1.9 Propositions as Types

The close correspondence between propositions and types, provided by dependent types and the Σ and Π type-constructors, allows one to do away completely with the distinction between propositions and types, regarding types themselves as propositions. Proof terms are promoted to full-fledged entities, namely terms of the corresponding types. The notion of a constructive proof is then identical with that of a construction of a suitable object. Moreover, the idea of function extraction from proofs becomes superfluous, since it is built in to the system of type theory by the identification of proofs with terms.

This conception is called "Propositions as types,"and was pioneered by Martin-Löf and others [?]. It is often formulated in the slogan:

A proposition is the type of its proofs.

One way to formalize this idea is to strngthen the elimination rule for Σ types, corresponding to the idea that proof terms are terms. Thus one takes instead of the usual elimination rule, the following rules of projections:

$$\frac{\Gamma \vdash t \in \Sigma x{\in}\sigma.\ \tau(x)}{\Gamma \vdash \mathbf{fst}(t) \in \sigma}\ \Sigma E_1$$

$$\frac{\Gamma \vdash t \in \Sigma x{\in}\sigma.\ \tau(x)}{\Gamma \vdash \mathbf{snd}(t) \in \tau(\mathbf{fst}(t))}\ \Sigma E_2$$

Note that this makes the rules for Σ types $\Sigma x \in \sigma.\ \tau$ parallel those for binary products $\sigma \times \tau$, in the way that the rules for dependent products $\Pi x \in \sigma.\ \tau$ parallel those for function types $\sigma \to \tau$.

A remarkable consequence of this point of view is the fact that the famous "Axiom of Choice"—which is unprovable in classical ZF set theory—*is* constructively provable! Intuitively, given types σ and τ and a type family $x \in \sigma, y \in \tau \vdash \rho(x, y)$, and given a "proof" (i.e. a term):

$$p \in \Pi x \in \sigma.\Sigma y \in \tau.\rho(x, y)$$

we can extract a function:

$$\overline{p} \in \sigma \to \tau.\Pi x \in \sigma.\ \rho(x, \overline{p}x)\ .$$

We therefore have a term of type:

$$\Sigma f \in \sigma \to \tau.\Pi x \in \sigma.\ \rho(x, fx)\ .$$

Moreover, since this argument is constructive, we can even find a term of type:

$$(\Pi x \in \sigma.\Sigma y \in \tau.\rho(x, y)) \to (\Sigma f \in \sigma \to \tau.\Pi x \in \sigma.\ \rho(x, fx))\ .$$

But this is exactly the type-theoretic statement of the Axiom of Choice. The formal proof is just as straightforward. We give it in linear notation, using the same notation for Σ and Π as for $\exists$ and $\forall$, respectively:

```
[ p: !x:s.?y:t. R(x,y);
 [ a: s;
  pa: ?y:t. R(a,y);
  fst(pa): t];
 fn x => fst(px): s => t;
 [ a: s;
  pa: ?y:t. R(a,y);
  snd(pa): R(a,fst(pa));
  snd(pa): R(a,(fn x’ => fst(px’))a)];
  fn x => snd(px): !x:s. R(x,(fn x’ => fst(px’))x);
 <fn x => fst(px), fn x => snd(px)>: ?f:s=>t. !x:s. R(x,fx)];
fn p => <fn x => fst(px), fn x => snd(px)>:%
 (!x:s.?y:t. R(x,y)) => (?f:s=>t. !x:s. R(x,fx))
```