For other details of concrete syntax and usage of the proof-checking program available for this course, please refer to the on-line documentation available through the course home page.

1.9 Normal Deductions

The strategy we have used so far in proof search is easily summarized: we reason with introduction rules from the bottom up and with elimination rules from the top down, hoping that the two will meet in the middle. This description is somewhat vague in that it is not obvious how to apply it to complex rules such as disjunction elimination which involve formulas other than the principal one whose connective is eliminated.

To make this precise we introduce two new judgments

$A\uparrow$ A has a normal proof
$A\downarrow$ A has a neutral proof

We are primarily interest in normal proofs, which are those that our strategy can find. Neutral proofs represent an auxiliary concept (sometimes called an *extraction proof*) necessary for the definition of normal proofs.

We will define these judgments via rules, trying to capture the following intuitions:

1. A normal proof is either neutral, or proceeds by applying introduction rules to other normal proofs.
2. A neutral proof proceeds by applying elimination rules to hypotheses or other neutral proofs.

By construction, every A which has a normal (or neutral) proof is true. The converse, namely that every true A has a normal proof also holds, but is not at all obvious. We may prove this property later on, at least for a fragment of the logic.

First, a general rule to express that every neutral proof is normal.

$$\frac{A\downarrow}{A\uparrow}\downarrow\uparrow$$

Conjunction. The rules for conjunction are easily annotated.

$$\frac{A\uparrow \qquad B\uparrow}{A\wedge B\uparrow}\wedge I \qquad \frac{A\wedge B\downarrow}{A\downarrow}\wedge E_L \qquad \frac{A\wedge B\downarrow}{B\downarrow}\wedge E_R$$

Truth. Truth only has an introduction rule and therefore no neutral proof constructor.

$$\frac{}{\top\uparrow}\top I$$

Implication. Implication first fixes the idea that hypotheses are neutral, so the introduction rule refers to both normal and neutral deductions.

$$\dfrac{\begin{array}{c}\overline{A\downarrow}^{\,u}\\ \vdots \\ B\uparrow\end{array}}{A\supset B\uparrow}\supset I^u \qquad \dfrac{A\supset B\downarrow \qquad A\uparrow}{B\downarrow}\supset E$$

The elimination rule is more difficult to understand. The principal premise (with the connective "⊃" we are eliminating) should have a neutral proof. The resulting derivation will once again be neutral, but we can only require the second premise to have a normal proof.

Disjunction. For disjunction, the introduction rules are straightforward. The elimination rule requires again the requires the principal premise to have a neutral proof. An the assumptions introduced in both branches are also neutral. In the end we can conclude that we have a normal proof of the conclusion, if we can find a normal proof in each premise.

$$\dfrac{A\uparrow}{A\vee B\uparrow}\vee I_L \qquad \dfrac{B\uparrow}{A\vee B\uparrow}\vee I_R \qquad \dfrac{A\vee B\downarrow \quad \begin{array}{c}\overline{A\downarrow}^{\,u}\\ \vdots \\ C\uparrow\end{array} \quad \begin{array}{c}\overline{B\downarrow}^{\,w}\\ \vdots \\ C\uparrow\end{array}}{C\uparrow}\vee E^{u,w}$$

Falsehood. Falsehood is analogous to the rules for disjunction. But since there are no introduction rules, there are no cases to consider in the elimination rule.

$$\dfrac{\bot\downarrow}{C\uparrow}\bot E$$

All the proofs we have seen so far in these notes are normal: we can easily annotate them with arrows using only the rules above. The following is an example of a proof which is not normal.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A\ \mathit{true}}\,w \qquad \dfrac{\dfrac{\dfrac{}{A\ \mathit{true}}\,u \quad \dfrac{}{\neg A\ \mathit{true}}\,w}{A \wedge \neg A\ \mathit{true}}\,\wedge I}{A\ \mathit{true}}\,\wedge E_L}{\bot\ \mathit{true}}\,\supset E}{\neg A \supset \bot\ \mathit{true}}\,\supset I^w}{A \supset \neg A \supset \bot\ \mathit{true}}\,\supset I^u$$

If we follow the process of annotation, we fail at only one place as indicated below.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A \downarrow}\,w \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \downarrow}\,u \quad \dfrac{}{\neg A \downarrow}\,w}{A \wedge \neg A\ ?}\,\wedge I}{A \downarrow}\,\wedge E_L}{A \uparrow}\,\downarrow\uparrow}{\bot \downarrow}\,\supset E}{\bot \uparrow}\,\downarrow\uparrow}{\neg A \supset \bot \uparrow}\,\supset I^w}{A \supset \neg A \supset \bot \uparrow}\,\supset I^u$$

The situation that prevents this deduction from being normal is that we introduce a connective (in this case, $A \wedge \neg A$) and then immediately eliminate it. This seems like a detour—why do it at all? In fact, we can just replace this little inference with the hypothesis $A \downarrow$ and obtain a deduction which is now normal.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg A \downarrow}\,w \qquad \dfrac{\dfrac{}{A \downarrow}\,u}{A \uparrow}\,\downarrow\uparrow}{\bot \downarrow}\,\supset E}{\bot \uparrow}\,\downarrow\uparrow}{\neg A \supset \bot \uparrow}\,\supset I^w}{A \supset \neg A \supset \bot \uparrow}\,\supset I^u$$

It turns out that the only reason a deduction may not be normal is an introduction followed by an elimination, and that we can always simplify such a derivation to (eventually) obtain a normal one. This process of simplification is directly connected to computation in a programming language. We only need to fix a particular simplification strategy. Under this interpretation, a proof corresponds to a program, simplification of the kind above corresponds to computation, and a normal proof corresponds to a value. It is precisely this correspondence which is the central topic of the next chapter.

We close this chapter with our first easy meta-theorem, that is, a theorem *about* a logical system rather than within it. We show that if a the proposition A has a normal proof then it must be true. In order to verify this, we also need the auxiliary property that if A has a neutral proof, it is true.

Theorem 1.1 (Soundness of Normal Proofs) *For natural deduction with logical constants* $\wedge$, $\supset$, $\vee$, $\top$ *and* $\bot$ *we have:*

1. *If* $A \uparrow$ *then* A *true, and*
2. *if* $A \downarrow$ *then* A *true.*

Proof: We replace every judgment $B \uparrow$ and $B \downarrow$ in the deduction of $A \uparrow$ or $A \downarrow$ by B *true* and B *true*. This leads to correct derivation that A *true* with one exception: the rule

$$\frac{B \downarrow}{B \uparrow} \downarrow\uparrow$$

turns into

$$\frac{B\ \mathit{true}}{B\ \mathit{true}}$$

We can simply delete this "inference" since premise and conclusion are identical. □

1.10 Exercises

Exercise 1.1 *Show the derivations for the rules* $\equiv I$, $\equiv E_L$ *and* $\equiv E_R$ *under the definition of* $A \equiv B$ *as* $(A \supset B) \wedge (B \supset A)$.