Now we can close the gap in the left-hand side by conjunction elimination.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \supset (B \wedge C)\ \mathit{true}}u \quad \dfrac{}{A\ \mathit{true}}w}{B \wedge C\ \mathit{true}}{\supset}E}{B\ \mathit{true}}{\wedge}E_L}{A \supset B\ \mathit{true}}{\supset}I^w \qquad \begin{array}{c}\dfrac{}{A \supset (B \wedge C)\ \mathit{true}}u \\ \vdots \\ A \supset C\ \mathit{true}\end{array}}{\dfrac{(A \supset B) \wedge (A \supset C)\ \mathit{true}}{(A \supset (B \wedge C)) \supset ((A \supset B) \wedge (A \supset C))\ \mathit{true}}{\supset}I^u}{\wedge}I$$

The right premise of the conjunction introduction can be filled in analogously. We skip the intermediate steps and only show the final derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \supset (B \wedge C)\ \mathit{true}}u \quad \dfrac{}{A\ \mathit{true}}w}{B \wedge C\ \mathit{true}}{\supset}E}{B\ \mathit{true}}{\wedge}E_L}{A \supset B\ \mathit{true}}{\supset}I^w \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \supset (B \wedge C)\ \mathit{true}}u \quad \dfrac{}{A\ \mathit{true}}v}{B \wedge C\ \mathit{true}}{\supset}E}{C\ \mathit{true}}{\wedge}E_R}{A \supset C\ \mathit{true}}{\supset}I^v}{\dfrac{(A \supset B) \wedge (A \supset C)\ \mathit{true}}{(A \supset (B \wedge C)) \supset ((A \supset B) \wedge (A \supset C))\ \mathit{true}}{\supset}I^u}{\wedge}I$$

1.3 Disjunction and Falsehood

So far we have explained the meaning of conjunction, truth, and implication. The disjunction "A *or* B" (written as $A \vee B$) is more difficult, but does not require any new judgment forms.

$$\dfrac{A\ \mathit{prop} \qquad B\ \mathit{prop}}{A \vee B\ \mathit{prop}}{\vee}F$$

Disjunction is characterized by two introduction rules: $A \vee B$ is true, if either A or B is true.

$$\dfrac{A\ \mathit{true}}{A \vee B\ \mathit{true}}{\vee}I_L \qquad \dfrac{B\ \mathit{true}}{A \vee B\ \mathit{true}}{\vee}I_R$$

Now it would be incorrect to have an elimination rule such as

$$\dfrac{A \vee B\ \mathit{true}}{A\ \mathit{true}}{\vee}E_L?$$

because even if we know that $A \vee B$ is true, we do not know whether the disjunct A or the disjunct B is true. Concretely, with such a rule we could derive the truth of *every* proposition A as follows:

$$\dfrac{\dfrac{\overline{B\ \textit{true}}\ u}{B \supset B\ \textit{true}}\supset I^u \qquad \dfrac{\dfrac{\dfrac{\overline{B \supset B\ \textit{true}}\ w}{A \lor (B \supset B)\ \textit{true}}\lor I_R}{A\ \textit{true}}\lor E_L?}{(B \supset B) \supset A\ \textit{true}}\supset I^w}{A\ \textit{true}}\supset E$$

Thus we take a different approach. If we know that $A \lor B$ is true, we must consider two cases: A *true* and B *true*. If we can prove a conclusion C *true* in both cases, then C must be true! Written as an inference rule:

$$\dfrac{A \lor B\ \textit{true} \qquad \begin{matrix}\overline{A\ \textit{true}}\ u \\ \vdots \\ C\ \textit{true}\end{matrix} \qquad \begin{matrix}\overline{B\ \textit{true}}\ w \\ \vdots \\ C\ \textit{true}\end{matrix}}{C\ \textit{true}}\lor E^{u,w}$$

Note that we use once again the mechanism of hypothetical judgments. In the proof of the second premise we may use the assumption A *true* labeled u, in the proof of the third premise we may use the assumption B *true* labeled w. Both are discharged at the disjunction elimination rule.

Let us justify the conclusion of this rule more explicitly. By the first premise we know $A \lor B$ *true*. The premises of the two possible introduction rules are A *true* and B *true*. In case A *true* we conclude C *true* by the substitution principle and the second premise: we substitute the proof of A *true* for any use of the assumption labeled u in the hypothetical derivation. The case for B *true* is symmetric, using the hypothetical derivation in the third premise.

Because of the complex nature of the elimination rule, reasoning with disjunction is more difficult than with implication and conjunction. As a simple example, we prove the commutativity of disjunction.

$$\begin{matrix}\vdots \\ (A \lor B) \supset (B \lor A)\ \textit{true}\end{matrix}$$

We begin with an implication introduction.

$$\dfrac{\begin{matrix}\overline{A \lor B\ \textit{true}}\ u \\ \vdots \\ B \lor A\ \textit{true}\end{matrix}}{(A \lor B) \supset (B \lor A)\ \textit{true}}\supset I^u$$

At this point we cannot use either of the two disjunction introduction rules. The problem is that neither B nor A follow from our assumption $A \vee B$! So first we need to distinguish the two cases via the rule of disjunction elimination.

$$\dfrac{\dfrac{}{A \vee B\ \mathit{true}}u \quad \begin{array}{c}\dfrac{}{A\ \mathit{true}}v \\ \vdots \\ B \vee A\ \mathit{true}\end{array} \quad \begin{array}{c}\dfrac{}{B\ \mathit{true}}w \\ \vdots \\ B \vee A\ \mathit{true}\end{array}}{\dfrac{B \vee A\ \mathit{true}}{(A \vee B) \supset (B \vee A)\ \mathit{true}} \supset I^u} \vee E^{v,w}$$

The assumption labeled u is still available for each of the two proof obligations, but we have omitted it, since it is no longer needed.

Now each gap can be filled in directly by the two disjunction introduction rules.

$$\dfrac{\dfrac{}{A \vee B\ \mathit{true}}u \quad \dfrac{\dfrac{}{A\ \mathit{true}}v}{B \vee A\ \mathit{true}}\vee I_R \quad \dfrac{\dfrac{}{B\ \mathit{true}}w}{B \vee A\ \mathit{true}}\vee I_L}{\dfrac{B \vee A\ \mathit{true}}{(A \vee B) \supset (B \vee A)\ \mathit{true}} \supset I^u} \vee E^{v,w}$$

This concludes the discussion of disjunction. Falsehood (written as $\bot$, sometimes called absurdity) is a proposition that should have no proof! Therefore there are no introduction rules, although we of course have the standard formation rule.

$$\dfrac{}{\bot\ \mathit{prop}}\bot F$$

Since there cannot be a proof of $\bot$ *true*, it is sound to conclude the truth of any arbitrary proposition if we know $\bot$ *true*. This justifies the elimination rule

$$\dfrac{\bot\ \mathit{true}}{C\ \mathit{true}}\bot E$$

We can also think of falsehood as a disjunction between zero alternatives. By analogy with the binary disjunction, we therefore have zero introduction rules, and an elimination rule in which we have to consider zero cases. This is precisely the $\bot E$ rule above.

From this is might seem that falsehood it useless: we can never prove it. This is correct, except that we might reason from contradictory hypotheses! We will see some examples when we discuss negation, since we may think of the proposition "*not* A" (written $\neg A$) as $A \supset \bot$. In other words, $\neg A$ is true precisely if the assumption A *true* is contradictory because we could derive $\bot$ *true*.

1.4 Notational Definition

The judgments, propositions, and inference rules we have defined so far collectively form a system of *natural deduction*. It is a minor variant of a system introduced by Gentzen [Gen35]. One of his main motivations was to devise rules that model mathematical reasoning as directly as possible, although clearly in much more detail than in a typical mathematical argument.

We now consider how to define negation. So far, the meaning of any logical connective has been defined by its introduction rules, from which we derived its elimination rules. The definitions for all the connectives are *orthogonal*: the rules for any of the connectives do not depend on any other connectives, only on basic judgmental concepts. Hence the meaning of a compound proposition depends only on the meaning of its constituent propositions. From the point of view of understanding logical connectives this is a critical property: to understand disjunction, for example, we only need to understand its introduction rules and not any other connectives.

A frequently proposed introduction rule for "*not* A" (written $\neg A$) is

$$\dfrac{\dfrac{}{A\ \mathit{true}}\,u \\ \vdots \\ \bot\ \mathit{true}}{\neg A\ \mathit{true}}\,\neg I^u?$$

In words: $\neg A$ is true if the assumption that A is true leads to a contradiction. However, this is not a satisfactory introduction rule, since the premise relies the meaning of $\bot$, violating orthogonality among the connectives. There are several approaches to removing this dependency. One is to introduce a new *judgment*, "A *is false*", and reason explicitly about truth and falsehood. Another employs schematic judgments, which we consider when we introduce universal and existential quantification.

Here we pursue a third alternative: for arbitrary propositions A, we think of $\neg A$ as a syntactic abbreviation for $A \supset \bot$. This is called a *notational definition* and we write

$$\neg A = A \supset \bot.$$

This notational definition is schematic in the proposition A. Implicit here is the formation rule

$$\dfrac{A\ \mathit{prop}}{\neg A\ \mathit{prop}}\,\neg F$$

We allow silent expansion of notational definitions. As an example, we prove that A and $\neg A$ cannot be true simultaneously.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \wedge \neg A\ \mathit{true}}\,u}{\neg A\ \mathit{true}}\,\wedge E_R \qquad \dfrac{\dfrac{}{A \wedge \neg A\ \mathit{true}}\,u}{A\ \mathit{true}}\,\wedge E_L}{\bot\ \mathit{true}}\,\supset E}{\neg(A \wedge \neg A)\ \mathit{true}}\,\supset I^u$$

We can only understand this derivation if we keep in mind that $\neg A$ stands for $A \supset \bot$, and that $\neg(A \wedge \neg A)$ stands for $(A \wedge \neg A) \supset \bot$.

As a second example, we show the proof that $A \supset \neg\neg A$ is true.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\ A\ \mathit{true}}\,w \qquad \dfrac{}{A\ \mathit{true}}\,u}{\bot\ \mathit{true}}\,\supset E}{\neg\neg A\ \mathit{true}}\,\supset I^w}{A \supset \neg\neg A\ \mathit{true}}\,\supset I^u$$

Next we consider $A \vee \neg A$, the so-called *"law" of excluded middle*. It claims that every proposition is either true or false. This, however, contradicts our definition of disjunction: we may have evidence neither for the truth of A, nor for the falsehood of A. Therefore we cannot expect $A \vee \neg A$ to be true unless we have more information about A.

One has to be careful how to interpret this statement, however. There are many propositions A for which it is indeed the case that we know $A \vee \neg A$. For example, $\top \vee (\neg\top)$ is clearly true because $\top$ *true*. Similarly, $\bot \vee (\neg\bot)$ is true because $\neg\bot$ is true. To make this fully explicit:

$$\dfrac{\dfrac{}{\top\ \mathit{true}}\,\top I}{\top \vee (\neg\top)\ \mathit{true}}\,\vee I_L \qquad \dfrac{\dfrac{\dfrac{}{\bot\ \mathit{true}}\,u}{\neg\bot\ \mathit{true}}\,\supset I^u}{\bot \vee (\neg\bot)\ \mathit{true}}\,\vee I_R$$

In mathematics and computer science, many basic relations satisfy the law of excluded middle. For example, we will be able to show that for any two numbers k and n, either $k < n$ or $\neg(k < n)$. However, this requires proof, because for more complex A propositions we may not know if A *true* or $\neg A$ *true*. We will return to this issue later in this course.

At present we do not have the tools to show formally that $A \vee \neg A$ should not be true for arbitrary A. A proof attempt with our generic proof strategy (reason from the bottom up with introduction rules and from the top down with elimination rules) fails quickly, no matter which introduction rule for disjunction we start with.

$$\dfrac{\begin{array}{c}\vdots\\ A\ \mathit{true}\end{array}}{A \lor \neg A\ \mathit{true}}\lor I_L \qquad\qquad \dfrac{\dfrac{\begin{array}{c}\overline{A\ \mathit{true}}^{\,u}\\ \vdots\\ \bot\ \mathit{true}\end{array}}{\neg A\ \mathit{true}}\supset I^u}{A \lor \neg A\ \mathit{true}}\lor I_R$$

We will see that this failure is in fact sufficient evidence to know that $A \lor \neg A$ is not true for arbitrary A.

1.5 Derived Rules of Inference

One popular device for shortening derivations is to introduce *derived rules of inference*. For example,

$$\dfrac{A \supset B\ \mathit{true} \qquad B \supset C\ \mathit{true}}{A \supset C\ \mathit{true}}$$

is a derived rule of inference. Its derivation is the following:

$$\dfrac{\dfrac{\dfrac{\overline{A\ \mathit{true}}^{\,u} \qquad A \supset B\ \mathit{true}}{B\ \mathit{true}}\supset E \qquad B \supset C\ \mathit{true}}{C\ \mathit{true}}\supset E}{A \supset C\ \mathit{true}}\supset I^u$$

Note that this is simply a hypothetical derivation, using the premises of the derived rule as assumptions. In other words, a derived rule of inference is nothing but an evident hypothetical judgment; its justification is a hypothetical derivation.

We can freely use derived rules in proofs, since any occurrence of such a rule can be expanded by replacing it with its justification.

A second example of notational definition is logical equivalence "*A if and only if B*" (written $A \equiv B$). We define

$$(A \equiv B) = (A \supset B) \land (B \supset A).$$

That is, two propositions A and B are logically equivalent if A implies B and B implies A. Under this definition, the following become derived rules of inference (see Exercise 1.1). They can also be seen as introduction and elimination rules