1.4 Notational Definition

The judgments, propositions, and inference rules we have defined so far collectively form a system of *natural deduction*. It is a minor variant of a system introduced by Gentzen [Gen35]. One of his main motivations was to devise rules that model mathematical reasoning as directly as possible, although clearly in much more detail than in a typical mathematical argument.

We now consider how to define negation. So far, the meaning of any logical connective has been defined by its introduction rules, from which we derived its elimination rules. The definitions for all the connectives are *orthogonal*: the rules for any of the connectives do not depend on any other connectives, only on basic judgmental concepts. Hence the meaning of a compound proposition depends only on the meaning of its constituent propositions. From the point of view of understanding logical connectives this is a critical property: to understand disjunction, for example, we only need to understand its introduction rules and not any other connectives.

A frequently proposed introduction rule for "*not* A" (written $\neg A$) is

$$\dfrac{\dfrac{}{A\ \mathit{true}}\,u \quad \begin{array}{c}\vdots\\ \bot\ \mathit{true}\end{array}}{\neg A\ \mathit{true}}\,\neg I^u?$$

In words: $\neg A$ is true if the assumption that A is true leads to a contradiction. However, this is not a satisfactory introduction rule, since the premise relies the meaning of $\bot$, violating orthogonality among the connectives. There are several approaches to removing this dependency. One is to introduce a new *judgment*, "A *is false*", and reason explicitly about truth and falsehood. Another employs schematic judgments, which we consider when we introduce universal and existential quantification.

Here we pursue a third alternative: for arbitrary propositions A, we think of $\neg A$ as a syntactic abbreviation for $A \supset \bot$. This is called a *notational definition* and we write

$$\neg A = A \supset \bot.$$

This notational definition is schematic in the proposition A. Implicit here is the formation rule

$$\dfrac{A\ \mathit{prop}}{\neg A\ \mathit{prop}}\,\neg F$$

We allow silent expansion of notational definitions. As an example, we prove that A and $\neg A$ cannot be true simultaneously.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge \neg A\ \mathit{true}}\,u}{\neg A\ \mathit{true}}\,\wedge E_R \qquad \dfrac{\dfrac{}{A \wedge \neg A\ \mathit{true}}\,u}{A\ \mathit{true}}\,\wedge E_L}{\bot\ \mathit{true}}\,\supset E}{\neg(A \wedge \neg A)\ \mathit{true}}\,\supset I^u$$

We can only understand this derivation if we keep in mind that $\neg A$ stands for $A \supset \bot$, and that $\neg(A \wedge \neg A)$ stands for $(A \wedge \neg A) \supset \bot$.

As a second example, we show the proof that $A \supset \neg\neg A$ is true.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\ A\ \mathit{true}}\,w \qquad \dfrac{}{A\ \mathit{true}}\,u}{\bot\ \mathit{true}}\,\supset E}{\neg\neg A\ \mathit{true}}\,\supset I^w}{A \supset \neg\neg A\ \mathit{true}}\,\supset I^u$$

Next we consider $A \vee \neg A$, the so-called *"law" of excluded middle*. It claims that every proposition is either true or false. This, however, contradicts our definition of disjunction: we may have evidence neither for the truth of A, nor for the falsehood of A. Therefore we cannot expect $A \vee \neg A$ to be true unless we have more information about A.

One has to be careful how to interpret this statement, however. There are many propositions A for which it is indeed the case that we know $A \vee \neg A$. For example, $\top \vee (\neg\top)$ is clearly true because $\top$ *true*. Similarly, $\bot \vee (\neg\bot)$ is true because $\neg\bot$ is true. To make this fully explicit:

$$\dfrac{\dfrac{}{\top\ \mathit{true}}\,\top I}{\top \vee (\neg\top)\ \mathit{true}}\,\vee I_L \qquad \dfrac{\dfrac{\dfrac{}{\bot\ \mathit{true}}\,u}{\neg\bot\ \mathit{true}}\,\supset I^u}{\bot \vee (\neg\bot)\ \mathit{true}}\,\vee I_R$$

In mathematics and computer science, many basic relations satisfy the law of excluded middle. For example, we will be able to show that for any two numbers k and n, either $k < n$ or $\neg(k < n)$. However, this requires proof, because for more complex A propositions we may not know if A *true* or $\neg A$ *true*. We will return to this issue later in this course.

At present we do not have the tools to show formally that $A \vee \neg A$ should not be true for arbitrary A. A proof attempt with our generic proof strategy (reason from the bottom up with introduction rules and from the top down with elimination rules) fails quickly, no matter which introduction rule for disjunction we start with.

$$\dfrac{\begin{array}{c}\vdots\\ A\ \mathit{true}\end{array}}{A \lor \neg A\ \mathit{true}}\lor I_L \qquad \dfrac{\dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \vdots\\ \bot\ \mathit{true}\end{array}}{\neg A\ \mathit{true}}\supset I^u}{A\lor\neg A\ \mathit{true}}\lor I_R$$

We will see that this failure is in fact sufficient evidence to know that $A \lor \neg A$ is not true for arbitrary A.

1.5 Derived Rules of Inference

One popular device for shortening derivations is to introduce *derived rules of inference*. For example,

$$\dfrac{A \supset B\ \mathit{true} \qquad B \supset C\ \mathit{true}}{A \supset C\ \mathit{true}}$$

is a derived rule of inference. Its derivation is the following:

$$\dfrac{\dfrac{\dfrac{\overline{A\ \mathit{true}}^{\,u} \qquad A \supset B\ \mathit{true}}{B\ \mathit{true}}\supset E \qquad B \supset C\ \mathit{true}}{C\ \mathit{true}}\supset E}{A \supset C\ \mathit{true}}\supset I^u$$

Note that this is simply a hypothetical derivation, using the premises of the derived rule as assumptions. In other words, a derived rule of inference is nothing but an evident hypothetical judgment; its justification is a hypothetical derivation.

We can freely use derived rules in proofs, since any occurrence of such a rule can be expanded by replacing it with its justification.

A second example of notational definition is logical equivalence "*A if and only if B*" (written $A \equiv B$). We define

$$(A \equiv B) = (A \supset B) \land (B \supset A).$$

That is, two propositions A and B are logically equivalent if A implies B and B implies A. Under this definition, the following become derived rules of inference (see Exercise 1.1). They can also be seen as introduction and elimination rules for logical equivalence (whence their names).

$$\dfrac{\begin{array}{cc} \overline{A\ \mathit{true}}^{\,u} & \overline{B\ \mathit{true}}^{\,w} \\ \vdots & \vdots \\ B\ \mathit{true} & A\ \mathit{true} \end{array}}{A \equiv B\ \mathit{true}} \equiv I^{u,w}$$

$$\dfrac{A \equiv B\ \mathit{true} \qquad A\ \mathit{true}}{B\ \mathit{true}} \equiv E_L \qquad\qquad \dfrac{A \equiv B\ \mathit{true} \qquad B\ \mathit{true}}{A\ \mathit{true}} \equiv E_R$$

1.6 Logical Equivalences

We now consider several classes of logical equivalences in order to develop some intuitions regarding the truth of propositions. Each equivalence has the form $A \equiv B$, but we consider only the basic connectives and constants ($\wedge$, $\supset$, $\vee$, $\top$, $\bot$) in A and B. Later on we consider negation as a special case. We use some standard conventions that allow us to omit some parentheses while writing propositions. We use the following operator precedences

$$\neg > \wedge > \vee > \supset > \equiv$$

where $\wedge$, $\vee$, and $\supset$ are right associative. For example

$$\neg A \supset A \vee \neg\neg A \supset \bot$$

stands for

$$(\neg A) \supset ((A \vee (\neg(\neg A))) \supset \bot)$$

In ordinary mathematical usage, $A \equiv B \equiv C$ stands for $(A \equiv B) \wedge (B \equiv C)$; in the formal language we do not allow iterated equivalences without explicit parentheses in order to avoid confusion with propositions such as $(A \equiv A) \equiv \top$.

Commutativity. Conjunction and disjunction are clearly commutative, while implication is not.

(C1) $A \wedge B \equiv B \wedge A$ *true*

(C2) $A \vee B \equiv B \vee A$ *true*

(C3) $A \supset B$ is not commutative

Idempotence. Conjunction and disjunction are idempotent, while self-implication reduces to truth.

(I1) $A \wedge A \equiv A$ *true*

(I2) $A \vee A \equiv A$ *true*

(I3) $A \supset A \equiv \top$ *true*

Interaction Laws. These involve two interacting connectives. In principle, there are left and right interaction laws, but because conjunction and disjunction are commutative, some coincide and are not repeated here.

(L1) $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$ *true*

(L2) $A \wedge \top \equiv A$ *true*

(L3) $A \wedge (B \supset C)$ do not interact

(L4) $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ *true*

(L5) $A \wedge \bot \equiv \bot$ *true*

(L6) $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$ *true*

(L7) $A \vee \top \equiv \top$ *true*

(L8) $A \vee (B \supset C)$ do not interact

(L9) $A \vee (B \vee C) \equiv (A \vee B) \vee C$ *true*

(L10) $A \vee \bot \equiv A$ *true*

(L11) $A \supset (B \wedge C) \equiv (A \supset B) \wedge (A \supset C)$ *true*

(L12) $A \supset \top \equiv \top$ *true*

(L13) $A \supset (B \supset C) \equiv (A \wedge B) \supset C$ *true*

(L14) $A \supset (B \vee C)$ do not interact

(L15) $A \supset \bot$ do not interact

(L16) $(A \wedge B) \supset C \equiv A \supset (B \supset C)$ *true*

(L17) $\top \supset C \equiv C$ *true*

(L18) $(A \supset B) \supset C$ do not interact

(L19) $(A \vee B) \supset C \equiv (A \supset C) \wedge (B \supset C)$ *true*

(L20) $\bot \supset C \equiv \top$ *true*

1.7 Summary of Judgments

Judgments.

A prop	A is a proposition
A true	Proposition A is true

Propositional Constants and Connectives. The following table summarizes the introduction and elimination rules for the propositional constants ($\top$, $\bot$) and connectives ($\wedge$, $\supset$, $\vee$). We omit the straightforward formation rules.

Introduction Rules | Elimination Rules

$$\frac{A\ \mathit{true} \qquad B\ \mathit{true}}{A \wedge B\ \mathit{true}}\wedge I \qquad\qquad \frac{A \wedge B\ \mathit{true}}{A\ \mathit{true}}\wedge E_L \qquad \frac{A \wedge B\ \mathit{true}}{B\ \mathit{true}}\wedge E_R$$

$$\frac{}{\top\ \mathit{true}}\top I \qquad\qquad \textit{no } \top E \textit{ rule}$$

$$\frac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \vdots \\ B\ \mathit{true}\end{array}}{A \supset B\ \mathit{true}}\supset I^u \qquad\qquad \frac{A \supset B\ \mathit{true} \qquad A\ \mathit{true}}{B\ \mathit{true}}\supset E$$

$$\frac{A\ \mathit{true}}{A \vee B\ \mathit{true}}\vee I_L \qquad \frac{B\ \mathit{true}}{A \vee B\ \mathit{true}}\vee I_R \qquad\qquad \frac{A \vee B\ \mathit{true} \quad \begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \vdots \\ C\ \mathit{true}\end{array} \quad \begin{array}{c}\overline{B\ \mathit{true}}^{\,w}\\ \vdots \\ C\ \mathit{true}\end{array}}{C\ \mathit{true}}\vee E^{u,w}$$

$$\textit{no } \bot I \textit{ rule} \qquad\qquad \frac{\bot\ \mathit{true}}{C\ \mathit{true}}\bot E$$

Notational Definitions. We use the following notational definitions.

$$\begin{array}{lcll} \neg A & = & A \supset \bot & \text{not } A \\ A \equiv B & = & (A \supset B) \wedge (B \supset A) & A \text{ if and only if } B \end{array}$$

1.8 A Linear Notation for Proofs

The two-dimensional format for rules of inference and deductions is almost universal in the literature on logic. Unfortunately, it is not well-suited for writing actual proofs of complex propositions, because deductions become very unwieldy. Instead with use a linearized format explained below. Furthermore, since logical symbols are not available on a keyboard, we use the following concrete syntax for propositions:

<table>
<tr><td>$A \equiv B$</td><td><code>A <=> B</code></td><td>A if and only if B</td></tr>
<tr><td>$A \supset B$</td><td><code>A => B</code></td><td>A implies B</td></tr>
<tr><td>$A \vee B$</td><td><code>A | B</code></td><td>A or B</td></tr>
<tr><td>$A \wedge B$</td><td><code>A & B</code></td><td>A and B</td></tr>
<tr><td>$\neg A$</td><td><code>~ A</code></td><td>not A</td></tr>
</table>

The operators are listed in order of increasing binding strength, and implication (`=>`), disjunction (`|`), and conjunction (`&`) associate to the right, just like the corresponding notation from earlier in this chapter.

The linear format is mostly straightforward. A proof is written as a sequence of judgments separated by semi-colon '`;`'. Later judgements must follow from earlier ones by simple applications of rules of inference. Since it can easily be verified that this is the case, explicit justifications of inferences are omitted. Since the only judgment we are interested in at the moment is the truth of a proposition, the judgment "A *true*" is abbreviated simply as "A".

The only additional notation we need is for hypothetical proofs. A hypothetical proof

$$\begin{array}{c} A\ \mathit{true} \\ \vdots \\ C\ \mathit{true} \end{array}$$

is written as `[A;...;C]`.

In other words, the hypothesis A is immediately preceded by a square bracket ('`[`'), followed by the lines representing the hypothetical proof of C, followed by a closing square bracket ('`]`'). So square brackets are used to delimit the scope of an assumption. If we need more than hypothesis, we nest this construct as we will see in the example below.

As an example, we consider the proof of $(A \supset B) \wedge (B \supset C) \supset (A \supset C)$ *true*. We show each stage in the proof during its natural construction, showing both the mathematical and concrete syntax, except that we omit the judgment "*true*" to keep the size of the derivation manageable. We write '`...`' to indicate that the following line has not yet been justified.

$$\begin{array}{c} \vdots \\ (A \supset B) \wedge (B \supset C) \supset (A \supset C) \end{array}$$

```
...
(A => B) & (B => C) => (A => C);
```

The first bottom-up step is an implication introduction. In the linear form, we use our notation for hypothetical judgments.

$$\dfrac{\begin{array}{c} \overline{(A \supset B) \wedge (B \supset C)}^{\,u} \\ \vdots \\ A \supset C \end{array}}{(A \supset B) \wedge (B \supset C) \supset (A \supset C)} \supset I^u$$

```
[ (A => B) & (B => C);
  ...
  A => C ];
(A => B) & (B => C) => (A => C);
```

Again, we proceed via an implication introduction. In the mathematical notation, the hypotheses are shown next to each other. In the linear notation, the second hypothesis `A` is nested inside the first, also making both of them available to fill the remaining gap in the proof.

$$
\dfrac{\dfrac{\begin{array}{cc}\overline{(A \supset B) \wedge (B \supset C)}^{\,u} & \overline{A}^{\,w} \\ \vdots & \\ C & \end{array}}{A \supset C}\,{\supset}I^w}{(A \supset B) \wedge (B \supset C) \supset (A \supset C)}\,{\supset}I^u
$$

```
[ (A => B) & (B => C);
  [ A;
    ...
    C ];
  A => C ];
(A => B) & (B => C) => (A => C);
```

Now that the conclusion is atomic and cannot be decomposed further, we reason downwards from the hypotheses. In the linear format, we write the new line `A => B;` immediately below the hypothesis, but we could also have inserted it directly below `A;`. In general, the requirement is that the lines representing the premise of an inference rule must all come before the conclusion. Furthermore, lines cannot be used outside the hypothetical proof in which they appear, because their proof could depend on the hypothesis.

$$
\dfrac{\dfrac{\begin{array}{cc}\dfrac{\overline{(A \supset B) \wedge (B \supset C)}^{\,u}}{A \supset B}\,{\wedge}E_L & \overline{A}^{\,w} \\ \vdots & \\ C & \end{array}}{A \supset C}\,{\supset}I^w}{(A \supset B) \wedge (B \supset C) \supset (A \supset C)}\,{\supset}I^u
$$

```
[ (A => B) & (B => C);
  A => B;
  [ A;
    ...
    C ];
  A => C ];
(A => B) & (B => C) => (A => C);
```

Nex we apply another straightforward top-down reasoning step. In this case, there is no choice on where to insert `B;`.

$$
\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{\overline{(A \supset B) \wedge (B \supset C)}^{\,u}}{A \supset B}\,{\wedge}E_L \quad \overline{A}^{\,w}}{B}\,{\supset}E \\ \vdots \\ C \end{array}}{A \supset C}\,{\supset}I^w}{(A \supset B) \wedge (B \supset C) \supset (A \supset C)}\,{\supset}I^u
$$

```
[ (A => B) & (B => C);
  A => B;
  [ A;
    B;
    ...
    C ];
  A => C ];
(A => B) & (B => C) => (A => C);
```

For the last two steps, we align the derivations vertically. The are both top-down steps (conjunction elimination followed by implication elimination).

$$\dfrac{}{(A \supset B) \wedge (B \supset C)}\,u \qquad \dfrac{\dfrac{}{(A \supset B) \wedge (B \supset C)}\,u}{A \supset B}\,\wedge E_L \qquad \dfrac{}{A}\,w$$

$$\dfrac{(A \supset B) \wedge (B \supset C)}{B \supset C}\,\wedge E_R \qquad \dfrac{A \supset B \qquad A}{B}\,\supset E$$

$$\vdots$$

$$\dfrac{\dfrac{C}{A \supset C}\,\supset I^w}{(A \supset B) \wedge (B \supset C) \supset (A \supset C)}\,\supset I^u$$

```
[ (A => B) & (B => C);
  A => B;
  B => C;
  [ A;
    B;
    ...
    C ];
  A => C ];
(A => B) & (B => C) => (A => C);
```

In the step above we notice that subproofs may be shared in the linearized format, while in the tree format they appear more than once. In this case it is only the hypothesis $(A \supset B) \wedge (B \supset C)$ which is shared.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{(A \supset B) \wedge (B \supset C)}\,u}{B \supset C}\,\wedge E_R \qquad \dfrac{\dfrac{\dfrac{}{(A \supset B) \wedge (B \supset C)}\,u}{A \supset B}\,\wedge E_L \qquad \dfrac{}{A}\,w}{B}\,\supset E}{C}\,\supset E}{\dfrac{A \supset C}{(A \supset B) \wedge (B \supset C) \supset (A \supset C)}\,\supset I^u}\,\supset I^w$$

```
[ (A => B) & (B => C);
  A => B;
  B => C;
  [ A;
    B;
    C ];
  A => C ];
(A => B) & (B => C) => (A => C);
```

In the last step, the linear derivation only changed in that we noticed that `C` already follows from two other lines and is therefore justified.