

OXFORD LOGIC GUIDES

*Series Editors*

D.M. GABBAY  
A.J. MACINTYRE  
D. SCOTT

## OXFORD LOGIC GUIDES

---

*Available books in the series:*

10. Michael Hallett: *Cantorian set theory and limitation of size*
17. Stewart Shapiro: *Foundations without foundationalism*
18. John P. Cleave: *A study of logics*
21. C. McLarty: *Elementary categories, elementary toposes*
22. R.M. Smullyan: *Recursion theory for metamathematics*
23. Peter Clote and Jan Krajíček: *Arithmetic, proof theory, and computational complexity*
24. A. Tarski: *Introduction to logic and to the methodology of deductive sciences*
25. G. Malinowski: *Many valued logics*
26. Alexandre Borovik and Ali Nesin: *Groups of finite Morley rank*
27. R.M. Smullyan: *Diagonalization and self-reference*
28. Dov M. Gabbay, Ian Hodkinson, and Mark Reynolds: *Temporal logic: Mathematical foundations and computational aspects: volume 1*
29. Saharon Shelah: *Cardinal arithmetic*
30. Erik Sandewall: *Features and fluents: Volume I: A Systematic approach to the representation of knowledge about dynamical systems*
31. T.E. Forster: *Set theory with a universal set: Exploring an untyped universe, second edition*
32. Anand Pillay: *Geometric stability theory*
33. Dov M. Gabbay: *Labelled deductive systems*
35. Alexander Chagrov and Michael Zakharyashev: *Modal logic*
36. G. Sambin and J. Smith: *Twenty-five years of Martin-Löf constructive type theory*
37. María Manzano: *Model theory*
38. Dov M. Gabbay: *Fibring logics*
39. Michael Dummett: *Elements of Intuitionism, second edition*
40. D.M. Gabbay, M.A. Reynolds, and Marcelo Finger: *Temporal logic: Mathematical foundations and computational aspects volume 2*
41. J.M. Dunn and G. Hardegree: *Algebraic Methods in Philosophical Logic*
42. H. Rott: *Change, Choice and Inference: A study of belief revision and non-monotonic reasoning*
43. P.T. Johnstone: *Sketches of an Elephant: A topos theory compendium: Volume 1*
44. P.T. Johnstone: *Sketches of an Elephant: A topos theory compendium: Volume 2*
45. David J. Pym and Eike Ritter: *Reductive Logic and Proof Search: Proof theory, semantics and control*
46. D.M. Gabbay and L. Maksimova: *Interpolation and Definability: Modal and Intuitionistic Logics*
47. John L. Bell: *Set Theory: Boolean-valued models and independence proofs, third edition*
48. Laura Crosilla and Peter Schuster: *From Sets and Types to Topology and Analysis: Towards practicable foundations for constructive mathematics*
49. Steve Awodey: *Category Theory*

# Category Theory

STEVE AWODEY  
*Carnegie Mellon University*

CLARENDON PRESS • OXFORD

2006

# OXFORD

UNIVERSITY PRESS

Great Clarendon Street, Oxford OX2 6DP

Oxford University Press is a department of the University of Oxford.  
It furthers the University's objective of excellence in research, scholarship,  
and education by publishing worldwide in

Oxford New York

Auckland Cape Town Dar es Salaam Hong Kong Karachi  
Kuala Lumpur Madrid Melbourne Mexico City Nairobi  
New Delhi Shanghai Taipei Toronto

With offices in

Argentina Austria Brazil Chile Czech Republic France Greece  
Guatemala Hungary Italy Japan Poland Portugal Singapore  
South Korea Switzerland Thailand Turkey Ukraine Vietnam

Oxford is a registered trade mark of Oxford University Press  
in the UK and in certain other countries

Published in the United States  
by Oxford University Press Inc., New York

© Steve Awodey, 2006

The moral rights of the author have been asserted  
Database right Oxford University Press (maker)

First published 2006

All rights reserved. No part of this publication may be reproduced,  
stored in a retrieval system, or transmitted, in any form or by any means,  
without the prior permission in writing of Oxford University Press,  
or as expressly permitted by law, or under terms agreed with the appropriate  
reprographics rights organization. Enquiries concerning reproduction  
outside the scope of the above should be sent to the Rights Department,  
Oxford University Press, at the address above

You must not circulate this book in any other binding or cover  
and you must impose this same condition on any acquirer

British Library Cataloguing in Publication Data  
Data available

Library of Congress Cataloging in Publication Data  
Data available

Typeset by Newgen Imaging Systems (P) Ltd., Chennai, India  
Printed in Great Britain  
on acid-free paper by  
Biddles Ltd., King's Lynn, Norfolk

ISBN 0-19-856861-4 978-0-19-856861-2

10 9 8 7 6 5 4 3 2 1

*in memoriam*  
Saunders Mac Lane

## PREFACE

Why write a new textbook on Category Theory, when we already have Mac Lane's *Categories for the Working Mathematician*? Simply put, because Mac Lane's book is for the working (and aspiring) mathematician. What is needed now, after 30 years of spreading into various other disciplines and places in the curriculum, is a book for everyone else.

This book has grown from my courses on Category Theory at Carnegie Mellon University over the last 10 years. In that time, I have given numerous lecture courses and advanced seminars to undergraduate and graduate students in Computer Science, Mathematics, and Logic. The lecture course based on the material in this book consists of two, 90-minute lectures a week for 15 weeks. The germ of these lectures was my own graduate student notes from a course on Category Theory given by Mac Lane at the University of Chicago. In teaching my own course, I soon discovered that the mixed group of students at Carnegie Mellon had very different needs than the Mathematics graduate students at Chicago and my search for a suitable textbook to meet these needs revealed a serious gap in the literature. My lecture notes evolved over a time to fill this gap, supplementing and eventually replacing the various texts I tried using.

The students in my courses often have little background in Mathematics beyond a course in Discrete Math and some Calculus or Linear Algebra or a course or two in Logic. Nonetheless, eventually, as researchers in Computer Science or Logic, many will need to be familiar with the basic notions of Category Theory, without the benefit of much further mathematical training. The Mathematics undergraduates are in a similar boat: mathematically talented, motivated to learn the subject by its evident relevance to their further studies, yet unable to follow Mac Lane because they still lack the mathematical prerequisites. Most of my students do not know what a free group is (yet), and so they are not illuminated to learn that it is an example of an adjoint.

This, then, is intended as a text and reference book on Category Theory, not only for students of Mathematics, but also for researchers and students in Computer Science, Logic, Linguistics, Cognitive Science, Philosophy, and any of the other fields that now make use of it. The challenge for me was to make the basic definitions, theorems, and proof techniques understandable to this readership, and thus without presuming familiarity with the main (or at least original) applications in algebra and topology. It will not do, however, to develop the subject in a vacuum, simply skipping the examples and applications. Material at this level of abstraction is simply incomprehensible without the applications and examples that bring it to life.

Faced with this dilemma, I have adopted the strategy of developing a few basic examples from scratch and in detail—namely posets and monoids—and

then carrying them along and using them throughout the book. This has several didactic advantages worth mentioning: both posets and monoids are themselves special kinds of categories, which in a certain sense represent the two “dimensions” (objects and arrows) that a general category has. Many phenomena occurring in categories can best be understood as generalizations from posets or monoids. On the other hand, the categories of posets (and monotone maps) and monoids (and homomorphisms) provide two further, quite different examples of categories in which to consider various concepts. The notion of a limit, for instance, can be considered both in a given poset and in the category of posets.

Of course, many other examples besides posets and monoids are treated as well. For example, the chapter on groups and categories develops the first steps of Group Theory up to kernels, quotient groups, and the homomorphism theorem, as an example of equalizers and coequalizers. Here, and occasionally elsewhere (e.g. in connection with Stone duality), I have included a bit more Mathematics than is strictly necessary to illustrate the concepts at hand. My thinking is that this may be the closest some students will ever get to a higher Mathematics course, so they should benefit from the labor of learning Category Theory by reaping some of the nearby fruits.

Although the mathematical prerequisites are substantially lighter than for Mac Lane, the standard of rigor has (I hope) not been compromised. Full proofs of all important propositions and theorems are given, and only occasional routine lemmas are left as exercises (and these are then usually listed as such at the end of the chapter). The selection of material was easy. There is a standard core that must be included: categories; functors; natural transformations; equivalence; limits and colimits; functor categories; representables; Yoneda’s Lemma; adjoints; and monads. That nearly fills a course. The only “optional” topic included here is cartesian closed categories and the lambda-calculus, which is a must for computer scientists, logicians, and linguists. Several other obvious further topics were purposely not included: 2-categories, toposes (in any depth), and monoidal categories. These topics are treated in Mac Lane, which the student should be able to read after having completed the course.

Finally, I take this opportunity to thank Wilfried Sieg for his exceptional support of this project; Peter Johnstone and Dana Scott for helpful suggestions and support; André Carus for advice and encouragement; Bill Lawvere for many very useful comments on the text; and the many students in my courses who have suggested improvements to the text, clarified the content with their questions, tested all of the exercises, and caught countless errors and typos. For the latter, I also thank the many readers who took the trouble to collect and send helpful corrections, particularly Brighten Godfrey, Peter Gumm, Bob Lubarsky and Dave Perkinson. Andrej Bauer and Kohei Kishida are to be thanked for providing Figures 9.1 and 8.1, respectively. Of course, Paul Taylor’s macros for commutative diagrams must also be acknowledged. And my dear Karin deserves thanks for too many things to mention. Finally, I wish to record here my debt of

gratitude to my mentor Saunders Mac Lane, not only for teaching me category theory, and trying to teach me how to write, but also for helping me to find my place in Mathematics. I dedicate this book to his memory.

*Steve Awodey  
Pittsburgh  
September 2005*

## CONTENTS

<b>Preface</b>	vi
<b>1 Categories</b>	1
1.1 Introduction	1
1.2 Functions of sets	3
1.3 Definition of a category	4
1.4 Examples of categories	5
1.5 Isomorphisms	11
1.6 Constructions on categories	13
1.7 Free categories	16
1.8 Foundations: large, small, and locally small	21
1.9 Exercises	22
<b>2 Abstract structures</b>	25
2.1 Epis and monos	25
2.2 Initial and terminal objects	28
2.3 Generalized elements	29
2.4 Sections and retractions	33
2.5 Products	34
2.6 Examples of products	36
2.7 Categories with products	41
2.8 Hom-sets	42
2.9 Exercises	45
<b>3 Duality</b>	47
3.1 The duality principle	47
3.2 Coproducts	49
3.3 Equalizers	54
3.4 Coequalizers	57
3.5 Exercises	63
<b>4 Groups and categories</b>	65
4.1 Groups in a category	65
4.2 The category of groups	68
4.3 Groups as categories	70
4.4 Finitely presented categories	73
4.5 Exercises	74
<b>5 Limits and colimits</b>	77
5.1 Subobjects	77
5.2 Pullbacks	80

5.3	Properties of pullbacks	84
5.4	Limits	89
5.5	Preservation of limits	94
5.6	Colimits	96
5.7	Exercises	103
<b>6</b>	<b>Exponentials</b>	<b>105</b>
6.1	Exponential in a category	105
6.2	Cartesian closed categories	108
6.3	Heyting algebras	113
6.4	Equational definition	118
6.5	$\lambda$ -calculus	119
6.6	Exercises	123
<b>7</b>	<b>Functors and naturality</b>	<b>125</b>
7.1	Category of categories	125
7.2	Representable structure	127
7.3	Stone duality	131
7.4	Naturality	133
7.5	Examples of natural transformations	135
7.6	Exponentials of categories	139
7.7	Functor categories	142
7.8	Equivalence of categories	146
7.9	Examples of equivalence	150
7.10	Exercises	155
<b>8</b>	<b>Categories of diagrams</b>	<b>159</b>
8.1	Set-valued functor categories	159
8.2	The Yoneda embedding	160
8.3	The Yoneda Lemma	162
8.4	Applications of the Yoneda Lemma	166
8.5	Limits in categories of diagrams	167
8.6	Colimits in categories of diagrams	168
8.7	Exponentials in categories of diagrams	172
8.8	Topoi	174
8.9	Exercises	176
<b>9</b>	<b>Adjoint</b> s	<b>179</b>
9.1	Preliminary definition	179
9.2	Hom-set definition	183
9.3	Examples of adjoints	187
9.4	Order adjoints	191
9.5	Quantifiers as adjoints	193
9.6	RAPL	197
9.7	Locally cartesian closed categories	202

CONTENTS

xi

9.8	Adjoint functor theorem	210
9.9	Exercises	210
<b>10</b>	<b>Monads and algebras</b>	<b>223</b>
10.1	The triangle identities	223
10.2	Monads and adjoints	225
10.3	Algebras for a monad	229
10.4	Comonads and coalgebras	234
10.5	Algebras for endofunctors	236
10.6	Exercises	244
	<b>References</b>	<b>249</b>
	<b>Index</b>	<b>251</b>

