xleftarrow{c_2} B$$

is a coproduct diagram just if for every object Z, the map

$$\begin{aligned} \mathrm{Hom}(C, Z) &\longrightarrow \mathrm{Hom}(A, Z) \times \mathrm{Hom}(B, Z) \\ f &\longmapsto \quad \langle f \circ c_1,\ f \circ c_2 \rangle \end{aligned}$$

is an isomorphism. Do this by using duality, taking the corresponding fact about products as given.

2. Show in detail that the free monoid functor M preserves coproducts: for any sets A, B,

$$M(A) + M(B) \cong M(A + B) \qquad \text{(canonically)}.$$

Do this as indicated in the text by using the UMPs of the coproducts $A+B$ and $M(A)+M(B)$ and of free monoids.

3. Verify that the construction given in the text of the coproduct of monoids $A+B$ as a quotient of the free monoid $M(|A|+|B|)$ really is a coproduct in the category of monoids.

4. Show that the product of two powerset boolean algebras $\mathcal{P}(A)$ and $\mathcal{P}(B)$ is also a powerset, namely of the coproduct of the sets A and B,

$$\mathcal{P}(A) \times \mathcal{P}(B) \cong \mathcal{P}(A+B).$$

(*Hint*: determine the projections $\pi_1 : \mathcal{P}(A+B) \to \mathcal{P}(A)$ and $\pi_2 : \mathcal{P}(A+B) \to \mathcal{P}(B)$, and check that they have the UMP of the product.)

5. Consider the category of proofs of a natural deduction system with disjunction introduction and elimination rules. Identify proofs under the equations

$$[p,q] \circ i_1 = p, \qquad [p,q] \circ i_2 = q$$
$$[r \circ i_1, r \circ i_2] = r$$

for any $p : A \to C$, $q : B \to C$, and $r : A+B \to C$. By passing to equivalence classes of proofs with respect to the equivalence relation generated by these equations (i.e. two proofs are equivalent if you can get one from the other by removing all such "detours"). Show that the resulting category does indeed have coproducts.

6. Verify that the category of monoids has all equalizers and finite products, then do the same for abelian groups.

7. Show that in any category with coproducts, the coproduct of two projectives is again projective.

8. Dualize the notion of projectivity to define an *injective* object in a category. Show that a map of posets is monic iff it is injective on elements. Give examples of a poset that is injective and one that is not injective.

9. Complete the proof of Proposition 3.24 in the text by showing that $\bar{h}$ is indeed a homomorphism.

10. In the proof of Proposition 3.24 in the text it is shown that any monoid M has a specific presentation $T^2M \rightrightarrows TM \to M$ as a coequalizer of free monoids. Show that coequalizers of this particular form are preserved by the forgetful functor **Mon** $\to$ **Sets**.

11. Prove that **Sets** has all coequalizers by constructing the coequalizer of a parallel pair of functions,

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \longrightarrow Q = B/(f=g)$$

by quotienting B by a suitable equivalence relation R on B, generated by the pairs $(f(x), g(x))$ for all $x \in A$. (Define R to be the intersection of all equivalence relations on B containing all such pairs.)

12. Verify the coproduct-coequalizer construction mentioned in the text for coproducts of rooted posets, i.e. posets with a least element 0 and monotone maps preserving 0. Specifically, show that the coproduct $P +_0 Q$ of two such posets can be constructed as a coequalizer in posets,

$$\begin{tikzcd} 1 \arrow[r, shift left, "0_P"] \arrow[r, shift right, "0_Q"'] & P + Q \arrow[r] & P +_0 Q. \end{tikzcd}$$

(You may assume as given the fact that the category of posets has all coequalizers.)

13. Show that the category of monoids has all coequalizers as follows.
1. Given any pair of monoid homomorphisms $f, g : M \to N$, show that the following equivalence relations on N agree:
   a) $n \sim n' \Leftrightarrow$ for all monoids X and homomorphisms $h : N \to X$, one has $hf = hg$ implies $hn = hn'$,
   b) the intersection of all equivalence relations $\sim$ on N satisfying $fm \sim gm$ for all $m \in M$ as well as:

$$n \sim n' \text{ and } m \sim m' \Rightarrow n \cdot m \sim n' \cdot m'$$

2. Taking $\sim$ to be the equivalence relation defined in (1), show that the quotient set $N/\sim$ is a monoid under $[n] \cdot [m] = [n \cdot m]$, and the projection $N \to N/\sim$ is the coequalizer of f and g.

14. Consider the category of sets.
   (a) Given a function $f : A \to B$, describe the equalizer of the functions $f \circ p_1, f \circ p_2 : A \times A \to B$ as a (binary) relation on A and show that it is an equivalence relation (called the *kernel* of f).
   (b) Show that the kernel of the quotient $A \to A/R$ by an equivalence relation R is R itself.
   (c) Given *any* binary relation $R \subseteq A \times A$, let $\langle R \rangle$ be the equivalence relation on A generated by R (the least equivalence relation on A containing R). Show that the quotient $A \to A/\langle R \rangle$ is the coequalizer of the two projections $R \rightrightarrows A$.
   (d) Using the foregoing, show that for any binary relation R on a set A, one can characterize the equivalence relation $\langle R \rangle$ generated by R as the kernel of the coequalizer of the two projections of R.

15. Construct coequalizers in **Top** as follows. Given a parallel pair of maps $f, g : X \rightrightarrows Y$, make a *quotient space* $q : Y \to Q$ by (i) taking the coequalizer of $|f|$ and $|g|$ in **Sets** to get the function $|q| : |Y| \to |Q|$, then (ii) equip $|Q|$

with the *quotient topology*, under which a set $V \subseteq Q$ is open iff $q^{-1}(V) \subseteq Y$ is open. This is plainly the finest topology on $|Q|$ which makes the projection $|q|$ continuous.